INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING, VOL. 18, 1565–1580 (1982)

SHORT COMMUNICATIONS

ON THE DISPLACEMENT FORMULATION OF TORSION OF SHAFTS WITH RECTANGULAR CROSS-SECTIONS

KLAUS-JÜRGEN BATHE† AND ANIL CHAUDHARY‡

Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts, U.S.A.

SUMMARY

The use of warping displacement functions for the torsional stiffness representation of beams with rectangular cross-sections is studied. These functions can directly be employed in the displacement-based formulation of Hermitian and isoparametric three-dimensional beam elements for linear, elastic–plastic or large displacement/large rotation analysis. The results of some studies are given to demonstrate the applicability and use of the proposed functions.

INTRODUCTION

The computer analysis of beam structures has certainly been given much attention during the past two decades (see, for example, Reference 1). At present, in by far the most analyses, the displacement method is employed. Using this technique in linear analysis, the 'exact' stiffness properties ('exact' within beam theory) of each beam element can be calculated (in static analysis) by the solution of the governing differential equations. Alternatively, if the 'exact' displacement variations within the beam element are known, the principle of virtual displacements can be employed to obtain these same stiffness coefficients.

In nonlinear analysis, however, and in the linear analysis of non-prismatic beams, it may be necessary, or it can be more effective, to use the principle of virtual displacements with only approximate displacement functions. In this case, the analysis corresponds to a displacement-based finite element solution with the stress–strain relations and compatibility requirements all satisfied exactly, but stress equilibrium within the beam element only satisfied approximately. This approach is, for example, used in the development of Hermitian beam elements[2] and isoparametric beam elements[3] for elastic–plastic and large displacement analysis.

Most of the developments available for nonlinear analysis of beam structures are only applicable to two-dimensional actions, and for this case the appropriate displacement functions are well established. However, a particularly difficult area can be the analysis of beams in three-dimensional action because the torsional rigidity must be properly taken into account.

The objective in this short communication is to propose appropriate displacement functions for the torsional displacements of beams of general rectangular cross-sections. These functions can directly be employed in the formulation of a 2-node Hermitian-based beam[2] and the formulation of a variable-number-nodes isoparametric beam[3] so as to make these elements with arbitrary rectangular cross-sections applicable to linear and nonlinear analysis. The warping displacement assumptions given in this paper were not used in References 2 and 3,

† Associate Professor.
‡ Research Assistant.

0029–5981/82/101565–04$01.00

Received 1 February 1982

but should be included in the formulations if the torsional action of a rectangular cross-section is important in the analysis.

DISPLACEMENT FUNCTIONS

Consider a shaft of general rectangular cross-section, sides a and b, as shown in Figure 1. When the beam is subjected to a torque, the components of displacements are

$$u = -\theta yz \tag{1}$$

$$v = \theta xz \tag{2}$$

$$w = f(x, y) \tag{3}$$

The assumptions for the displacements u and v are contained in the usual displacement assumptions,[2,3] and since the warping displacement is zero for a circular section no further considerations are necessary in that case. However, when the beam is of rectangular cross-section, the warping displacement is an important ingredient in the calculation of the torsional stiffness.

Considering the general rectangular cross-section in Figure 1, we want to use a warping displacement assumption that is as close as possible to the actual warping displacements

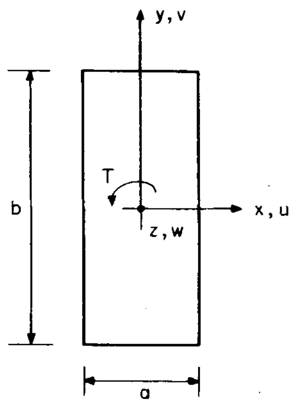

Figure 1. Rectangular section subjected to torque T resulting in rotation per unit length θ

without over-complicating the analysis. Using the theory of elasticity, for a square section (case $a = b$) the warping displacement is

$$w = \alpha_0 xy\,(x^2 - y^2) \tag{4}$$

where α_0 is a constant, which depends on the torque, the shaft dimension and the shear modulus of the material. In case $b \gg a$ (a very narrow section) the warping displacement is

$$w = \beta_0 xy \tag{5}$$

where β_0 is constant. Hence, it appears reasonable to use in the displacement-based finite element formulation the following displacement assumption for a beam of general rectangular cross-section:

$$w = \alpha_1 xy + \beta_1 xy(x^2 - y^2) \tag{6}$$

This warping displacement assumption can be employed in linear analysis, in small displacement materially nonlinear (elastic–plastic) analysis and in large displacement/large rotation elastic or elastic–plastic analysis using an updated or total Lagrangian formulation.[2,3]

Note that using the warping displacement in equation (6) for the complete beam element, it is assumed that the torque is constant in the element. With equation (6) the usual strain–displacement matrices of the Hermitian and isoparametric beam elements can directly be amended for the degrees-of-freedom α_1 and β_1. The resulting stiffness matrices then correspond to the usual beam degrees-of-freedom plus the degrees-of-freedom α_1 and β_1, which may be eliminated by static condensation prior to the assemblage of the element matrices into the global structural stiffness matrix. This approach means that continuity in the warping displacements between elements is not enforced.

SAMPLE PROBLEMS

The above analysis procedure has been implemented in the computer program ADINA for the 2-node Hermitian beam and the 2- to 4-node isoparametric beam.[5] The objective in this section is to summarize briefly some simple, demonstrative analysis results. Since, in a local co-ordinate system, the torsional action is decoupled from the flexural and shear actions, only the torsional behaviour is studied.

Linear elastic torsional stiffness of rectangular sections

In this analysis we study the applicability of the functions in equation (6) to describe the warping displacements. Table I compares the torsional stiffnesses calculated using ADINA

Table I. Torsion constant k in formula, $T = k\,G\theta\,a^3 b$

	k	
$\frac{b}{a}$	Timoshenko[4]	ADINA
1·0	0·141	0·141
2·0	0·229	0·230
4·0	0·281	0·289
10·0	0·312	0·323
100·0	0·333	0·333

and by the theory of elasticity (Reference 4, p. 312). It is seen that the torsional stiffness predicted by ADINA has a maximum discrepancy of about 3.5 per cent to the values given by Timoshenko.

Elastic–plastic torsional stiffness of a square cross-section

In this analysis the square section shown in Figure 2 was analysed. The solution was performed to compare the results with those given by Greenberg *et al.*[6] These authors assumed a Ramberg–Osgood material law and performed a finite difference analysis with a 24×24 finite difference grid. With ADINA, a bilinear elastic–plastic material assumption that approximates as closely as possible the Ramberg–Osgood material law was used, and 4×4 Gauss integration was employed. In both analyses the von Mises flow theory was used. Figure 2 shows good correspondence in the analysis results.

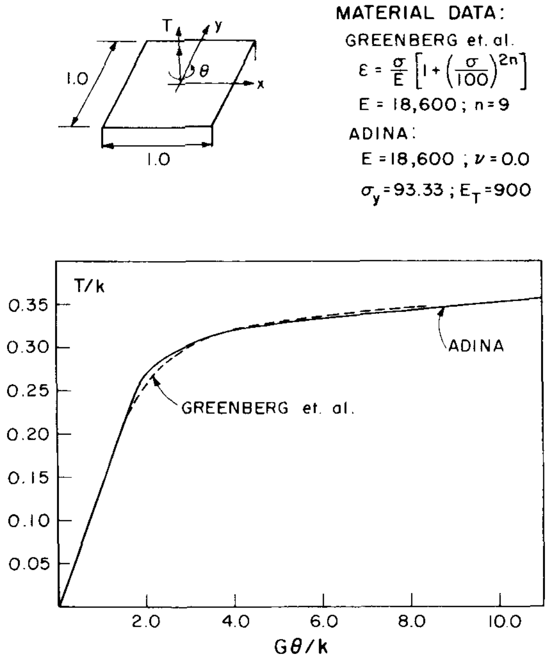

Figure 2. Elastic–plastic analysis of torsion problem ($k = 100/\sqrt{3}$, θ = rotation per unit length)

CONCLUDING REMARKS

Appropriate interpolation functions to represent the warping displacements of general rectangular sections have been studied. These functions can directly be employed in linear and nonlinear analyses. Also, since the St. Venant torsional stiffnesses of I, L and T-sections can be calculated as the sum of the St. Venant torsional stiffnesses of the component rectangular sections, the total stiffness of such a section can be obtained by modelling it as an assemblage of rectangular sections (with the use of constraint equations to eliminate the dependent degrees-of-freedom).

ACKNOWLEDGEMENTS

We are grateful to the ADINA users' group for the financial support of this work.

REFERENCES

1. J. S. Przemieniecki, *Theory of Matrix Structural Analysis*, McGraw-Hill, New York, 1968.
2. K. J. Bathe and S. Bolourchi, 'Large displacement analysis of three dimensional beam structures', *Int. J. num. Meth. Engng*, **14**, 961–986 (1979).
3. K. J. Bathe, *Finite Element Procedures in Engineering Analysis*, Prentice-Hall, N.J., 1982.
4. S. P. Timoshenko and J. N. Goodier, *Theory of Elasticity*, McGraw-Hill, New York, 1970.
5. 'ADINA-A finite element program for automatic dynamic incremental nonlinear analysis', *Report AE 81-1*, ADINA Engineering, September 1981.
6. H. J. Greenberg, W. S. Dorn and E. H. Wetherell, 'A comparison of flow and deformation theories in plastic torsion of a square cylinder', in *Plasticity*, Proc. 2nd Symp. on Naval Structural Mechanics (E. H. Lee and P. S. Symonds, Eds.), Pergamon Press, New York, 1960.