Rend. Sem. Mat. Univ. Pol. Torino
Fascicolo Speciale 1991
Numerical Methods

A.L. Eterovic - K.J. Bathe

ON LARGE STRAIN ELASTO-PLASTIC ANALYSIS WITH FRICTIONAL CONTACT CONDITIONS

Abstract. This paper presents a framework for finite element analysis of large deformation elasto-plastic problems with frictional contact conditions. The elasto-plastic constitutive formulation is derived from basic principles. The formulation is hyperelastic-based, uses the logarithmic strain tensor, and allows for both isotropic and kinematic hardening. The interface formulation enforces all contact conditions explicitly in the governing equations. The proposed methods are aimed to provide consistent, reliable and efficient solution procedures for engineering analysis of large deformation elasto-plastic problems.

1 - Introduction

The finite element analysis of large strain elasto-plastic problems involving contact conditions has attracted much interest during the recent years. Such analysis capabilities are for example needed for the numerical prediction of the forces and strains in metal forming problems. If the forming of metals can be predicted in a *reliable* and *accurate* manner, considerable resources can be saved in the design of manufacturing equipment and the production of components, as encountered for example in the automotive and aircraft industries.

Although a number of finite element programs have been developed and applied to large strain elasto-plastic problems involving contact, there is still considerable need to further increase the general applicability, reliability and efficiency of the available solution schemes. Large strain finite element analysis requires a physically correct continuum mechanics formulation, appropriate finite element discretization and an efficient solution of the governing equations [1]. For the reliability of the complete solution scheme, the methods used must be mechanistically clear and numerically well-founded.

The objective in this paper is to present a consistent formulation for large strain elasto-plastic analysis [2] and to briefly introduce a new approach for the analysis of contact problems. The elasto-plastic formulation is already in wide use, and the proposed contact procedure might provide a significant extension of a Lagrange multiplier method also widely used already for contact problems [3,4,5].

The major attributes of the procedures we discuss herein are that they are consistent and reliable from the continuum mechanics and algorithmic points of view, and that they are intended for general engineering analysis.

2 - Material formulation

In this section we derive a set of elasto-plastic constitutive equations that characterize the material response. Let $\mathbf{X} = D_{o_X}\mathbf{x}$ be the deformation gradient and $\mathbf{L} = D_{t_X}\mathbf{v} = \dot{\mathbf{X}}\mathbf{X}^{-1}$ be the velocity gradient.

2.1 - *The reduced dissipation inequality*

Consider the field equations associated with the first and second law of thermodynamics

$$\rho\dot{\varepsilon} = \mathbf{T}\cdot\mathbf{D} + \rho r - \nabla\cdot\mathbf{q} \tag{1}$$

$$\rho\dot{\eta} \geq \rho\frac{r}{\theta} - \nabla\cdot\frac{\mathbf{q}}{\theta} \tag{2}$$

where ρ is the density in the current configuration, ε is the internal energy per unit mass, η is the entropy per unit mass, $\mathbf{D} =$ sym $\mathbf{L}$ is the stretching tensor, r is the heat supply per unit mass, $\mathbf{q}$ is the heat flux vector, and θ is the temperature. Expanding the divergence operator in (2) and using (1) to eliminate r we obtain

$$-\rho(\dot{\varepsilon} - \theta\dot{\eta}) + \mathbf{T}\cdot\mathbf{D} - \frac{\mathbf{q}}{\theta}\cdot\nabla\theta \geq 0\ . \tag{3}$$

We next perform a transformation to eliminate the entropy rate and introduce the rate of change of temperature. For this purpose, let $\psi = \varepsilon - \theta\eta$

be the free energy per unit mass. Substitution in (3) yields the reduced dissipation inequality

$$-\rho(\dot{\psi}+\eta\dot{\theta})+\mathbf{T}\cdot\mathbf{D}-\frac{\mathbf{q}}{\theta}\cdot\nabla\theta\geq 0\ . \tag{4}$$

We restrict ourselves to isothermal processes where the temperature field is constant over space and time. In this case $\dot{\theta}=0$, $\nabla\theta=0$ and (4) reduces to

$$-\rho\dot{\psi}+\mathbf{T}\cdot\mathbf{D}\geq 0\ . \tag{5}$$

2.2 - *Description of plastic flow*

We use the product decomposition $\mathbf{X} = \mathbf{X}^e\mathbf{X}^p$ of the deformation gradient [6], where $\mathbf{X}^e$ and $\mathbf{X}^p$ are respectively the elastic and plastic deformation gradients. The plastic deformation gradient takes the reference configuration into an intermediate configuration, obtained conceptually by unloading a neighborhood of each particle from the current configuration to a state of zero stress in such a way that no inelastic process takes place during the deformation [7, 8, 9, 10]. We have $\dot{\mathbf{X}} = \dot{\mathbf{X}}^e\mathbf{X}^p + \mathbf{X}^e\dot{\mathbf{X}}^p$ and $\mathbf{X}^{-1} = (\mathbf{X}^p)^{-1}(\mathbf{X}^e)^{-1}$, so the velocity gradient can be written

$$\mathbf{L}=\mathbf{L}^e+\mathbf{L}^p \tag{6}$$

with $\mathbf{L}^e = \dot{\mathbf{X}}^e(\mathbf{X}^e)^{-1}$ and $\mathbf{L}^p = \mathbf{X}^e\dot{\mathbf{X}}^p(\mathbf{X}^p)^{-1}(\mathbf{X}^e)^{-1}$. The elastic and plastic stretching tensors are, respectively, given by $\mathbf{D}^e = \operatorname{sym}\mathbf{L}^e$ and $\mathbf{D}^p = \operatorname{sym}\mathbf{L}^p$.

Since plastic deformation is considered isochoric $\det\mathbf{X}^p = 1$, so $\det\mathbf{X}^e = \det\mathbf{X} = J > 0$ and the elastic deformation gradient admits the polar decomposition

$$\mathbf{X}^e=\mathbf{R}^e\mathbf{U}^e \tag{7}$$

where $\mathbf{R}^e$ is the elastic rotation tensor and $\mathbf{U}^e$ is the elastic right stretch tensor. We use the Hencky strain tensor $\mathbf{E}^e = \ln\mathbf{U}^e$ and its elastic work conjugate stress tensor $\overline{\mathbf{T}}$, defined by [1]

$$\overline{\mathbf{T}}\cdot\dot{\mathbf{E}}^e=J\mathbf{T}\cdot\mathbf{D}^e\ . \tag{8}$$

Equation (8) yields $\overline{\mathbf{T}} = \mathcal{T}(\mathbf{U}^e)\mathbf{T}$, where $\mathcal{T}(\mathbf{U}^e)$ is a linear operator acting on $\mathbf{T}$ that depends on $\mathbf{U}^e$ [11].

[1] Note that this definition differs from that in reference [2] by the factor J. While both definitions can be equally used, for small elastic strains the resulting difference in $\overline{\mathbf{T}}$ is negligible and definition (8) simplifies the introduction of the free enthalpy function.

Multiplying (5) by J, recalling that by mass conservation $\rho J = {}^0\rho$, where ${}^0\rho$ is the density in the reference configuration, and using (6) and (8) we obtain for the reduced dissipation inequality

$$-{}^0\rho\dot{\psi} + \overline{\mathbf{T}} \cdot \dot{\mathbf{E}}^e + J\mathbf{T} \cdot \mathbf{D}^p \geq 0 \ . \tag{9}$$

It is now convenient to exchange the strain rate for the stress rate. For that purpose we define the free enthaply per unit reference volume as $u = {}^0\rho\psi - \overline{\mathbf{T}} \cdot \mathbf{E}^e$. Differentiating with respect to time and substituting in (9) we obtain

$$-\dot{u} - \mathbf{E}^e \cdot \dot{\overline{\mathbf{T}}} + J\mathbf{T} \cdot \mathbf{D}^p \geq 0 \ . \tag{10}$$

2.3 - *State variables*

We consider the plastic state of the material as characterized by the two internal variables σ and $\mathbf{B}$. The "deformation resistance" σ is a scalar that represents an average macroscopic resistance to plastic flow. The "back stress" $\mathbf{B}$ is a symmetric second order tensor with zero trace and represents an average intensity of the microscopic residual stresses [12]. Both σ and $\mathbf{B}$ have the dimension of stress. To be consistent with the choice of stress measure in (8), we define a referential back stress by $\overline{\mathbf{B}} = \mathcal{T}(\mathbf{U}^e)\mathbf{B}$.

The list of state variables that characterize the elasto-plastic process is then taken to be $\tau = (\overline{\mathbf{T}}, \sigma, \overline{\mathbf{B}})$. We seek a stress-strain law for $\mathbf{E}^e$ and evolution equations for $\mathbf{D}^p$, σ and $\overline{\mathbf{B}}$ in terms of τ and $\dot{\tau}$. Since the free enthalpy is a function of the state, we have $u = u(\tau)$, and equation (10) can be written as

$$-\left(\mathbf{E}^e + \frac{\partial u}{\partial \overline{\mathbf{T}}}\right) \cdot \dot{\overline{\mathbf{T}}} - \frac{\partial u}{\partial \sigma}\dot{\sigma} - \frac{\partial u}{\partial \overline{\mathbf{B}}} \cdot \dot{\overline{\mathbf{B}}} + J\mathbf{T} \cdot \mathbf{D}^p \geq 0 \ . \tag{11}$$

This equation is interpreted as to hold for all stress rates $\dot{\overline{\mathbf{T}}}$, see for example reference [13]. Necessary conditions are then

$$\mathbf{E}^e = -\frac{\partial u}{\partial \overline{\mathbf{T}}} \tag{12}$$

$$\mathcal{D}(\tau, \dot{\tau}) = -\frac{\partial u}{\partial \sigma}\dot{\sigma} - \frac{\partial u}{\partial \overline{\mathbf{B}}} \cdot \dot{\overline{\mathbf{B}}} + J\mathbf{T} \cdot \mathbf{D}^p \geq 0 \tag{13}$$

where $\mathcal{D}(\tau, \dot{\tau})$ is the dissipation function.

2.4 - *Stress-strain law*

We adopt the following special form of the free enthalpy function

$$u(\tau) = u(\overline{\mathbf{T}}, \sigma, \overline{\mathbf{B}}) = -\frac{1}{2}\overline{\mathbf{T}} \cdot \mathcal{C}\,\overline{\mathbf{T}} + \frac{\sigma^2}{2\beta H} + \frac{3\,\overline{\mathbf{B}} \cdot \overline{\mathbf{B}}}{4(1-\beta)H} \tag{14}$$

where H is the plastic hardening modulus, $\beta \in (0,1)$ is a fixed number and $\mathcal{C}$ is the compliance tensor, given by

$$\mathcal{C} = \frac{1}{2\mu}\mathbf{I} + \left(\frac{1}{9\kappa} - \frac{1}{6\mu}\right) \mathbf{1} \otimes \mathbf{1} \quad . \tag{15}$$

In this equation $\mathbf{1}$ and $\mathbf{I}$ are respectively the second and fourth order identity tensors, and μ and κ are taken to be respectively the shear modulus and the bulk modulus of small-strain elasticity.

Using (14) in (12) we obtain $\mathbf{E}^e = \mathcal{C}\overline{\mathbf{T}}$, or equivalently

$$\overline{\mathbf{T}} = \mathcal{L}\,\mathbf{E}^e \tag{16}$$

where $\mathcal{L} = \mathcal{C}^{-1}$ is the isotropic elastic moduli tensor, given by

$$\mathcal{L} = 2\mu\mathbf{I} + \left(\kappa - \frac{2}{3}\mu\right) \mathbf{1} \otimes \mathbf{1} \ . \tag{17}$$

For the isotropic stress-strain law (16) the stress and strain tensors commute, and $\overline{\mathbf{T}}$ and $\overline{\mathbf{B}}$ take the simple form

$$\overline{\mathbf{T}} = J(\mathbf{R}^e)^T\mathbf{T}\mathbf{R}^e \tag{18}$$

$$\overline{\mathbf{B}} = J(\mathbf{R}^e)^T\mathbf{B}\mathbf{R}^e \ . \tag{19}$$

In this case we call $\overline{\mathbf{T}}$ the "rotated stress tensor" and $\overline{\mathbf{B}}$ the "rotated back stress tensor".

In view of the symmetry of the Cauchy stress tensor, the definition of $\mathbf{D}^p$ below (6), equation (18), and the polar decomposition (7), the last term in (13) can be written as

$$J\mathbf{T} \cdot \mathbf{L}^p = J\mathbf{T} \cdot \mathbf{X}^e\overline{\mathbf{L}}^p(\mathbf{X}^e)^{-1} = \mathbf{U}^e\overline{\mathbf{T}}(\mathbf{U}^e)^{-1} \cdot \overline{\mathbf{L}}^p = \overline{\mathbf{T}} \cdot \overline{\mathbf{L}}^p \tag{20}$$

where by definition $\overline{\mathbf{L}}^p = (\mathbf{X}^e)^{-1}\mathbf{L}^p\mathbf{X}^e = \dot{\mathbf{X}}^p(\mathbf{X}^p)^{-1}$ and the last equality follows from the fact that $\overline{\mathbf{T}}$ commutes with $\mathbf{U}^e$. Using (14) and (20) we obtain for the dissipation function

$$\mathcal{D}(\tau, \dot{\tau}) = -\frac{\sigma\dot{\sigma}}{\beta H} - \frac{3\,\overline{\mathbf{B}} \cdot \dot{\overline{\mathbf{B}}}}{2(1-\beta)H} + \overline{\mathbf{T}} \cdot \overline{\mathbf{D}}^p \geq 0 \ . \tag{21}$$

2.5 - *Yield surface*

Define the "effective stress tensor" by $\overline{\mathbf{S}} = \overline{\mathbf{T}}' - \overline{\mathbf{B}}$, where $\overline{\mathbf{T}}'$ is the deviatoric part of $\overline{\mathbf{T}}$, and the "effective stress" s by

$$s = \sqrt{\frac{3}{2}\overline{\mathbf{S}} \cdot \overline{\mathbf{S}}} \, . \tag{22}$$

The standard yield surface for combined isotropic-kinematic hardening is given by

$$\phi(\tau) = \sigma - s = 0 \, . \tag{23}$$

In rate-independent plasticity, the state variables $\tau = (\overline{\mathbf{T}}, \sigma, \overline{\mathbf{B}})$ are constrained to satisfy $\phi(\tau) \geq 0$. Furthermore, no plastic deformation takes place if $\phi(\tau) > 0$. Recalling that for any second order tensor $\overline{\mathbf{T}}$

$$\frac{\partial \overline{\mathbf{T}}'}{\partial \overline{\mathbf{T}}} = \mathbf{I} - \frac{1}{3}\mathbf{1} \otimes \mathbf{1} \tag{24}$$

we have for the derivatives of $\phi(\tau)$

$$\frac{\partial \phi}{\partial \overline{\mathbf{T}}} = -\sqrt{\frac{3}{2}}\overline{\mathbf{N}} \tag{25}$$

$$\frac{\partial \phi}{\partial \sigma} = 1 \tag{26}$$

$$\frac{\partial \phi}{\partial \overline{\mathbf{B}}} = \sqrt{\frac{3}{2}}\overline{\mathbf{N}} \tag{27}$$

where $\overline{\mathbf{N}}$ is the unit normal to the yield surface

$$\overline{\mathbf{N}} = \sqrt{\frac{3}{2}}\frac{\overline{\mathbf{S}}}{s} \, . \tag{28}$$

2.6 - *Evolution equations*

So far we have derived the condition that for all state variables τ and all rates $\dot{\tau}$ the dissipation function $\mathcal{D}(\tau, \dot{\tau})$ has to be nonnegative. We now strengthen this condition with the principle of maximum plastic dissipation [14, 15] as follows. For fixed rates $\dot{\tau}$, the actual state variables τ are such that plastic dissipation is maximized subject to the constraint $\phi(\tau) \geq 0$, i.e. τ is determined from the problem

$$\max_{\tau} \mathcal{D}(\tau, \dot{\tau}) \text{ such that } \phi(\tau) \geq 0 \, . \tag{29}$$

Denoting by $\dot{e}^p$ the Lagrange multiplier associated with the inequality constraint in (29), the first order necessary conditions for a maximum are [16]

$$\frac{\partial \mathcal{D}}{\partial \tau} + \dot{e}^p \frac{\partial \phi}{\partial \tau} = 0 \tag{30}$$

$$\dot{e}^p \geq 0 \quad \phi \geq 0 \quad \dot{e}^p \phi = 0 \,. \tag{31}$$

Using equations (21) and (25-27) we obtain

$$\overline{\mathbf{D}}^p - \dot{e}^p \sqrt{\frac{3}{2}}\, \overline{\mathbf{N}} = \mathbf{0} \tag{32}$$

$$-\frac{\dot{\sigma}}{\beta H} + \dot{e}^p = 0 \tag{33}$$

$$-\frac{3}{2(1-\beta)H} \dot{\overline{\mathbf{B}}} + \dot{e}^p \sqrt{\frac{3}{2}}\, \overline{\mathbf{N}} = \mathbf{0} \tag{34}$$

or equivalenty

$$\overline{\mathbf{D}}^p = \sqrt{\frac{3}{2}}\, \dot{e}^p \overline{\mathbf{N}} \tag{35}$$

$$\dot{\sigma} = \beta H\, \dot{e}^p \tag{36}$$

$$\dot{\overline{\mathbf{B}}} = \sqrt{\frac{2}{3}} (1-\beta) H\, \dot{e}^p \overline{\mathbf{N}} \,. \tag{37}$$

These, together with the assumption $\overline{\mathbf{W}}^p = \operatorname{skw} \overline{\mathbf{L}}^p = \mathbf{0}$, constitute the evolution equations for the plastic variables. Note that for $\beta \rightarrow 1$ we recover the isotropic hardening case while for $\beta \rightarrow 0$ the purely kinematic hardening limit is obtained. It follows from (35) that $\dot{e}^p$ satisfies

$$\dot{e}^p = \sqrt{\frac{2}{3} \overline{\mathbf{D}}^p \cdot \overline{\mathbf{D}}^p} \,. \tag{38}$$

This justifies the denomination "effective plastic strain rate" for $\dot{e}^p$.

We summarize below our constitutive equation for further reference

$$\overline{\mathbf{T}} = \mathcal{L}\, \mathbf{E}^e \tag{39}$$

$$\dot{\mathbf{X}}^p = \overline{\mathbf{D}}^p \mathbf{X}^p \tag{40}$$

$$\overline{\mathbf{D}}^p = \sqrt{\frac{3}{2}}\, \dot{e}^p \overline{\mathbf{N}} \tag{41}$$

$$\dot{\sigma} = \beta H\, \dot{e}^p \tag{42}$$

$$\dot{\overline{\mathbf{B}}} = \frac{2}{3} (1-\beta)\, H \overline{\mathbf{D}}^p \tag{43}$$

where for convenience (43) combines equations (35) and (37).

The above equations are solved in the ADINA program using the effective-stress-function algorithm [2, 17].

3 - Contact formulation

Consider two bodies and a system of loads such that various contact conditions are established during the motion. Let Γ^I and Γ^J be the part of the boundaries where contact between body I and body J may occur. For convenience, we call Γ^I the "contactor surface" and Γ^J the "target surface".

Let x be a point on the contactor surface. Let y^* be a point on the target surface that minimizes the distance to $\mathbf{x}$,

$$||\mathbf{x} - \mathbf{y}^*|| = \min\{||\mathbf{x} - \mathbf{y}|| \ : \ \mathbf{y} \in \Gamma^J\} \ . \tag{44}$$

We define the gap function g on the contactor surface by

$$g(\mathbf{x}) = (\mathbf{x} - \mathbf{y}^*) \cdot \mathbf{n} \tag{45}$$

where $\mathbf{n}$ is the normal to the target surface at $\mathbf{y}^*$.

Let us decompose the contact tractions onto body I into a scalar normal component λ and a vector tangent component $\mathbf{t}$. Actually, $\mathbf{t}$ is a scalar or a vector depending on whether two or three-dimensional problems are considered. The conditions for normal contact can then be summarized by the complementarity conditions

$$g \geq 0 \quad \lambda \geq 0 \quad g\lambda = 0 \ . \tag{46}$$

If there is no contact, the gap g is greater than zero, and the third condition in (46) implies that the normal contact traction λ must vanish. Conversely, if the normal contact traction is strictly positive then the gap must be zero.

We assume that the classical Coulomb's law of friction holds pointwise on the contact region. This law states that the frictional resistance $\mu\lambda$, where μ is the coefficient of friction, is always greater than or equal to the norm of the frictional force $\mathbf{t}$. If $\mu\lambda$ is strictly greater than $||\mathbf{t}||$ we have sticking contact, and the relative velocity $\dot{\mathbf{u}}^{IJ}$ between the two bodies is zero. If $\mu\lambda$ is equal to $||\mathbf{t}||$ we have sliding contact, and $\dot{\mathbf{u}}^{IJ}$ must be in the direction of the frictional force, i.e. $\dot{\mathbf{u}}^{IJ} = \gamma\mathbf{t}$ for some $\gamma \geq 0$. We note that we here define $\dot{\mathbf{u}}^{IJ}$ to represent the velocity of point $\mathbf{y}$ on body J relative to point $\mathbf{x}$ on body I, where points $\mathbf{y}$ and $\mathbf{x}$ are in contact at time t, and $\mathbf{t}$ to represent the tangential traction acting onto body I.

It is possible to restate these conditions in terms of a set of complementarity equations as follows. Define

$$h = \mu\lambda - \|\mathbf{t}\| \tag{47}$$

then Coulomb's law of friction can be written as

$$h \geq 0 \quad \gamma \geq 0 \quad h\gamma = 0 \tag{48}$$

$$\lambda(\dot{\mathbf{u}}^{IJ} - \gamma\mathbf{t}) = \mathbf{0} \; . \tag{49}$$

If there is no contact, then $g > 0$ and $\lambda = 0$, and it follows from (47) and the first condition in (48) that $\mathbf{t} = \mathbf{0}$. Also, equation (49) is trivially satisfied for any value of $\dot{\mathbf{u}}^{IJ}$. Conversely, suppose that $\lambda > 0$. If $h > 0$, the frictional resistance exceeds the frictional force and the third condition in (48) and (49) imply $\dot{\mathbf{u}}^{IJ} = \mathbf{0}$. Hence, in this case there is no relative motion. If on the other hand sliding does occur, i.e. $\dot{\mathbf{u}}^{IJ} \neq \mathbf{0}$, it follows from (49) and the second condition in (48) that $\gamma > 0$. The third condition in (48) then gives $h = 0$, meaning that the norm of the frictional force equals the frictional resistance.

In order to enforce all inequalities arising from the contact conditions we use the following approach. Let $w(x, y)$ be a continuously differentiable function such that $w(x, y) = 0$ if and only if $x \geq 0$, $y \geq 0$ and $xy = 0$. Then conditions (46) and (48) are equivalent to

$$w(g, \lambda) = 0 \tag{50}$$

$$w(h, \gamma) = 0 \; . \tag{51}$$

Let $\mathbf{s}_1$ and $\mathbf{s}_2$ be orthogonal unit tangent vectors on the target surface, and let the frictional force be written as

$$\mathbf{t} = t_1\mathbf{s}_1 + t_2\mathbf{s}_2 \tag{52}$$

where $t_1 = \mathbf{t}\cdot\mathbf{s}_1$ and $t_2 = \mathbf{t}\cdot\mathbf{s}_2$. Then (49) is equivalent to the scalar equations

$$\lambda(\dot{\mathbf{u}}^{IJ} \cdot \mathbf{s}_1 - \gamma t_1) = 0 \tag{53}$$

$$\lambda(\dot{\mathbf{u}}^{IJ} \cdot \mathbf{s}_2 - \gamma t_2) = 0 \; . \tag{54}$$

We use equations (50-51) and (53-54) to complete the formulation of the equilibrium equations.

4 - Equilibrium equations

For the solution of large strain elasto-plastic contact problems we combine the developments of the last two sections. We partition the relevant time interval into a sequence of time-steps. Assuming quasistatic conditions, inertia forces are neglected and the relative interface velocity $\dot{\mathbf{u}}^{IJ}$ at time $t+\Delta t$ is approximated by $\Delta\mathbf{u}^{IJ}/\Delta t$, where $\Delta\mathbf{u}^{IJ}$ is the change in the relative interface displacement from time t to time $t+\Delta t$. With these simplifications, given the solution at time t we seek displacements and contact tractions at time $t+\Delta t$ that satisfy

$$\begin{aligned}\int_{^0V} J\mathbf{T}\mathbf{X}^{-T}\cdot \text{Grad}\ \overline{\mathbf{u}}\,{}^0dV - \int_{^0V} {}^0\rho\mathbf{b}\cdot\overline{\mathbf{u}}\,{}^0dV - \int_{^0\partial V^f} {}^0\mathbf{f}\cdot\overline{\mathbf{u}}\,{}^0dS \\ - \int_{\partial V^c} \mathbf{f}\cdot\overline{\mathbf{u}}\,dS = 0\end{aligned} \tag{55}$$

$$\begin{aligned}\int_{\partial V^c} [w(g,\lambda)\overline{\lambda} + w(h,\gamma)\overline{\gamma} + \lambda(\mathbf{s}_1\cdot\Delta\mathbf{u}^{IJ}/\Delta t - \gamma t_1)\overline{t}_1 \\ + \lambda(\mathbf{s}_2\cdot\Delta\mathbf{u}^{IJ}/\Delta t - \gamma t_2)\overline{t}_2]dS = 0\end{aligned} \tag{56}$$

where $\overline{\mathbf{u}}$ is a virtual displacement field, Grad indicates the gradient with respect to the original coordinates, $\mathbf{b}$ is the body force per unit mass, ${}^0\mathbf{f}$ is the traction per unit reference surface, and $\overline{\lambda}$, $\overline{\gamma},\overline{t}_1$ and $\overline{t}_2$ are virtual fields associated with the contact traction degrees of freedom λ,γ,t_1 and t_2, respectively.

We note that the relation in (55) is of course the principle of virtual work including the unknown contact tractions on ∂V^c [1]. These forces are given in terms of $\lambda\mathbf{n}$ and $\mathbf{t}$. For the finite element solution, the first integral in (55) is still modified to correspond to the u/p formulation to allow for almost incompressible analysis conditions [18, 19].

The relation in (56) expresses the contact constraints as obtained in the previous section.

A two point implicit time integration algorithm is used to solve equations (39-43) for the updated Cauchy stresses and plastic variables. The relations in (55) and (56) are then solved by a Newton-Raphson iteration with line searching.

Note that the specific difference in this contact formulation to previously proposed formulations is that all contact constraints are explicitly enforced in the governing equations.

5 - Conclusions

The objective in this paper was to present a material model and an interface model for the formulation of large-strain elasto-plastic finite element analysis with frictional contact conditions.

The large strain elasto-plastic model has been presented starting from the basic laws of thermodynamics and clearly delineating the assumptions used. The contact formulation represents a new approach in that all contact conditions are explicitly enforced in the governing equations.

The formulations given are intended for wide use in engineering applications such as the analysis of metal forming problems.

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Adrian Luis ETEROVIC - Klaus-Jürgen BATHE
Massachusetts Institute of Technology
Cambridge MA – 02139 – U.S.A