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ON THE CONVERGENCE OF A FOUR-NODE PLATE BENDING ELEMENT BASED ON MINDLIN/REISSNER PLATE THEORY AND A MIXED INTERPOLATION

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1. INTRODUCTION

The aim of this paper is to analyze, from a theoretical point of view, a four-node element for Mindlin/Reissner plates recently introduced by Bathe and Dvorkin [3]. We recall that the Mindlin/Reissner formulation for "moderately thick" plates amounts to assume that the "in-plane" displacements u_1 and u_2 have the form [2]

$$u_1(x,y,z) = -z\beta_1(x,y); \; u_2(x,y,z) = -z\beta_2(x,y) \tag{1.1}$$

In the Mindlin/Reissner theory the unknowns are $\beta_1(x,y)$, $\beta_2(x,y)$ and u_3 which is assumed as

$$u_3(x,y,z) = w(x,y) \tag{1.2}$$

The unknowns $\underline{\beta}$ and w are defined in $\Omega \subseteq \mathbb{R}^2$, and the undeformed configuration of the plate occupies the region $\Omega\times]-t/2,t/2[$ (hence, t is the plate thickness). The corresponding strain field is therefore

$$\varepsilon_{11} = -z\partial\beta_1/\partial x \qquad \varepsilon_{22} = -z\partial\beta_2/\partial y \qquad \varepsilon_{33} = 0 \tag{1.3}$$

$$2\varepsilon_{12} = -z(\partial\beta_1/\partial y + \partial\beta_2/\partial x) \quad 2\varepsilon_{13} = \partial w/\partial x-\beta_1 \quad 2\varepsilon_{23} = \partial w/\partial y-\beta_2$$

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and the corresponding stress field is

$$\sigma_{11} = (\varepsilon_{11}+\nu\varepsilon_{22})\ E/(1-\nu^2) \quad \sigma_{22} = (\varepsilon_{22}+\nu\varepsilon_{11})\ E/(1-\nu^2)$$

$$\sigma_{ij} = E\ \varepsilon_{ij}/(1+\nu) \quad i,j = 1,2,3;\ i\neq j \tag{1.4}$$

where E is the Young's modulus and ν is the Poisson's ratio. If $p(x,y)$ is the transverse loading per unit area, the total potential energy is given by

$$\Pi = \frac{1}{2}\ t^3 a\ (\underline{\beta},\underline{\beta}) + \frac{1}{2}\lambda t \int_\Omega |\nabla w-\underline{\beta}|^2\ d\Omega - \int_\Omega pw d\Omega \tag{1.5}$$

where

$$a(\underline{\beta},\underline{\beta}) = \frac{E}{12(1-\nu^2)}\int_\Omega [(\partial\beta_1/\partial x+\nu\partial\beta_2/\partial y)\partial\beta_1/\partial x + (\nu\partial\beta_1/\partial x$$
$$+\partial\beta_2/\partial y)\ \partial\beta_2/\partial y + 2^{-1}(1-\nu)(\partial\beta_1/\partial y+\partial\beta_2/\partial x)^2]dxdy \tag{1.6}$$

$$\lambda = Ek/(2(1+\nu)) \tag{1.7}$$

and k is a constant to account for the actual nonuniformity of the shearing stresses (namely, since $\sigma_{13}=\sigma_{23}=0$ on the lower and upper surfacesof the plate, we must "correct" (1.3) which assumes ε_{13} and ε_{23} to be constant with z; for more details, see e.g. [2]).

We assume here, for the sake of simplicity(†) that our plate is clamped along the entire boundary $\partial\Omega$; hence we look for $(\underline{\beta}, w)$ in the space

$$V = \{(\underline{\eta},\zeta)\ |\ \underline{\eta}\in(H^1_0\ (\Omega))^2,\ \ \zeta\in H^1_0\ (\Omega)\} \tag{1.8}$$

The well-known Korn's inequality (see e.g. [6]) states that

$$\exists\ c>0 \text{ such that } \forall\ \underline{\eta}\in(H^1_0\ (\Omega))^2$$
$$a(\underline{\eta},\underline{\eta}) \geq c\ \|\underline{\eta}\|^2_1 \tag{1.9}$$

This implies that, from the mathematical point of view, for any $t>0$ the functional Π in (1.5) is strictly convex and hence has a unique minimiser in V. However, it is also well known that many finite element discretizations of (1.5) fail when the

(†)Although best tractable for mathematical analysis, in actual computations, the clamped plate problem usually yields the worst convergence properties.

thickness t of the plate is "too small". In order to understand this phenomenon, we thought it necessary to construct a sequence of problems $(P_t)_{t>0}$ such that the corresponding solutions $(\underline{\beta}(t), w(t))$ stay bounded for $t\to 0$. Such a sequence is meant as a test for the numerical discretizations and our analysis will concentrate on the behaviour of the discretization in [3] when applied to the sequence. For a similar analysis on several beam elements we refer to the very good paper by Arnold [1]. Unfortunately, our (two-dimensional) case is much harder, so that the generality of the results of [1] on beams is (at this time) out of reach for plates.

An outline of the paper is as follows. In Section 2 we introduce the sequence of problems (P_t) and we study the behaviour of the corresponding solutions for $t\to 0$. In Section 3 we recall the definition of the element in [3] and we show that, when it is applied to the sequence of problems (P_t), the corresponding discrete solutions satisfy

$$||\underline{\beta}(t)-\underline{\beta}_h(t)||_1 + ||w(t)-w_h(t)||_1 < ch(||\underline{\beta}(t)||_3+||\underline{\gamma}(t)||_2) \quad (1.10)$$

with c independent of h and t. We also have the (non-optimal) result

$$t||\underline{\gamma}(t)-\underline{\gamma}_h(t)||_0 \leq ch \quad (1.11)$$

where $\underline{\gamma} = t^{-2}(\underline{\nabla} w-\underline{\beta})$ and $\underline{\gamma}_h = t^{-2}(\underline{\nabla} w_h-\underline{\beta}_h)$ are the continuous and discrete "shear strains". For the sake of simplicity, the analysis of Section 3 is carried out on the particular case of a rectangular Ω and for a decomposition into rectangles. The analysis of more general cases (using the element in shell problems [7]) is not a straight-forward generalization of the present case.

2. THE SEQUENCE OF PROBLEMS $(P_t)_{t>0}$.

We construct our test sequence in the following way. We assume that Ω, E, ν are kept constant, while the pressure $p_t(x,y)$ varies as

$$p_t(x,y) \equiv t^3 f(x,y) \quad (2.1)$$

where f(x,y) is a given function, say, in $L^2(\Omega)$, which obviously does not change with t. By dividing the potential energy in (1.5) by t^3 we have therefore the following sequence of problems $(P_t)_{t>0}$

$$\left\{\begin{array}{l}\text{Minimize, for } (\underline{\beta},w)\in V \text{ the functional}\\ \frac{1}{2}\, a(\underline{\beta},\underline{\beta}) + \frac{\lambda}{2} t^{-2}\, ||\underline{\nabla} w-\underline{\beta}||_0^2 - (f,w)\end{array}\right\} \quad (2.2)$$

where, as usual, $(\cdot\,,\,)$ denotes the $L^2(\Omega)$ inner product.

Again Korn's inequality (1.9) ensures that (2.2) has a unique solution, $(\underline{\beta}(t), w(t))$, for all $t>0$. It is also an easy matter to check that (say, for $0<t<t_0$)

$$\|\underline{\beta}(t)\|_1^2 + \|w(t)\|_1^2 + t^2\|\underline{\gamma}(t)\|_0^2 \leq \text{c indep. of t} \tag{2.3}$$

where $\underline{\gamma}(t)$ is defined by

$$\underline{\gamma}(t) := t^{-2}(\underline{\nabla}w(t)-\underline{\beta}(t)) \tag{2.4}$$

In order to analyze the behaviour of the solution for $t\to 0$ it is convenient to consider the Euler equations associated with (2.2)

$$\left\{\begin{array}{l} \text{find } (\underline{\beta}(t), w(t)) \in V \text{ such that} \\ a(\underline{\beta}(t),\underline{\eta}) + \lambda(\underline{\gamma}(t),\underline{\nabla}\zeta-\underline{\eta}) = (f,\zeta) \;\forall\; (\underline{\eta},\zeta)\in V \\ \underline{\gamma}(t) = t^{-2}(\underline{\nabla}w(t)-\underline{\beta}(t)) \end{array}\right\} \tag{2.5}$$

It is clear that in order to study the behaviour of the solutions for $t\to 0$ we need estimates on $\underline{\gamma}(t)$ rather than on $t\underline{\gamma}(t)$ as in (2.3). For this we introduce the space

$$H_0(\mathrm{rot};\Omega) = \{\underline{\chi} \mid \underline{\chi} \in (L^2)^2,\ \mathrm{rot}\underline{\chi} \in L^2(\Omega),\ \underline{\chi}\cdot\underline{\tau} = 0\} \tag{2.6}$$

where

$$\mathrm{rot}(\chi_1,\chi_2) := \partial\chi_1/\partial y - \partial\chi_2/\partial x \tag{2.7}$$

$$\underline{\tau} = \text{counterclockwise tangent unit vector to } \partial\Omega \tag{2.8}$$

$$\|\underline{\chi}\|^2_{H_0(\mathrm{rot};\Omega)} := \|\underline{\chi}\|_0^2 + \|\mathrm{rot}\underline{\chi}\|_0^2 =: \|\underline{\chi}\|_H^2 \tag{2.9}$$

and the space

$$\Gamma := (H_0(\mathrm{rot};\Omega))' = \text{dual space of } H_0(\mathrm{rot};\Omega) \tag{2.10}$$

__Theorem 2.1__ __We have__

$$\|\underline{\gamma}(t)\|_\Gamma \leq c \text{ \underline{independent of} } t \tag{2.11}$$

__Proof__. From (2.10) we have that there exists $\underline{\chi}\in H_0(\mathrm{rot};\Omega)$ such that

$$(\underline{\gamma}(t), \underline{\chi}) = \|\underline{\gamma}(t)\|_{\Gamma}^{2} = \|\underline{\chi}\|_{H}^{2} \tag{2.12}$$

We shall show that it is possible to find $(\underline{\eta}, \zeta) \in V$ such that

$$\underline{\chi} = \underline{\nabla}\zeta - \underline{\eta} \tag{2.13}$$

$$\|\underline{\eta}\|_1 + \|\zeta\|_1 \leq c\|\underline{\chi}\|_H \tag{2.14}$$

with c independent of $\underline{\chi}$. If this is true, then from (2.3), (2.5) and (2.12)-(2.14) we get

$$\begin{aligned} \|\underline{\gamma}\|_{\Gamma} \ \|\underline{\chi}\|_H &= (\underline{\gamma}, \underline{\chi}) = (f, \zeta) - a(\underline{\beta}, \underline{\eta}) \\ &\leq c(\|\zeta\|_1 + \|\underline{\eta}\|_1) \leq c\ \|\underline{\chi}\|_H \end{aligned} \tag{2.15}$$

which implies (2.11). Hence we have just to show (2.13), (2.14). We first choose $\theta \in (H_0^1(\Omega))^2$ such that

$$\operatorname{div} \underline{\theta} = -\operatorname{rot} \underline{\chi} \tag{2.16}$$

$$\|\underline{\theta}\|_1 \leq c\ \|\operatorname{rot} \underline{\chi}\|_0 \tag{2.17}$$

This is always possible (see e.g. [9]) since rot $\underline{\chi}$ has zero mean value on Ω. Then we set

$$\underline{\eta} \equiv (\eta_1, \eta_2) := (-\theta_2, \theta_1) \tag{2.18}$$

so that from (2.16) - (2.18) we have

$$\operatorname{rot} \underline{\eta} = -\operatorname{rot} \underline{\chi} \tag{2.19}$$

$$\|\underline{\eta}\|_1 \leq c\|\operatorname{rot} \underline{\chi}\|_0 \leq c\|\underline{\chi}\|_H \tag{2.20}$$

Now we choose ζ as the unique solution of: $\Delta\zeta = \operatorname{div} \underline{\gamma} + \operatorname{div} \underline{\eta}$, $\zeta \in H_0^1(\Omega)$. Clearly

$$\begin{aligned} \|\zeta\|_1 &\leq c(\|\operatorname{div}\underline{\chi} + \operatorname{div} \underline{\eta}\|_{-1}) \leq c(\|\underline{\chi}\|_0 + \|\underline{\eta}\|_0) \\ &\leq c\|\underline{\chi}\|_H \end{aligned} \tag{2.21}$$

We have now

$$\left\{\begin{aligned} &\operatorname{div}(\underline{\nabla}\zeta - \underline{\eta}) = \operatorname{div}\underline{\chi};\ \operatorname{rot}(\underline{\nabla}\zeta - \underline{\eta}) = -\operatorname{rot}\underline{\eta} = \operatorname{rot}\underline{\chi} \\ &(\underline{\nabla}\zeta - \underline{\eta}) \cdot \underline{\tau} = \underline{\chi} \cdot \underline{\tau} = 0 \text{ on } \partial\Omega \end{aligned}\right\} \tag{2.22}$$

which easily imply (2.13). On the other hand (2.14) follows from (2.20) (2.21).

■

We are now able to analyze the behaviour of $\underline{\beta}(t)$, $w(t)$ for $t \to 0$.

Theorem 2.2 We have, for $t \to 0$,

$$\begin{cases} \underline{\beta}(t) \rightharpoonup \underline{\beta}_0, \; w(t) \rightharpoonup w_0 \text{ in } H_0^1(\Omega) \\ \underline{\gamma}(t) \rightharpoonup \underline{\gamma}_0 \text{ in } \Gamma \end{cases} \tag{2.23}$$

where $\underline{\beta}_0$, w_0, $\underline{\gamma}_0$ satisfy:

$$a(\underline{\beta}_0,\underline{\eta}) + \lambda < \underline{\gamma}_0, \underline{\nabla}\zeta - \underline{\eta} > = (f,\zeta) \;\; \forall \; (\underline{\eta},\zeta) \in V \tag{2.24}$$

$$\underline{\nabla} w_0 = \underline{\beta}_0 \tag{2.25}$$

$$E\, \Delta^2 w_0 = 12(1-\nu^2) f \tag{2.26}$$

Proof. (2.23) is obvious from (2.3) and (2.11). Then (2.24), (2.25) follow immediately passing to the limit in (2.5). Now, taking $\zeta=0$ in (2.24) we have

$$-A\, \underline{\beta}_0 = \lambda\, \underline{\gamma}_0 \tag{2.27}$$

where $-A$ is the (2-d) linear elasticity operator (associated with the bilinear form $a(\;,\;)$). Taking instead $\underline{\eta}=0$ in (2.24) we get

$$\text{div}\, \underline{\gamma}_0 = -f \tag{2.28}$$

and (2.26) follows from (2.25), (2.27), (2.28). ∎

Remark. Let $w_K(t)$ be the solution of the Kirchhoff model

$$\frac{E\, t^3}{12(1-\nu^2)} \Delta^2 w_K(t) = p_t = t^3 f \tag{2.29}$$

corresponding to the same values of E, ν, and load p_t of our test sequence of Mindlin plates. Then theorem 2.2 states that the Mindlin solution $w(t)$ converges to the Kirchhoff solution w_K (which actually is independent of t). Note also that this is independent of the choice of the correction factor k in (1.7). For additional results on this convergence question see e.g. Destuynder [5].

Remark. It can be shown that the space Γ defined in (2.10) could also be written as

$$\Gamma = \{\underline{\kappa} \mid \underline{\kappa} \in (H^{-1}(\Omega))^2, \; \text{div}\, \underline{\kappa} \in H^{-1}(\Omega)\} \tag{2.30}$$

It seems natural to guess that optimal error estimates for $\underline{\gamma}(t)$ should be given in the space Γ. However, for the case analyzed in the next section we were not able to do so. The task is much easier in the one-dimensional case where the space corresponding to Γ is just L^2.

Remark. The main reason for proving theorem 2.2 is to show that the sequence $(\underline{\beta}(t), w(t))$ does not tend to zero when $t \to 0$. The test would not be serious otherwise.

3. CONVERGENCE OF THE FOUR-NODE ELEMENT

We assume now that Ω is a rectangle and that we are given a sequence $\{\mathcal{T}_h\}_h$ of uniform decompositions of Ω into rectangles R; if h_1 and h_2 are the lengths of the edges of R, we set

$$|h| = \max h_i \tag{3.1}$$

and we assume as usual that there exist two constants c_1, $c_2 > 0$ such that

$$c_2 |h| \le h_1/h_2 \le c_1 \; |h| \tag{3.2}$$

for all $\mathcal{T}_h$.

We define, for all $\mathcal{T}_h$,

$$Q_h = \{\phi \,|\, \phi \in H_0^1(\Omega);\ \phi_{|R} \in Q_1 \ \forall\, R \in \mathcal{T}_h\} \tag{3.3}$$

where Q_1 is the space of bilinear functions. We also introduce

$$\Gamma_h = \{\underline{\chi} \,|\, \underline{\chi} \in H_0(\mathrm{rot};\Omega),\ \underline{\chi}_{|R} \in (Q_{01}, Q_{10}) \ \forall\, R \in \mathcal{T}_h\} \tag{3.4}$$

where $Q_{01} = \mathrm{span}\,\{1,y\}$ and $Q_{10} = \mathrm{span}\,\{x,1\}$. Note that the condition $\underline{\chi} \in H_0(\mathrm{rot};\Omega)$ means in this case that $\underline{\chi} \cdot \underline{\tau} = 0$ on $\partial\Omega$, χ_1 is continuous on the horizontal edges, and that χ_2 is continuous on the vertical edges. Hence the degrees of freedom in Γ_h are: values of χ_1 on the internal horizontal edges and values of χ_2 on the internal vertical edges.

For any vector $\underline{\eta}_h \in (Q_h)^2$ we define now its interpolant $\overset{*}{\underline{\eta}}_h$ in Γ_h by the formulas

$$\left.\begin{aligned} \overset{*}{\eta}_{h,1} &= \eta_{h,1} \text{ at the midpoints of horizontal edges} \\ \overset{*}{\eta}_{h,2} &= \eta_{h,2} \text{ at the midpoints of vertical edges} \end{aligned}\right\} \tag{3.5}$$

We finally set

$$V_h = (Q_h)^2 \times Q_h \tag{3.6}$$

Applying the four-node element introduced in [3] to each problem (2.2) we have the following sequence of problems:

$$\left.\begin{array}{l}\text{Minimize, for } (\underline{\beta}_h, w_h) \in V_h \text{ the functional} \\ \frac{1}{2}\, a(\underline{\beta}_h, \underline{\beta}_h) + \frac{\lambda t^{-2}}{2} \;||\, \underline{\nabla}\, w_h - \overset{*}{\underline{\beta}}_h ||_0^2 \;-\; (f,w)\end{array}\right\} \tag{3.7}$$

It is easy to check that each problem (3.7), for $t>0$, has a unique solution $(\underline{\beta}_h(t), w_h(t))$. The dependence on t will not be made explicit in the notation when unnecessary. Our aim is to estimate the error $||\underline{\beta}(t) - \underline{\beta}_h(t)||_1 + ||w(t) - w_h(t)||_1$ uniformly in t. As in the continuous problem, it will be convenient to introduce

$$\underline{\gamma}_h(t) = t^{-2}\; (\underline{\nabla} w_h(t) - \overset{*}{\underline{\beta}}_h(t)) \tag{3.8}$$

and to write the Euler equations of (3.7) in the form

$$\left.\begin{array}{l}\text{find } (\underline{\beta}_h, w_h) \in V_h \text{ such that} \\ a(\underline{\beta}_h, \underline{\eta}_h) + \lambda\; (\underline{\gamma}_h, \underline{\nabla}\zeta_h - \overset{*}{\underline{\eta}}_h) = (f, \zeta_h) \;\; \forall (\underline{\eta}_h, \zeta_h) \in V_h \\ \underline{\gamma}_h = t^{-2}\; (\underline{\nabla} w_h - \overset{*}{\underline{\beta}}_h)\end{array}\right\} \tag{3.9}$$

The following lemma summarizes a few properties that can be proved by simple algebraic manipulations or standard numerical integration techniques.

<u>Lemma 3.1</u> <u>We have, for all</u> $(\underline{\eta}_h, w_h)$ <u>in</u> V_h <u>and for all</u> $\underline{\chi}_h \in \Gamma_h$:

$$\underline{\nabla} w_h \in \Gamma_h \tag{3.10}$$

$$(\underline{\chi}_h, \underline{\eta}_h) = (\underline{\chi}_h, \overset{*}{\underline{\eta}}_h) \tag{3.11}$$

$$\mathrm{rot}\, \overset{*}{\underline{\eta}}_{h|R} = \frac{1}{|R|} \int_R \mathrm{rot}\, \underline{\eta}_h \; dxdy \quad \forall\, R \in \mathcal{T}_h \tag{3.12}$$

The following lemma provides some asymptotic estimates that will be used later on.

<u>Lemma 3.2</u> <u>We have</u>

$$||\zeta - \zeta^I||_{1,R} \le c|h| (||\partial^2\zeta/\partial x^2||_{0,R}^2 + ||\partial^2\zeta/\partial y^2||_{0,R}^2)^{1/2} \quad \forall\, R \in \mathcal{T}_h \tag{3.13}$$

<u>for</u> ζ <u>smooth and</u> ζ^I = <u>interpolant of</u> ζ <u>in</u> Q_h

$$\left.\begin{aligned} &||\underline{\eta}_h - \overset{*}{\underline{\eta}}_h||_0 \le c|h|\ ||\underline{\eta}_h||_1 \\ &\text{for } \underline{\eta}_h \in (Q_h)^2 \end{aligned}\right\} \tag{3.14}$$

<u>Proof</u>. The inequality (3.13) is well known (see e.g. [4] [11]). The proof of (3.14) is an easy exercise.

■

We now give the main theorem.

<u>Theorem 3.1</u> <u>Let</u> $(\underline{\beta}(t), w(t))$ <u>and</u> $(\underline{\beta}_h(t), w_h(t))$ <u>be the solutions of</u> (2.5) <u>and</u> (3.9) <u>respectively</u>. <u>We have</u>

$$\begin{aligned} &||\underline{\beta}(t)-\underline{\beta}_h(t)||_1 + ||w(t)-w_h(t)||_1 + t||\underline{\gamma}(t)-\underline{\gamma}_h(t)||_0 \\ &\le c|h|\ (||\underline{\beta}(t)||_3 + ||\underline{\gamma}(t)||_2) \end{aligned} \tag{3.15}$$

<u>with</u> c <u>independent of</u> h <u>and</u> t, <u>and</u> $\underline{\gamma}(t)$, $\underline{\gamma}_h(t)$ <u>given in</u> (2.5) <u>and</u> (3.9).

<u>Proof</u>. Let $\underline{\beta}_I \in (Q_h)^2$ be such that

$$||\underline{\beta} - \underline{\beta}_I||_1 \le c|h|\ ||\underline{\beta}||_3 \tag{3.16}$$

Other requirements on $\underline{\beta}_I$ will be given later on. Using the Korn's inequality (1.9) we have

$$\begin{aligned} c||\underline{\beta}-\underline{\beta}_h||_1^2 &\le a(\underline{\beta}-\underline{\beta}_h, \underline{\beta}-\underline{\beta}_h) \\ &= a(\underline{\beta}-\underline{\beta}_h, \underline{\beta}-\underline{\beta}_I) + a(\underline{\beta}-\underline{\beta}_h, \underline{\beta}_I-\underline{\beta}_h) \\ &\le c|h|\ ||\underline{\beta}||_3 + a(\underline{\beta}-\underline{\beta}_h, \underline{\beta}_I-\underline{\beta}_h) \end{aligned} \tag{3.17}$$

On the other hand, from (2.5) and (3.9) we obtain

$$\begin{aligned} a(\underline{\beta}-\underline{\beta}_h, \underline{\beta}_I-\underline{\beta}_h) &= \lambda(\underline{\gamma}, \underline{\beta}_I-\underline{\beta}_h) - \lambda(\underline{\gamma}_h, \overset{*}{\underline{\beta}}_I-\overset{*}{\underline{\beta}}_h) \\ &\le \lambda(\underline{\gamma}-\underline{\gamma}_h, \overset{*}{\underline{\beta}}_I-\overset{*}{\underline{\beta}}_h) + ||\underline{\gamma}||_0\ c|h|\ ||\underline{\beta}_I-\underline{\beta}_h||_1 \end{aligned} \tag{3.18}$$

where we have used (3.14) for $\underline{\eta}_h = \underline{\beta}_I - \underline{\beta}_h$. Now we have

$$\begin{aligned} \lambda(\underline{\gamma}-\underline{\gamma}_h, \overset{*}{\underline{\beta}}_I-\overset{*}{\underline{\beta}}_h) &= \lambda t^2(\underline{\gamma}-\underline{\gamma}_h, \underline{\gamma}_h-\underline{\gamma}) - \lambda(\underline{\gamma}-\underline{\gamma}_h, t^2\underline{\gamma}_h + \overset{*}{\underline{\beta}}_h) \\ &\quad + \lambda(\underline{\gamma}-\underline{\gamma}_h, t^2\underline{\gamma} + \overset{*}{\underline{\beta}}_I) \end{aligned} \tag{3.19}$$

Note now that from (2.5) and (3.9) we have

$$(\underline{\gamma}-\underline{\gamma}_h, \underline{\nabla}\zeta_h) = 0 \qquad \forall\ \zeta_h \in Q_h \tag{3.20}$$

so that, since $t^2\underline{\gamma}_h + \overset{*}{\underline{\beta}}_h = \underline{\nabla} w_h$, (3.19) becomes

$$\lambda(\underline{\gamma}-\underline{\gamma}_h, \overset{*}{\underline{\beta}}_I - \underline{\beta}_h) = -\lambda t^2 ||\underline{\gamma}-\underline{\gamma}_h||_0^2 + \lambda(\underline{\gamma}-\underline{\gamma}_h, t^2\underline{\gamma}+\overset{*}{\underline{\beta}}_I)$$

Combining (3.17), (3.18), (3.20) we have

$$c||\underline{\beta}-\underline{\beta}_h||_1^2 + \lambda t^2 ||\underline{\gamma}-\underline{\gamma}_h||_0^2 \le c|h| \, (||\underline{\beta}||_3 \; ||\underline{\beta}-\underline{\beta}_h||_1 + ||\underline{\gamma}||_0 \; ||\underline{\beta}_I-\underline{\beta}_h||_1) + \lambda(\underline{\gamma}-\underline{\gamma}_h, t^2\underline{\gamma}+\overset{*}{\underline{\beta}}_I) \tag{3.21}$$

Here we need more help from $\overset{*}{\underline{\beta}}_I$. More precisely we require that

$$\text{rot}\,\overset{*}{\underline{\beta}}_I\Big|_R = -\frac{1}{|R|}\int_R t^2 \text{rot}\underline{\gamma} = \frac{-1}{|R|}\int_R \text{rot}\,\underline{\beta} \tag{3.22}$$

for all R in $\mathcal{T}_h$. The existence of a $\underline{\beta}_I$ satisfying both (3.16) and (3.22) is proved in lemma 3.3 below. Next we choose $\underline{q} \in (H_0^1(\Omega))^2$ such that

$$\text{rot}\,\underline{q}\big|_R = t^2 \,\text{rot}\underline{\gamma} + \text{rot}\overset{*}{\underline{\beta}}_I \tag{3.23}$$

It follows from (3.22) that $\underline{q}$ may be chosen such that

$$||\underline{q}||_1 \le c|h|t^2 \; ||\text{rot}\underline{\gamma}||_1 \tag{3.24}$$

(see (2.19), (2.20) for a similar argument, plus the standard bound $||\text{rot}\underline{\gamma} - \frac{1}{R}\int_R \text{rot}\underline{\gamma}||_0 \le c|h| \; ||\text{rot}\underline{\gamma}||_1$). Hence $t^2\underline{\gamma}+\overset{*}{\underline{\beta}}_I-\underline{q}$ has zero rotation, so that we may write

$$t^2\underline{\gamma} + \overset{*}{\underline{\beta}}_I - \underline{q} = \underline{\nabla}\zeta \tag{3.25}$$

for some $\zeta \in H_0^1(\Omega)$. Using (3.25) and (3.20) we have

$$\begin{aligned}\lambda(\underline{\gamma}-\underline{\gamma}_h, t^2\underline{\gamma}+\overset{*}{\underline{\beta}}_I) &= \lambda(\underline{\gamma}-\underline{\gamma}_h, \underline{q} + \underline{\nabla}\zeta) \\ &= \lambda(\underline{\gamma}-\underline{\gamma}_h, \underline{q}) + \lambda(\underline{\gamma}-\underline{\gamma}_h, \underline{\nabla}\zeta - \underline{\nabla}\zeta^I)\end{aligned} \tag{3.26}$$

We have finally from (3.13) and (3.25)

$$\begin{aligned}||\zeta-\zeta_I||_{1,R} &\le c|h| \, (||\partial^2\zeta/\partial x^2||_{0,R}^2 + ||\partial^2\zeta/\partial y^2||_{0,R}^2)^{1/2} \\ &\le c|h| \, (t^2||\underline{\gamma}||_1 + ||\underline{q}||_1)\end{aligned} \tag{3.27}$$

and collecting (3.26), (3.24) and (3.27) we get

$$\lambda(\underline{\gamma}-\underline{\gamma}_h, t^2\underline{\gamma}-\overset{*}{\underline{\beta}}_I) \le c|h|t^2 \; ||\underline{\gamma}-\underline{\gamma}_h||_0 \; ||\underline{\gamma}||_2 \tag{3.28}$$

Combining now (3.21) and (3.28) we obtain

$$||\underline{\beta}-\underline{\beta}_h||_1 + \lambda t^2 \, ||\underline{\gamma}-\underline{\gamma}_h||_0^2 \le c|h| \, \{||\underline{\beta}||_3 \, ||\underline{\beta}-\underline{\beta}_h||_1 + ||\underline{\gamma}||_0 \, ||\underline{\beta}_I-\underline{\beta}_h||_1 + t^2||\underline{\gamma}-\underline{\gamma}_h||_0 \, ||\underline{\gamma}||_2\} \tag{3.29}$$

and hence easily (for fixed λ)

$$||\underline{\beta}-\underline{\beta}_h||_1 + t||\underline{\gamma}-\underline{\gamma}_h||_0 \le c|h| \, (||\underline{\beta}||_3 + ||\underline{\gamma}||_2) \tag{3.30}$$

Finally

$$\underline{\underline{\nabla}}w-\underline{\nabla}w_h = \underline{\beta}-\underline{\overset{*}{\beta}}_h + t^2\underline{\gamma} - t^2\underline{\gamma}_h \tag{3.31}$$

and (3.15) follows from (3.30), (3.31) using again (3.14), now with $\underline{\eta}_h = \underline{\beta}_h$. ■

The proof of theorem 3.1 used the existence of a function $\underline{\beta}_I$ satisfying (3.16) and (3.22). The proof of existence of $\underline{\beta}_I$ is the object of the following final lemma.

Lemma 3.3 For any $\underline{\beta} \in (H^3 \cap H_0^1)^2$ there exists $\underline{\beta}_I \in (Q_h)^2$ such that (3.16) and (3.22) are satisfied.

Proof. Let us first set $\theta_1 := -\beta_2$ and $\theta_2 := \beta_1$ so that $\mathrm{rot}\underline{\beta} = \mathrm{div}\underline{\theta}$. Next consider the auxiliary problem

$$\left.\begin{aligned} -\Delta\underline{\overline{\theta}} + \underline{\nabla}p &= -\Delta\underline{\theta} \\ \mathrm{div}\underline{\overline{\theta}} &= \mathrm{div}\underline{\theta} \end{aligned}\right\} \tag{3.32}$$

which obviously has the unique solution $\underline{\overline{\theta}} = \underline{\theta}$ and $p = 0$. Next consider its finite element approximation

$$\begin{aligned} \int \underline{\nabla}\underline{\theta}_h \cdot \underline{\nabla}\underline{\eta}_h - \int p_h \, \mathrm{div}\underline{\eta}_h &= \int \underline{\nabla}\underline{\theta}\cdot\underline{\nabla}\underline{\eta}_h \quad \forall \, \underline{\eta}_h \in (Q_h)^2 \\ \int_R \mathrm{div}\underline{\theta}_h &= \int_R \mathrm{div}\underline{\theta} \quad \forall \, R \in \mathcal{T}_h \end{aligned} \tag{3.33}$$

where $\underline{\theta}_h$ is sought in $(Q_h)^2$ and p_h among piecewise constants. Following [8], [10] (and their notations) we obtain that (3.33) has a unique solution, which satisfies

$$||\pi p_h||_0 + h||p_h||_0 \le c \, ||\underline{\theta}-\underline{\theta}_h||_1 \tag{3.34}$$

where π is a projection operator that filters out the checker board modes by blocks of four elements. Hence we have, for all $\underline{\theta}_I \in (Q_h)^2$,

$$\|\underline{\nabla\theta}-\underline{\nabla\theta}_h\|_0^2 = \int(\underline{\nabla\theta}-\underline{\nabla\theta}_h)\cdot(\underline{\nabla\theta}-\underline{\nabla\theta}_I) + \int(\underline{\nabla\theta}-\underline{\nabla\theta}_h)\cdot(\underline{\nabla\theta}_I-\underline{\nabla\theta}_h) \tag{3.35}$$

On the other hand

$$\int(\underline{\nabla\theta}-\underline{\nabla\theta}_h)\cdot(\underline{\nabla\theta}_I-\underline{\nabla\theta}_h) = \int p_h \operatorname{div}(\underline{\theta}_I-\underline{\theta}_h) = \int p_h \operatorname{div}(\underline{\theta}_I-\underline{\theta}) = \int \pi p_h \operatorname{div}(\underline{\theta}_I-\underline{\theta}) + \int (I-\pi)p_h \operatorname{div}(\underline{\theta}_I-\underline{\theta}) \tag{3.36}$$

If we choose now θ_I to be the interpolant of $\underline{\theta}$ by blocks of four elements ([8]) we have $\int (I-\pi p_h) \operatorname{div}\underline{\theta}_I=0$. Then

$$\int \pi p_h \operatorname{div}(\underline{\theta}_I-\underline{\theta}) \leq c\|\underline{\theta}-\underline{\theta}_h\|_1 \|\underline{\theta}_I-\underline{\theta}\|_1 \tag{3.37}$$

$$\int (I-\pi)p_h \operatorname{div}\underline{\theta} \leq c\|p_h\|_0 |h|^2 \|\operatorname{div}\underline{\theta}\|_2 \tag{3.38}$$

because of the shape of $(I-\pi)p_h$. Combining (3.35)-(3.38) we have

$$\|\underline{\theta}-\underline{\theta}_h\|_1 \leq c|h| \|\underline{\theta}\|_3 \tag{3.39}$$

Now we rotate back, setting

$$\beta_{I,1} := \theta_{h,2}, \quad \beta_{I,2} = -\theta_{h,1} \tag{3.40}$$

Therefore (3.39) implies (3.16) and (3.33) with (3.12) gives (3.22).

Remark. In the proof of theorem 3.1, we discard at several points, information (for instance an estimate of order $|h|t$ would be enough in (3.28) instead of the $|h|t^2$). This is due to two reasons. Firstly, our estimate is not optimal. The optimal estimate to be expected should be, for instance

$$\|\underline{\beta}-\underline{\beta}_h\|_1 + \|w-w_h\|_1 + \|\underline{\gamma}-\underline{\gamma}_h\|_\Gamma \leq c|h|$$

for $\underline{\beta}$, w, $\underline{\gamma}$ smooth enough. Secondly, since the estimate is already non-optimal, we did not endeavor to reduce the regularity required on $\underline{\beta}$ and $\underline{\gamma}$. An improvement of (3.15) in this respect should also be possible.

Remark. We used the assumption that the mesh $\mathcal{T}_h$ is uniform only in proof of lemma 3.3 (namely in (3.38)). Actually a general rectangular mesh can be allowed, provided that each rectangle is then split into sixteen equal subrectangles, see [10] for more details.

4. CONCLUSIONS

We analyzed from the mathematical point of view the finite element discretization proposed in [3] for Mindlin plates. At least for particular cases (like a uniform rectangular mesh) we proved that the element is uniformly stable with respect to the thickness parameter t and that it converges with optimal rate $O(|h|)$ in H^1, uniformly in t. We did not prove uniform stability of the "shear strains" $\underline{\gamma}_h = t^{-2} (\underline{\nabla} w_h - \underline{\beta}_h^*)$ nor uniform convergence. Actually, we have $\|\underline{\gamma} - \underline{\gamma}_h\|_0 \leq c|h|t^{-1}$ which is basically unsatisfactory. Probably a filtering procedure should be applied to $\underline{\gamma}_h$ in order to have L^2 stability and an optimal rate of convergence uniformly in t.

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