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ON MIXED ELEMENTS FOR ACOUSTIC FLUID-STRUCTURE INTERACTIONS

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In this paper we investigate the performance of some mixed finite elements used in the displacement/pressure ($\mathbf{u}/p$) and displacement-pressure-vorticity moment ($\mathbf{u}$-p-$\mathbf{\Lambda}$) formulations for acoustic fluid-structure interactions. In particular, we show that certain elements pass a numerical inf-sup test and are valuable for general applications. Also considered are macroelements that are based on simple four-node elements.

1. Introduction

For the past two decades, much research effort has gone into the application of finite element procedures to fluid flows and fluid-structure interactions.[1] With the ever-increasing availability of high speed and large capacity computers, the primitive variable formulations show great promise in general applications to the solution of a broad range of fluid-structure interaction problems.

As part of these research efforts, it has been widely reported that the displacement-based fluid elements employed in frequency or dynamic analyses of acoustic fluid-structure interaction problems exhibit spurious *nonzero* frequency circulation modes.[5,6,8] Various approaches have been introduced to obtain improved formulations, however, only recently, the true origins of the spurious nonzero frequency rotational modes have been identified.[2,10] It was concluded that the reported nonzero frequency spurious modes are caused either by the pure displacement formulation, the use of mixed elements which do not satisfy the inf-sup condition, or the improper treatment of the boundary conditions. Therefore, a proper way of eliminating the nonzero frequency spurious modes is to use the displacement/pressure based mixed formulation with elements satisfying the inf-sup condition. In addition to the traditional $\mathbf{u}/p$ formulation (as used for solids with zero shear modulus),[10] an effective three-field mixed finite element formulation ($\mathbf{u}$-p-$\mathbf{\Lambda}$) was presented in Ref. 2.

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In this paper, we briefly review the $\mathbf{u}/p$ and $\mathbf{u}$-p-$\mathbf{\Lambda}$ formulations, and we then study numerically the stability and effectiveness of various elements. We achieve this by investigating whether the elements pass a numerical inf-sup test and by solving a paradigm of acoustic fluid-structure interactions. We conclude that various valuable elements are available. However, although two macroelements constructed with 4/1 elements satisfy the inf-sup condition in the $\mathbf{u}/p$ formulation,[3,7] we see numerically that the corresponding macroelements with 4-1-1 elements do not satisfy the inf-sup condition in the $\mathbf{u}$-p-$\mathbf{\Lambda}$ formulation.

2. Governing Equations of Solid

Consider first a continuous solid medium in a Cartesian coordinate system. The momentum conservation equation is commonly written as

$$\rho \ddot{u}_i = \tau_{ij,j} + f_i^B \tag{1}$$

where ρ, $\mathbf{u}$, $\boldsymbol{\tau}$ and $\mathbf{f}^B$ are the mass density, displacement vector, stress tensor and body force vector.

In linear elasticity, the components of the strain tensor ε and the deviatoric strain tensor ε' are defined as

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{2}$$

$$\varepsilon'_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij} \tag{3}$$

and the constitutive relation can be written as

$$\tau_{ij} = \kappa\varepsilon_{kk}\delta_{ij} + 2G\varepsilon'_{ij}, \tag{4}$$

where κ and G are the bulk and shear moduli.

The pure displacement-based formulation can be written as follows.

Find $\mathbf{u} \in V = (H^1_{0,S_u}(\Omega))^n$ such that

$$a(\mathbf{u},\mathbf{v}) = (\mathbf{f},\mathbf{v}), \quad \forall\, \mathbf{v} \in V$$

where Ω is the $\mathcal{R}^n$ domain with a continuous boundary $\partial\Omega = S$ (which consists of the Dirichlet boundary S_u and the Neumann boundary S_f), and

$$a(\mathbf{u},\mathbf{v}) = 2G\int_\Omega \varepsilon'(\mathbf{u}) : \varepsilon'(\mathbf{v}) d\Omega + \kappa\int_\Omega (\nabla\cdot\mathbf{u})(\nabla\cdot\mathbf{v}) d\Omega,$$

$$(\mathbf{f},\mathbf{v}) = \int_\Omega \mathbf{f}^B\cdot\mathbf{v} d\Omega + \int_{S_f} \mathbf{f}^{S_f}\cdot\mathbf{v}^{S_f} dS,$$

$$S = S_f \cup S_u, \quad S_f \cap S_u = \emptyset,$$

$$H^1_{0,S_u}(\Omega) = \{\mathbf{v}|\mathbf{v}\in H^1(\Omega), \quad \mathbf{v}|_{S_u} = \mathbf{0}\}.$$

Of course, for finite element discretization, the above pure displacement-based formulation is not appropriate when general incompressible (or almost incompressible) conditions are considered; instead, the following mixed formulation is effective.[1,3]

Find $\mathbf{u} \in V = (H^1_{0,S_u}(\Omega))^n$ and $p \in Q = L^2(\Omega)$ such that

$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}), \quad \forall\, \mathbf{v} \in V \tag{5}$$

$$b(\mathbf{u}, q) = 0, \quad \forall\, q \in Q \tag{6}$$

where

$$b(\mathbf{v}, q) = -\int_\Omega q\left(\frac{p}{\kappa} + \nabla \cdot \mathbf{v}\right) d\Omega\,.$$

In finite element analysis, we search for the solution within the spaces $V_h \subset V$ and $Q_h \subset Q$. For the mixed finite element discretization to be reliable and effective, the following inf-sup condition should be satisfied[1,3]

$$\inf_{q_h \in Q_h} \sup_{\mathbf{v}_h \in V_h} \frac{\displaystyle\int_\Omega q_h \nabla \cdot \mathbf{v}_h d\Omega}{\|q_h\|\,\|\mathbf{v}_h\|} \geq c > 0\,, \tag{7}$$

where c is a constant independent of h and the bulk modulus.

3. Acoustic Fluid Model

The irrotational and isentropic acoustic fluid model can be seen as an elastic continuous medium with a zero shear modulus. Since the coupling between fluids and solids is directly imposed by the element assemblage process, no special interface conditions are considered. We assume that the slipping boundary conditions are imposed as presented in Refs. 2 and 10.

3.1. u/p *formulation*

We define a variational indicator

$$\Pi = \int_\Omega \left\{\frac{p^2}{2\kappa} - \mathbf{u} \cdot \mathbf{f}^B - \lambda_p\left(\frac{p}{\kappa} + \nabla \cdot \mathbf{u}\right)\right\} d\Omega + \int_{S_f} \overline{p}\, u_n^S\, dS\,,$$

where the variables are p, $\mathbf{u}$, and the Lagrange multiplier λ_p.

Invoking the stationarity of Π, we identify the Lagrange multiplier λ_p to be the pressure p and obtain the governing equations

$$\nabla p - \mathbf{f}^B = \mathbf{0}\,, \tag{8}$$

$$\nabla \cdot \mathbf{u} + \frac{p}{\kappa} = 0\,, \tag{9}$$

with the inertia force $-\rho\ddot{\mathbf{u}}$ included in $\mathbf{f}^B$ and the boundary conditions

$$\begin{aligned} \mathbf{u} \cdot \mathbf{n} &= \overline{u}_n \qquad \text{on } S_u\,, \\ p &= \overline{p} \qquad \text{on } S_f\,. \end{aligned} \tag{10}$$

It is clear that Eqs. (8) and (9) are the momentum and mass conservation equations. The pressure $\overline{p}$ is commonly assigned to be zero on the free surface if we do not include effects of surface gravity waves. To introduce such effects, we simply add a surface wave potential term $\int_{S_s} \frac{1}{2}\rho g u_s^2 dS$, where S_s stands for the free surface.

Using the standard procedure of interpolating, and considering a typical element, we have

$$\begin{aligned} \mathbf{u} &= \mathbf{H}\widehat{\mathbf{U}}\,, \\ p &= \mathbf{H}_p\widehat{\mathbf{P}}\,, \\ \nabla \cdot \mathbf{u} &= (\nabla \cdot \mathbf{H})\widehat{\mathbf{U}} = \mathbf{B}\widehat{\mathbf{U}}\,. \end{aligned}$$

The matrix equations for the formulation are

$$\begin{pmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{Bmatrix} \ddot{\widehat{\mathbf{U}}} \\ \ddot{\widehat{\mathbf{P}}} \end{Bmatrix} + \begin{pmatrix} \mathbf{0} & \mathbf{L} \\ \mathbf{L}^{\mathrm{T}} & \mathbf{A} \end{pmatrix} \begin{Bmatrix} \widehat{\mathbf{U}} \\ \widehat{\mathbf{P}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{R} \\ \mathbf{0} \end{Bmatrix}, \tag{11}$$

where

$$\begin{aligned} \mathbf{M} &= \int_\Omega \rho \mathbf{H}^{\mathrm{T}}\mathbf{H} d\Omega\,, & \mathbf{L} &= -\int_\Omega \mathbf{B}^{\mathrm{T}}\mathbf{H}_p d\Omega\,, \\ \mathbf{A} &= -\int_\Omega \frac{1}{\kappa}\mathbf{H}_p^{\mathrm{T}}\mathbf{H}_p d\Omega\,, & \mathbf{R} &= -\int_{S_f} \mathbf{H}_n^{S^{\mathrm{T}}}\overline{p} dS\,. \end{aligned}$$

3.2. u-p-Λ *formulation*

For the isentropic and irrotational acoustic fluid model with conservative body forces, we can derive from Eq. (8) an irrotationality constraint[5,8]

$$\nabla \times \mathbf{u} = \mathbf{0}\,. \tag{12}$$

In order to impose this constraint, we introduce

$$\nabla \times \mathbf{u} = \frac{\boldsymbol{\Lambda}}{\alpha}\,, \tag{13}$$

where $\mathbf{\Lambda}$ is a "vorticity moment". The magnitude of $\mathbf{\Lambda}$ shall be small while α is a constant of large value.[2]

A variational indicator including the constraint of Eq. (13) is given as

$$\Pi = \int_{\Omega} \left\{ \frac{p^2}{2\kappa} - \mathbf{u}\cdot\mathbf{f}^B - \lambda_p \left(\frac{p}{\kappa} + \nabla\cdot\mathbf{u}\right) + \frac{\mathbf{\Lambda}\cdot\mathbf{\Lambda}}{2\alpha} - \lambda_{\mathbf{\Lambda}}\cdot\left(\frac{\mathbf{\Lambda}}{\alpha} - \nabla\times\mathbf{u}\right)\right\} d\Omega + \int_{S_f} \overline{p}\, u_n \, dS\,, \tag{14}$$

where the variables are p, $\mathbf{u}$, $\mathbf{\Lambda}$, and the Lagrange multipliers λ_p and $\lambda_{\mathbf{\Lambda}}$.

Invoking the stationarity of Π, we identify the Langrange multipliers λ_p and $\lambda_{\mathbf{\Lambda}}$ to be the pressure p and vorticity moment $\mathbf{\Lambda}$, respectively, and we obtain the field equation

$$\nabla p - \mathbf{f}^B + \nabla\times\mathbf{\Lambda} = \mathbf{0}\,, \tag{15}$$

$$\nabla\cdot\mathbf{u} + \frac{p}{\kappa} = 0\,, \tag{16}$$

$$\nabla\times\mathbf{u} - \frac{\mathbf{\Lambda}}{\alpha} = \mathbf{0}\,, \tag{17}$$

with the inertia force $-\rho\ddot{\mathbf{u}}$ included in $\mathbf{f}^B$ and the boundary conditions

$$\begin{aligned} \mathbf{u}\cdot\mathbf{n} &= \overline{u}_n && \text{on } S_u\,,\\ p &= \overline{p} && \text{on } S_f\,,\\ \mathbf{\Lambda} &= \mathbf{0} && \text{on } S\,. \end{aligned} \tag{18}$$

Hence we have the additional terms for an element

$$\nabla\times\mathbf{u} = (\nabla\times\mathbf{H})\widehat{\mathbf{U}} = \mathbf{D}\widehat{\mathbf{U}}\,,$$

$$\mathbf{\Lambda} = \mathbf{H}_{\Lambda}\widehat{\mathbf{\Lambda}}\,,$$

and then obtain

$$\begin{pmatrix} \mathbf{M} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{pmatrix}\begin{Bmatrix}\ddot{\widehat{\mathbf{U}}}\\ \ddot{\widehat{\mathbf{P}}}\\ \ddot{\widehat{\mathbf{\Lambda}}}\end{Bmatrix} + \begin{pmatrix} \mathbf{0} & \mathbf{L} & \mathbf{Q}\\ \mathbf{L}^{\mathrm{T}} & \mathbf{A} & \mathbf{0}\\ \mathbf{Q}^{\mathrm{T}} & \mathbf{0} & \mathbf{G}\end{pmatrix}\begin{Bmatrix}\widehat{\mathbf{U}}\\ \widehat{\mathbf{P}}\\ \widehat{\mathbf{\Lambda}}\end{Bmatrix} = \begin{Bmatrix}\mathbf{R}\\ \mathbf{0}\\ \mathbf{0}\end{Bmatrix}, \tag{19}$$

where

$$\mathbf{M} = \int_\Omega \rho \mathbf{H}^{\mathrm{T}} \mathbf{H} d\Omega\,, \qquad \mathbf{L} = -\int_\Omega \mathbf{B}^{\mathrm{T}} \mathbf{H}_p d\Omega\,,$$

$$\mathbf{Q} = \int_\Omega \mathbf{D}^{\mathrm{T}} \mathbf{H}_\Lambda d\Omega\,, \qquad \mathbf{A} = -\int_\Omega \frac{1}{\kappa} \mathbf{H}_p^{\mathrm{T}} \mathbf{H}_p d\Omega\,,$$

$$\mathbf{G} = -\int_\Omega \frac{1}{\alpha} \mathbf{H}_\Lambda^{\mathrm{T}} \mathbf{H}_\Lambda d\Omega\,, \qquad \mathbf{R} = -\int_{S_f} \mathbf{H}_n^{S^{\mathrm{T}}} \overline{p}\, dS\,.$$

4. Stability and Number of Zero Frequencies

Considering now an assemblage of elements, using Eqs. (11) and (19), we have the general governing equation

$$\begin{pmatrix} (\mathbf{M}_{uu})_h & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{Bmatrix} \ddot{\widehat{\mathbf{U}}}_h \\ \ddot{\widehat{\mathbf{S}}}_h \end{Bmatrix} + \begin{pmatrix} (\mathbf{K}_{uu})_h & (\mathbf{K}_{us})_h \\ (\mathbf{K}_{us})_h^{\mathrm{T}} & (\mathbf{K}_{ss})_h \end{pmatrix} \begin{Bmatrix} \widehat{\mathbf{U}}_h \\ \widehat{\mathbf{S}}_h \end{Bmatrix} = \begin{Bmatrix} \mathbf{R}_h \\ \mathbf{0} \end{Bmatrix}, \tag{20}$$

where now $\widehat{\mathbf{U}}_h$ lists all the unknown nodal point displacements and $\widehat{\mathbf{S}}_h$ lists all the pressure and, if applicable, vorticity moment variables. In our formulations for the acoustic fluid model, we have

for the $\mathbf{u}/p$ *formulation*:

$$(\mathbf{K}_{us})_h = (\mathbf{L})_h\,, \qquad (\mathbf{K}_{ss})_h = (\mathbf{A})_h\,,$$

for the $\mathbf{u}$-p-$\mathbf{\Lambda}$ *formulation*:

$$(\mathbf{K}_{us})_h = (\mathbf{L} \quad \mathbf{Q})_h\,, \qquad (\mathbf{K}_{ss})_h = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} \end{pmatrix}_h.$$

Since $(\mathbf{M}_{uu})_h$ is positive definite and $(\mathbf{K}_{ss})_h$ is invertible, based on our inviscid acoustic fluid model, in frequency analysis, we will encounter exact zero frequencies corresponding to $(\mathbf{K}_{uu})_h = \mathbf{0}$. In a transient direct step-by-step solution, at each time step, we have

$$\begin{pmatrix} (\mathbf{K}^*_{uu})_h & (\mathbf{K}_{us})_h \\ (\mathbf{K}_{us})_h^{\mathrm{T}} & (\mathbf{K}_{ss})_h \end{pmatrix} \begin{Bmatrix} \widehat{\mathbf{U}}_h \\ \widehat{\mathbf{S}}_h \end{Bmatrix} = \begin{Bmatrix} \widehat{\mathbf{R}}_h \\ \mathbf{0} \end{Bmatrix}, \tag{21}$$

where $\widehat{\mathbf{R}}_h$ is an effective load vector,

$$(\mathbf{K}^*_{uu})_h = \mathcal{C}(\mathbf{M}_{uu})_h \tag{22}$$

and the constant $\mathcal{C}$ is a positive number related to the direct time integration scheme used (for instance, $\mathcal{C} = 4/\Delta t^2$ for the trapezoidal rule).[1] Therefore $(\mathbf{K}^*_{uu})_h$ is always a positive-definite matrix.

For stability of our formulations, we need to choose spaces of displacements and pressure (and vorticity moment) such that we satisfy the following ellipticity and inf-sup conditions corresponding to Eq. (21).
ellipticity condition:

$$\exists\, c_1 > 0 \quad \text{such that} \quad \widehat{\mathbf{V}}_h^{\mathrm{T}}(\mathbf{K}_{uu}^*)_h\widehat{\mathbf{V}}_h \geq c_1\|\widehat{\mathbf{V}}_h\|^2 \quad \forall\, \widehat{\mathbf{V}}_h \in K \tag{23}$$

where $K = \mathrm{Ker}((\mathbf{K}_{us})_h^{\mathrm{T}})$

$$K = \{\widehat{\mathbf{V}}_h | \widehat{\mathbf{V}}_h \in \mathcal{R}^n\,, \quad (\mathbf{K}_{us})_h^{\mathrm{T}}\widehat{\mathbf{V}}_h = \mathbf{0}\}\,; \tag{24}$$

inf-sup condition:

$$\inf_{\widehat{\mathbf{S}}_h}\sup_{\widehat{\mathbf{U}}_h}\frac{\widehat{\mathbf{U}}_h^{\mathrm{T}}(\mathbf{K}_{us})_h\widehat{\mathbf{S}}_h}{\|\widehat{\mathbf{U}}_h\|\,\|\widehat{\mathbf{S}}_h\|} \geq c_2 > 0\,, \tag{25}$$

where the constant c_2 is independent of the mesh size h, the material property κ and the constant α.

The usual inf-sup condition is for the $\mathbf{u}/p$ formulation and a number of reliable finite elements have been proposed. However, little knowledge is available for the inf-sup condition given in Eq. (25) when the vorticity moment variables are included.

Since $(\mathbf{A})_h$ and $(\mathbf{G})_h$ are negative-definite matrices, we need only to consider the stiffness matrix $(\mathbf{K}_{us})_h(\mathbf{K}_{ss}^{-1})_h(\mathbf{K}_{us})_h^{\mathrm{T}}$ in order to understand the zero frequencies in our formulations. Assume that we have n displacement unknowns and m pressure (and vorticity moment) unknowns. Using elements that satisfy the inf-sup condition, the rank of $(\mathbf{K}_{us})_h$ is m and the number of zero frequencies is $n - m$. We assume here that the physical constant pressure mode arising with the boundary condition, if applicable,[1]

$$\mathbf{u}\cdot\mathbf{n} = 0 \quad \text{on } S \tag{26}$$

has been eliminated.

Of course, as stated earlier, the boundary conditions on the discretized configuration have to be imposed properly.[2,10]

5. Mixed Elements

For the mathematical models considered in this paper, working in $\mathcal{R}^2$ rather than $\mathcal{R}^3$ is not restrictive, and we will therefore consider two-dimensional solutions in this paper. The application to three dimensions can directly be achieved.[1]

The key to the success of our mixed formulations is to choose appropriate interpolations for the displacements, pressure and vorticity moment. Appropriate interpolations for the displacement/pressure formulation were summarized in Refs. 1 and 3. The $\mathbf{u}/p$ elements correspond to continuous displacements and discontinuous pressure interpolations whereas the $\mathbf{u}/p$-c elements correspond to continuous displacements and pressures across the element boundaries. Two proposed elements, the 9/3 and 9/4-c elements for the $\mathbf{u}/p$ formulation are schematically depicted in Fig. 1. Of course, there are other elements available, and in particular also the macroelements constructed with 4/1 elements shown in Fig. 2.

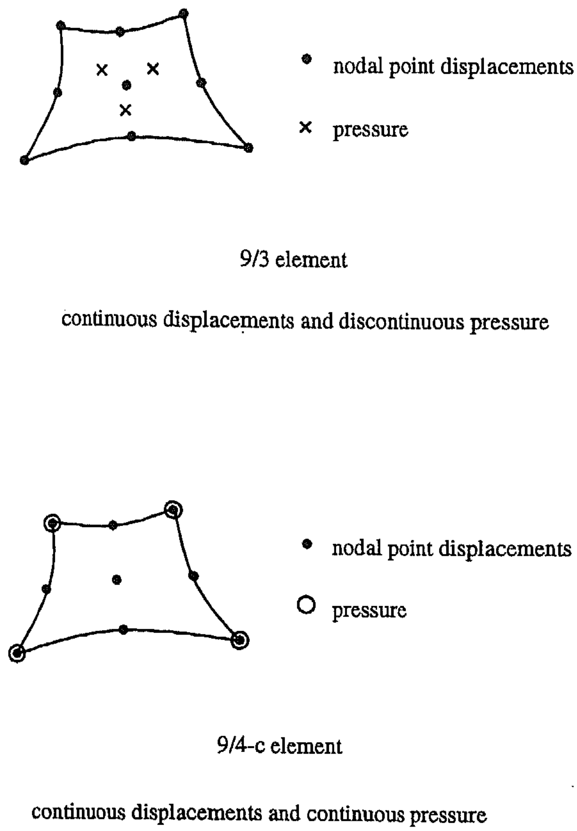

Fig. 1. Two elements for the $\mathbf{u}/p$ formulation. Full numerical integration is used (i.e. 3×3 Gauss integration).

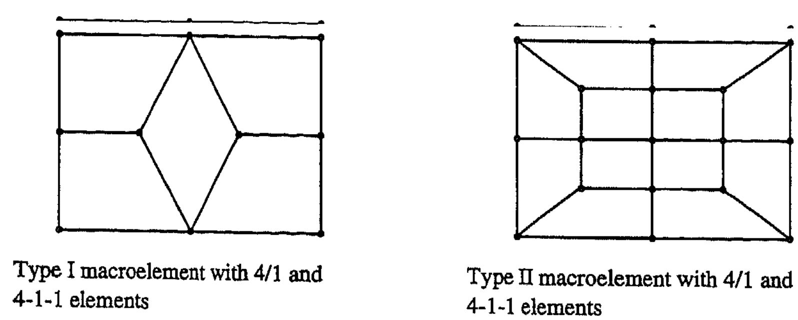

Fig. 2. Two macroelements for the $\mathbf{u}/p$ and $\mathbf{u}$-p-Λ formulations. Full numerical integration is used (i.e. 2×2 Gauss integration).

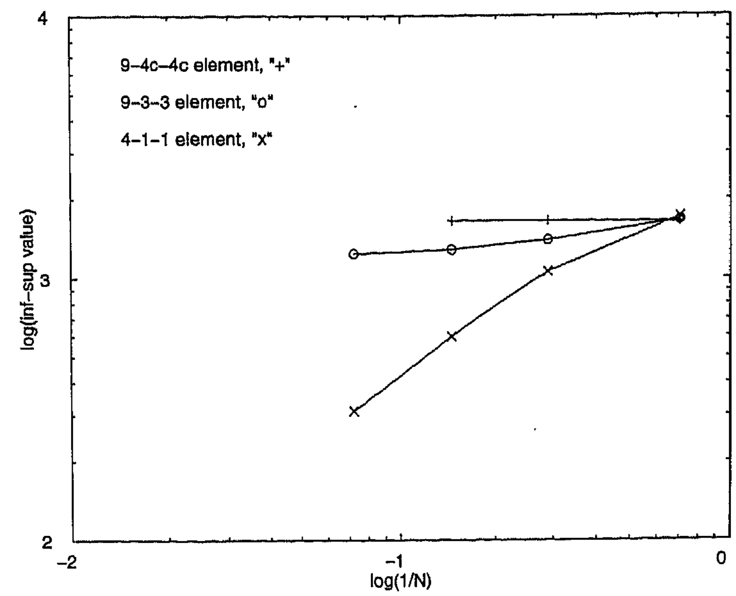

Fig. 3. Numerical inf-sup test of 9-3-3, 9-4c-4c and 4-1-1 elements of the u-p-$\mathbf{\Lambda}$ formulation (N is equal to the square root of the number of elements used).

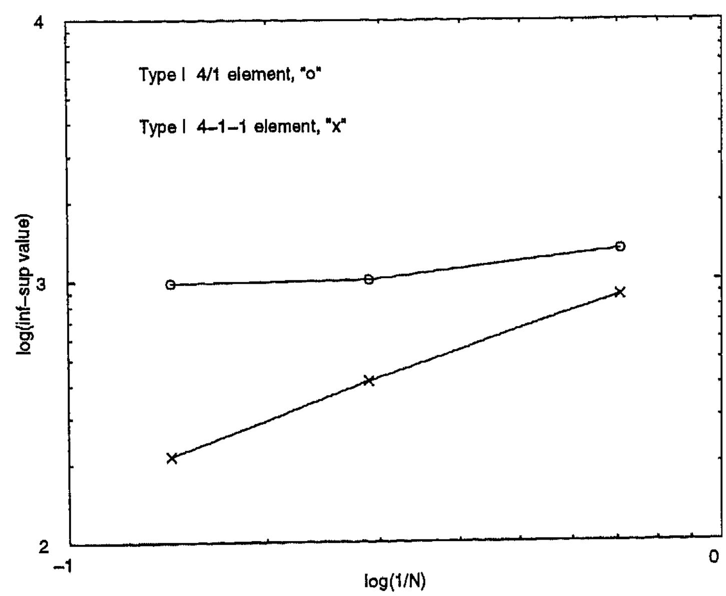

Fig. 4. Numerical inf-sup test of the Type I 4/1 and 4-1-1 macroelements of the $\mathbf{u}/p$ and u-p-$\mathbf{\Lambda}$ formulations (N is equal to the square root of the number of elements used).

Considering the u-p-$\mathbf{\Lambda}$ formulation, as pointed out in Ref. 2, the linear constraints of irrotationality and incompressibility are similar. Hence, we may endeavor to interpolate in the formulation the pressure and vorticity moments with the same functions, and with those used for stable elements in the $\mathbf{u}/p$ formulation. Natural elements to use are then the 9-3-3 and 9-4*c*-4*c* elements. However, while it is easy to show that the 9-3-1 element satisfies the inf-sup condition, see Ref. 1, the proof that the 9-3-3 and 9-4*c*-4*c* elements satisfy the inf-sup condition is not immediate. For this reason, we performed the numerical inf-sup test of Refs. 1 and 4 by considering the problem of an acoustic fluid in a rigid cavity. The results in Fig. 3 show that both elements pass this test. In addition, this figure shows the results of the numerical inf-sup test performed for the 4-1-1 element, and as expected, the test is failed. Figures 4 and 5 show the results of the numerical inf-sup test performed for the macroelements of Fig. 2. We note that, as expected, the macroelements of the 4/1 element pass the test, but the macroelements of the 4-1-1 element do not pass the test. These observations are also confirmed by the example numerical solutions given in the next section.

We note that for all elements full numerical integration is employed.

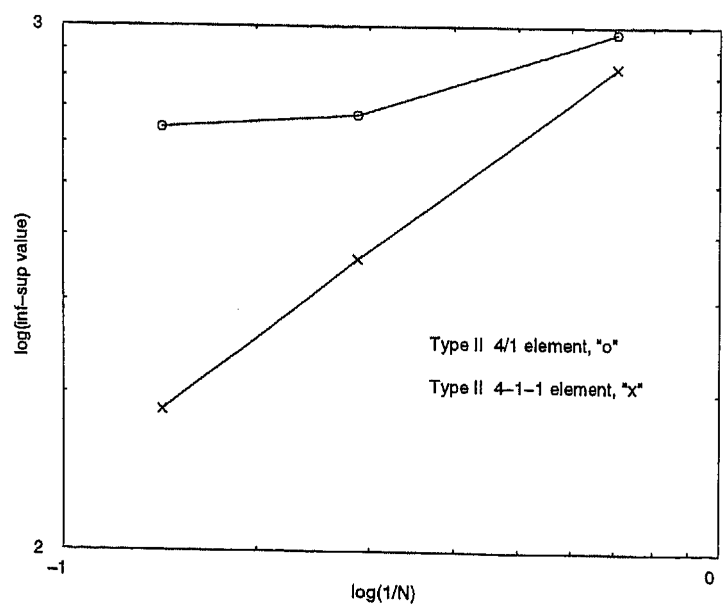

Fig. 5. Numerical inf-sup test of the Type II 4/1 and 4-1-1 macroelements of the $\mathbf{u}/p$ and $\mathbf{u}$-p-$\mathbf{\Lambda}$ formulations (N is equal to the square root of the number of elements used).

6. Example with Solutions

Figure 6 describes the paradigm of a rigid ellipse suspended from a spring in an acoustic fluid. We use this generic problem to test some mixed elements of the $\mathbf{u}/p$ and $\mathbf{u}$-p-$\mathbf{\Lambda}$ formulations. In each case, we evaluate the lowest frequencies of the complete system and give the number of zero frequencies present in the analyses.

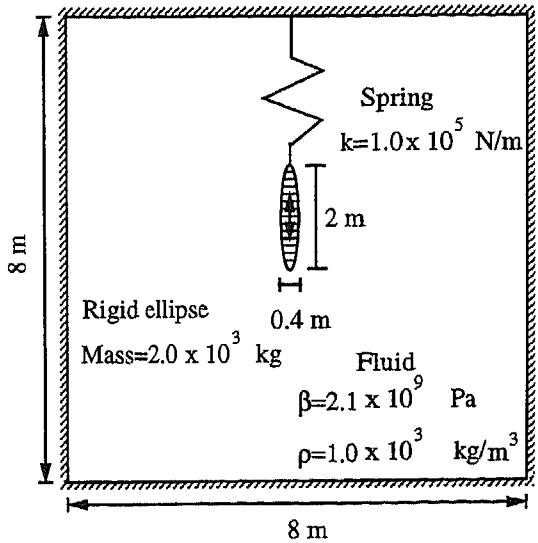

Fig. 6. A rigid ellipse vibrating in an acoustic cavity.

Table 1. Convergence in the test problem using the $\mathbf{u}/p$, and $\mathbf{u}$-p-$\boldsymbol{\Lambda}$ formulations.

Formulations	Mesh no. of elements	Number of zero frequencies	Frequencies (rad/sec) First	Second	Third	Fourth
$\mathbf{u}/p$ (9/3 elements)	2	7	6.192	697.8	1216	1269
	8	37	6.755	591.7	1229	1235
	32	157	6.848	572.6	1157	1178
$\mathbf{u}$-p-$\boldsymbol{\Lambda}$ (9-3-3 elements)	2	1	5.511	629.9	932.6	1246
	8	13	6.612	589.1	1220	1232
	32	61	6.827	572.1	1155	1176
$\mathbf{u}/p$ (9/4-c element)	2	7	6.300	700.3	1016	1244
	8	46	6.769	577.7	1158	1184
	32	208	6.841	567.0	1154	1171
$\mathbf{u}$-p-$\boldsymbol{\Lambda}$ (9-4c-4c element)	2	1	3.814	572.5	661.2	1241
	8	31	6.598	576.9	1156	1183
	32	163	6.819	566.9	1154	1170
$\mathbf{u} - \phi$ (nine-node element)	32	0	7.071	563.2	1138	1158

We consider a displacement-velocity potential ($\mathbf{u} - \phi$) formulation a very reliable procedure[9] and include the results calculated with this formulation as reference.

Table 1 lists the results obtained using the $\mathbf{u}/p$, $\mathbf{u}$-p-$\mathbf{\Lambda}$ and $\mathbf{u} - \phi$ formulations. The meshes used in these solutions have been derived by starting with coarse meshes and subdividing in each refinement each element into two or four elements. As shown in Table 1, the 9/3 and 9/4-c mixed elements for the $\mathbf{u}/p$ formulation and 9-3-3 and 9-4c-4c mixed elements for the $\mathbf{u}$-p-$\mathbf{\Lambda}$ formulation perform well. As is also shown in the numerical tests, the $\mathbf{u}$-p-$\mathbf{\Lambda}$ formulation gives less zero frequency modes than

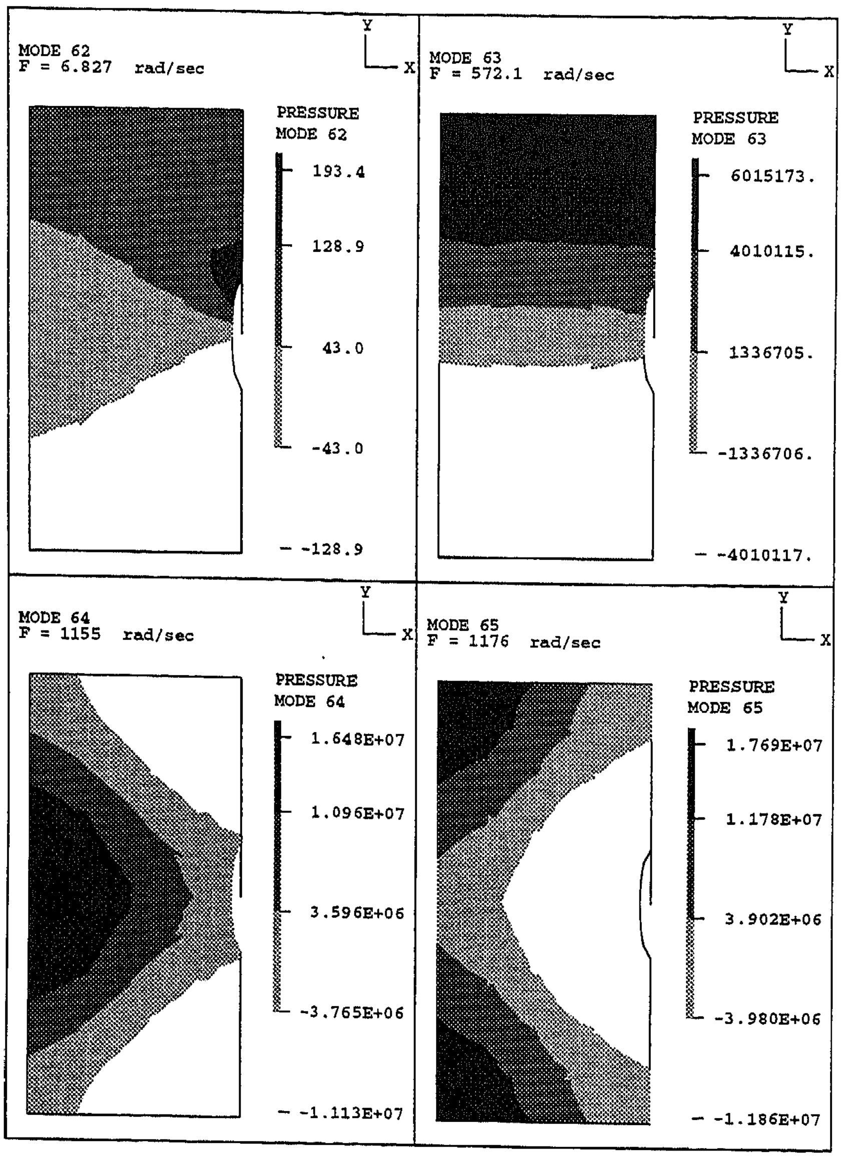

Fig. 7. Pressure bands of the first four modes of the rigid ellipse problem. (Mesh of 32 9-3-3 elements.)

the u/p formulation and the difference is given by the number of vorticity moment unknowns.

Figure 7 shows typical (unsmoothed) pressure bands obtained with a 9-3-3 element mesh. The bands are reasonably smooth and hence indicate that the solution error is small. On the other hand, considering the (unsmoothed) pressure bands of the solution using the 4-1-1 elements, given in Fig. 8, these bands show a

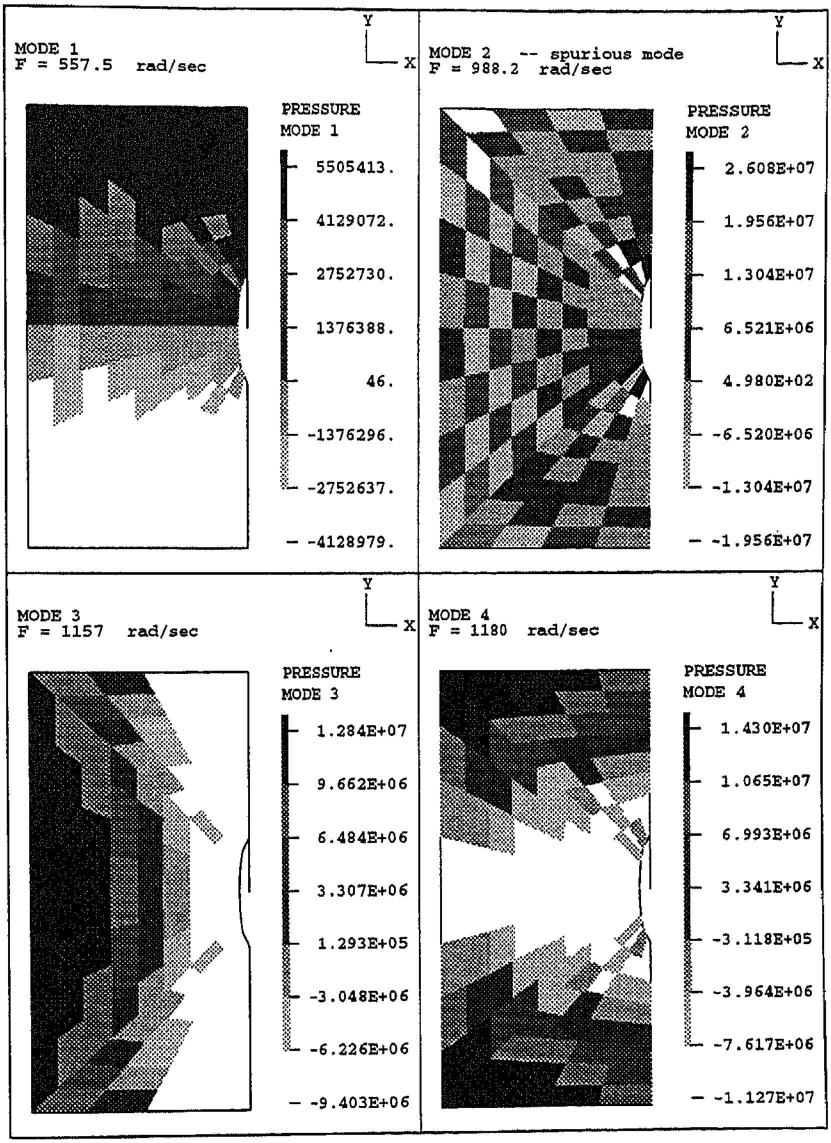

Fig. 8. Pressure bands of the first four modes of the rigid ellipse problem. (Mesh of 128 4-1-1 elements.)

strong checkerboarding in the second mode. This mode actually corresponds to a spurious nonzero frequency. The solution result is not surprising because the 4-1-1 element failed the inf-sup test, see Fig. 3.

Finally, we consider the solution of the problem using the macroelements in Fig. 2. Table 2 gives the results. We see that the macroelements using the 4/1 element perform well, whereas the macroelements using the 4-1-1 element yield very inaccurate solutions. These results correspond to the observations given in Figs. 4 and 5 regarding the inf-sup test.

Table 2. Solutions of the test problem using different macroelements in the **u**/p and **u**-p-$\mathbf{\Lambda}$ formulations.

Mixed formulations	Macroelement type	Number of macroelements	Frequencies (rad/sec)			
			First	Second	Third	Fourth
u/p	I	32	6.802	575.6	1167	1210
	II	32	6.781	579.0	1165	1195
u-p-$\mathbf{\Lambda}$	I	32	560.8	1035	1167	1194
	II	32	559.2	1155	1163	1191

7. Conclusions

We considered in this paper the **u**/p and **u**-p-$\mathbf{\Lambda}$ formulations for the acoustic fluid and showed that **u**/p elements known to satisfy the inf-sup condition yield accurate solutions for a difficult problem. We also showed that the 9-3-3 and 9-4c-4c elements of the **u**-p-$\mathbf{\Lambda}$ formulation are effective. However, the 4-1-1 element and the macroelements used are not recommended for the **u**-p-$\mathbf{\Lambda}$ formulation.

The results given are all based on numerical evaluations and it would be valuable to study analytically the convergence properties of the **u**-p-$\mathbf{\Lambda}$ elements.

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