EARTHQUAKE ENGINEERING AND STRUCTURAL DYNAMICS, VOL. 1, 283–291 (1973)

STABILITY AND ACCURACY ANALYSIS OF DIRECT INTEGRATION METHODS

K. J. BATHE* AND E. L. WILSON†
Department of Civil Engineering, University of California, Berkeley, U.S.A.

SUMMARY

A systematic procedure is presented for the stability and accuracy analysis of direct integration methods in structural dynamics. Amplitude decay and period elongation are used as the basic parameters in order to compare various integration methods. The specific methods studied are the Newmark generalized acceleration scheme, the Houbolt method and the Wilson θ-method. The advantages of each of these methods are discussed. In addition, it is shown how the direct integration of the equations of motion is related to the mode superposition analysis.

INTRODUCTION

Many integration methods are currently used for the direct integration of the equations of motion of lumped parameter structural systems.[1–5] Some investigators have concluded that a particular method is superior for a certain type of problem. However, a procedure is lacking which can be used to compare the merits of these methods in practical application for complex structural systems.

The stability of time integration methods can be proven by invoking one of the established theorems.[6,7] Also, various methods have been compared by studying a single degree-of-freedom system.[7] However, since accuracy is not required in all modes of a complex structure this is not an adequate basis for comparison. In fact, the participation of all modes in the solution is not desirable in most dynamic problems.

The objective of this paper is to present a systematic and fundamental procedure for the stability and accuracy analysis of direct integration schemes and to apply the technique to the Newmark, the Houbolt and the Wilson θ-method. The θ-method is optimized with respect to stability and accuracy. The integration methods are compared and the relationship between integration and mode superposition analysis is discussed. Based on the results of the analysis, guidelines can be established to select an appropriate time step for a given problem.

Further research is required to develop better integration operators for linear and, in particular, non-linear problems. The direct procedure of stability and accuracy analysis presented in this paper can be very effective in this research. However, it should be recognized that the efficiency of an integration algorithm also depends on other factors—for example, the number of numerical operations required for solution—which are not discussed here.

PROBLEM DEFINITION

In the dynamic response analysis of an n-degree of freedom structural system we are concerned with the solution of the equation

$$\mathbf{M}\ddot{\mathbf{u}}+\mathbf{C}\dot{\mathbf{u}}+\mathbf{K}\mathbf{u}=\mathbf{R} \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, stiffness and damping matrices, all of order n; the vectors $\mathbf{u}$ and $\mathbf{R}$ store the displacements and forces, respectively, and a dot denotes a time derivative.[8] This equation arises, in particular, in the finite element analysis of continuous systems.[9] In this paper we assume that the system is linear, in which case the elements in $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are constant.

* Assistant Research Engineer.

† Associate Professor.

Received 20 April 1972

Mode superposition analysis

If the damping is assumed to be of a restricted form,[10] the quadratic eigenvalue problem is avoided and the solution to equation (1) can be obtained by conventional mode superposition. In this analysis we consider first free vibration conditions with damping neglected

$$\mathbf{M}\ddot{\mathbf{u}}+\mathbf{K}\mathbf{u}=0 \tag{2}$$

Substituting $\mathbf{u}=\boldsymbol{\phi}\sin\omega t$, we obtain the generalized eigenvalue problem

$$\mathbf{K}\boldsymbol{\phi}=\omega^2\mathbf{M}\boldsymbol{\phi} \tag{3}$$

The n solutions of equation (3) can be written as

$$\mathbf{K}\boldsymbol{\Phi}=\mathbf{M}\boldsymbol{\Phi}\boldsymbol{\Omega}^2 \tag{4}$$

where the columns in $\boldsymbol{\Phi}$ are the M-orthonormalized eigenvectors (free vibration modes) $\boldsymbol{\phi}_1\ldots\boldsymbol{\phi}_n$ and $\boldsymbol{\Omega}^2$ is a diagonal matrix listing the eigenvalues (free vibration frequencies squared) $\omega_1^2\ldots\omega_n^2$.

The next step is to write the equations of dynamic equilibrium in the basis of eigenvectors; using $\mathbf{u}=\boldsymbol{\Phi}\mathbf{X}$ we obtain

$$\ddot{\mathbf{X}}+\boldsymbol{\Delta}\dot{\mathbf{X}}+\boldsymbol{\Omega}^2\mathbf{X}=\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{R} \tag{5}$$

where $\boldsymbol{\Delta}=\operatorname{diag}(2\omega_i\,\xi_i)$ and ξ_i is the damping ratio in the i'th mode of vibration. Equation (5) consists of n uncoupled equations which can be solved 'exactly' using the Duhamel integral. Alternatively we may use numerical integration. Because the periods of vibration T_i, $i=1,\ldots,n$, where $T_i=2\pi/\omega_i$, are known we can choose in the step-by-step integration of each equation a time step Δt which assures a required level of accuracy.

The most time consuming phase of the analysis is the solution of the eigenvalue problem. If the order of the matrices is large, the computer time required to solve all eigenvalues and vectors can be enormous. However, it may be sufficiently accurate to include in the analysis only the lowest eigenvalues and associated vectors because the higher modes do not participate in the response. Also, in comparison to the continuous problem the highest modes of the discrete element system should be expected to be in error, so that there may be little justification to include them in the analysis.

Direct step-by-step integration

An alternative procedure to obtain the solution to equation (1) is by direct integration.[8] In this case the step-by-step integration is performed directly on equation (1) without first representing the equilibrium relations in the basis of eigenvectors. Whereas in the solution of the uncoupled equations a different time step can be chosen for each equation to insure integration accuracy, in the direct integration one time step is used and the response in all modes is integrated simultaneously. This is equivalent to choosing a common time step Δt in the integration of all n uncoupled equations. Accuracy in this integration can be obtained only in the evaluation of those response components for which Δt is a small fraction of the period. The other modal response components will not be evaluated accurately, but the errors will be unimportant if the amplitudes are small; however, we need integration stability for all modes. This means that the initial conditions for the equations with a large value $\Delta t/T_i$ must not be amplified artificially and thus make the accurate integration of the response in the lower modes worthless. Stability also means that any errors in the displacements, velocities and accelerations at time t which may be due to round-off do not grow in the integration. Naturally, stability is assured if the time step is small enough to integrate accurately the response in the highest frequency component. But this may require a very small timestep and, as has been pointed out, the accurate integration of this response is usually not necessary.

DIRECT INTEGRATION SCHEMES

Because the direct integration of equation (1) is equivalent to the integration of equation (5) with a common time step Δt, we only need to study the integration of a typical row in equation (5), which may be written as

$$\ddot{x}+2\xi\omega\dot{x}+\omega^2 x = r \tag{6}$$

This is the equation governing motion of a single degree of freedom system with free vibration period T, damping ratio ξ and applied load r.

In the step-by-step solutions considered here an approximation operator and a load operator are used which relate explicitly the unknown required variables at time $t+\Delta t$ to previously calculated quantities.

The Wilson θ-method

Let the acceleration, velocity and displacement at time t, i.e. $\ddot{x}_t$, $\dot{x}_t$ and x_t, where the subscript denotes time t, be known quantities. For solution of $\ddot{x}_{t+\Delta t}$, $\dot{x}_{t+\Delta t}$ and $x_{t+\Delta t}$ we assume that the acceleration varies linearly during the time interval $\theta\Delta t$, where $\theta \geqslant 1$. The parameter θ shall be chosen to obtain accuracy and stability in the integration. When $\theta = 1$ we have the linear acceleration method which is known to be only conditionally stable. In Wilson's averaging method θ equals 2 and the integration is unconditionally stable. However, without losing unconditional stability, θ can be selected to obtain a scheme which has less integration error.

Let τ denote the increase in time, where $0 \leqslant \tau \leqslant \theta\Delta t$, then for the time interval t to $t+\theta\Delta t$ we have

$$\ddot{x}_{t+\tau} = \ddot{x}_t + (\ddot{x}_{t+\Delta t} - \ddot{x}_t)\frac{\tau}{\Delta t} \tag{7}$$

$$\dot{x}_{t+\tau} = \dot{x}_t + \ddot{x}_t\tau + (\ddot{x}_{t+\Delta t} - \ddot{x}_t)\frac{\tau^2}{2\Delta t} \tag{8}$$

$$x_{t+\tau} = x_t + \dot{x}_t\tau + \tfrac{1}{2}\ddot{x}_t\tau^2 + (\ddot{x}_{t+\Delta t} - \ddot{x}_t)\frac{\tau^3}{6\Delta t} \tag{9}$$

At time $t+\Delta t$ we have

$$\dot{x}_{t+\Delta t} = \dot{x}_t + (\ddot{x}_{t+\Delta t} + \ddot{x}_t)\frac{\Delta t}{2} \tag{10}$$

$$x_{t+\Delta t} = x_t + \dot{x}_t\Delta t + (2\ddot{x}_t + \ddot{x}_{t+\Delta t})\frac{\Delta t^2}{6} \tag{11}$$

Equation (6) shall be satisfied at time $t+\theta\Delta t$, which gives

$$\ddot{x}_{t+\theta\Delta t} + 2\xi\omega\dot{x}_{t+\theta\Delta t} + \omega^2 x_{t+\theta\Delta t} = r_{t+\theta\Delta t} \tag{12}$$

Using equations (7)–(9) at time $\tau = \theta\Delta t$ to substitute into equation (12) an equation is obtained with $\ddot{x}_{t+\Delta t}$ as the only unknown. Solving for $\ddot{x}_{t+\Delta t}$ and substituting into equations (10) and (11) the following relationship is established

$$\begin{bmatrix} \ddot{x}_{t+\Delta t} \\ \dot{x}_{t+\Delta t} \\ x_{t+\Delta t} \end{bmatrix} = \mathbf{A} \begin{bmatrix} \ddot{x}_t \\ \dot{x}_t \\ x_t \end{bmatrix} + \mathbf{L} r_{t+\theta\Delta t} \tag{13}$$

where **A** is the approximation operator and **L** is the load operator; both are given in Table I. This recurrence relation can be used to study the stability and accuracy of the integration scheme, where we note that the solution at time $t+n\Delta t$ with n an integer is given by

$$\begin{bmatrix} \ddot{x}_{t+n\Delta t} \\ \dot{x}_{t+n\Delta t} \\ x_{t+n\Delta t} \end{bmatrix} = \mathbf{A}^n \begin{bmatrix} \ddot{x}_t \\ \dot{x}_t \\ x_t \end{bmatrix} + \mathbf{A}^{n-1}\mathbf{L} r_{t+\Delta t+(\theta-1)\Delta t} + \ldots + \mathbf{L} r_{t+n\Delta t+(\theta-1)\Delta t} \tag{14}$$

Table I. Approximation and load operator of the Wilson method

$$\mathbf{A} = \begin{bmatrix} \left(1-\dfrac{\beta\theta^2}{3}-\dfrac{1}{\theta}-\kappa\theta\right) & \dfrac{1}{\Delta t}(-\beta\theta-2\kappa) & \dfrac{1}{\Delta t^2}(-\beta) \\ \Delta t\left(1-\dfrac{1}{2\theta}-\dfrac{\beta\theta^2}{6}-\dfrac{\kappa\theta}{2}\right) & \left(1-\dfrac{\beta\theta}{2}-\kappa\right) & \dfrac{1}{\Delta t}\left(\dfrac{-\beta}{2}\right) \\ \Delta t^2\left(\dfrac{1}{2}-\dfrac{1}{6\theta}-\dfrac{\beta\theta^2}{18}-\dfrac{\kappa\theta}{6}\right) & \Delta t\left(1-\dfrac{\beta\theta}{6}-\dfrac{\kappa}{3}\right) & \left(1-\dfrac{\beta}{6}\right) \end{bmatrix}; \quad \mathbf{L} = \begin{bmatrix} \dfrac{\beta}{\omega^2\Delta t^2} \\ \dfrac{\beta}{2\omega^2\Delta t} \\ \dfrac{\beta}{6\omega^2} \end{bmatrix}$$

where

$$\beta = \left(\frac{\theta}{\omega^2\Delta t^2}+\frac{\xi\theta^2}{\omega\Delta t}+\frac{\theta^3}{6}\right)^{-1}; \quad \kappa = \frac{\xi\beta}{\omega\Delta t}$$

The Newmark generalized acceleration method

In this integration scheme it is assumed that

$$\dot{x}_{t+\Delta t} = \dot{x}_t + [(1-\delta)\,\ddot{x}_t + \delta\ddot{x}_{t+\Delta t}]\,\Delta t \tag{15}$$

and

$$x_{t+\Delta t} = x_t + \dot{x}_t\Delta t + [(\tfrac{1}{2}-\alpha)\,\ddot{x}_t + \alpha\ddot{x}_{t+\Delta t}]\,\Delta t^2 \tag{16}$$

The parameters δ and α can be chosen to obtain integration stability and accuracy. When $\delta = \frac{1}{2}$ and $\alpha = \frac{1}{6}$ we have the equations of the linear acceleration method. Newmark proposed as an unconditionally stable scheme the constant average acceleration method, in which case $\delta = \frac{1}{2}$ and $\alpha = \frac{1}{4}$.

Table II gives the approximation operator and the load operator of the Newmark method, which are obtained using equations (15) and (16) together with equation (12) when $\theta = 1$. Note the close relationship between this approximation operator and the operator of Wilson's method.

Table II. Approximation and load operator of the Newmark method

$$\mathbf{A} = \begin{bmatrix} [-(\tfrac{1}{2}-\alpha)\,\beta-2(1-\delta)\,\kappa] & \dfrac{1}{\Delta t}(-\beta-2\kappa) & \dfrac{1}{\Delta t^2}(-\beta) \\ \Delta t[1-\delta-(\tfrac{1}{2}-\alpha)\,\delta\beta-2(1-\delta)\,\delta\kappa] & (1-\beta\delta-2\delta\kappa) & \dfrac{1}{\Delta t}(-\beta\delta) \\ \Delta t^2[\tfrac{1}{2}-\alpha-(\tfrac{1}{2}-\alpha)\,\alpha\beta-2(1-\delta)\,\alpha\kappa] & \Delta t(1-\alpha\beta-2\alpha\kappa) & (1-\alpha\beta) \end{bmatrix}; \quad \mathbf{L} = \begin{bmatrix} \dfrac{\beta}{\omega^2\Delta t^2} \\ \dfrac{\beta\delta}{\omega^2\Delta t} \\ \dfrac{\alpha\beta}{\omega^2} \end{bmatrix}$$

where

$$\beta = \left(\frac{1}{\omega^2\Delta t^2}+\frac{2\xi\delta}{\omega\Delta t}+\alpha\right)^{-1}; \quad \kappa = \frac{\xi\beta}{\omega\Delta t}$$

The Houbolt method

In the Houbolt integration scheme two backward difference formulae are used for the acceleration and velocity at time $t+\Delta t$, namely

$$\ddot{x}_{t+\Delta t} = \frac{1}{\Delta t^2}(2x_{t+\Delta t}-5x_t+4x_{t-\Delta t}-x_{t-2\Delta t}) \tag{17}$$

$$\dot{x}_{t+\Delta t} = \frac{1}{6\Delta t}(11x_{t+\Delta t}-18x_t+9x_{t-\Delta t}-2x_{t-2\Delta t}) \tag{18}$$

Substituting equations (17) and (18) into equation (12) when $\theta = 1$, we can establish the relation

$$\begin{bmatrix} x_{t+\Delta t} \\ x_t \\ x_{t-\Delta t} \end{bmatrix} = \mathbf{A}\begin{bmatrix} x_t \\ x_{t-\Delta t} \\ x_{t-2\Delta t} \end{bmatrix} + \mathbf{L}r_{t+\Delta t} \tag{19}$$

where $\mathbf{A}$ and $\mathbf{L}$ are given in Table III.

To calculate the displacements and velocities at times between the discrete time points, we may use in the Wilson method equations (8) and (9) and in the Newmark method equations (15) and (16) with Δt being

Table III. Approximation and load operator of the Houbolt method

$$\mathbf{A} = \begin{bmatrix} \left(\frac{5\beta}{\omega^2 \Delta t^2} + 6\kappa\right) & -\left(\frac{4\beta}{\omega^2 \Delta t^2} + 3\kappa\right) & \left(\frac{\beta}{\omega^2 \Delta t^2} + \frac{2\kappa}{3}\right) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}; \quad \mathbf{L} = \begin{bmatrix} \frac{\beta}{\omega^2} \\ 0 \\ 0 \end{bmatrix}$$

where

$$\beta = \left(\frac{2}{\omega^2 \Delta t^2} + \frac{11\xi}{3\omega\Delta t} + 1\right)^{-1}; \quad \kappa = \frac{\xi\beta}{\omega\Delta t}$$

replaced by the increment in time τ, i.e. $0 \leqslant \tau \leqslant \Delta t$. In the Houbolt method an interpolating polynomial of order three which fits the displacements at the four discrete time points in equation (19) was used.

STABILITY

Consider equation (14) with r equal to zero. An integration method is unconditionally stable if the solution for any initial conditions does not grow without bound for any time step Δt, in particular when $\Delta t/T$ is large. We should note then, that any error in the displacements, velocities and accelerations at time t, for example due to round-off in the computer, does not grow. The method is only conditionally stable if the same only holds provided $\Delta t/T$ is smaller than a certain number. In discrete element analysis of continuous systems we may have very high (infinite) frequencies,[11] and an unconditionally stable scheme is needed.

To investigate the stability of a method we realize that $\mathbf{A} = \mathbf{P}^{-1}\mathbf{J}\mathbf{P}$ and therefore in equation (14)

$$\mathbf{A}^n = \mathbf{P}^{-1}\mathbf{J}^n\mathbf{P} \tag{20}$$

where $\mathbf{P}$ is the matrix of eigenvectors and $\mathbf{J}$ is the Jordan form of $\mathbf{A}$ with the eigenvalues λ_i of $\mathbf{A}$ on its diagonal.[12] Let $\rho(\mathbf{A})$ be the spectral radius of $\mathbf{A}$, defined as

$$\rho(\mathbf{A}) = \max|\lambda_i|, \quad i = 1, 2, 3 \tag{21}$$

then $\mathbf{J}^n$ is bounded for $n \to \infty$ if and only if $\rho(\mathbf{A}) \leqslant 1$. This is the stability criterion. Furthermore, $\mathbf{J}^n \to 0$ if $\rho(\mathbf{A}) < 1$ and the smaller $\rho(\mathbf{A})$ the more rapid is the convergence.

Before the eigenvalues of $\mathbf{A}$ are calculated it can be convenient to apply a similarity transformation $\mathbf{D}^{-1}\mathbf{A}\mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix with $d_{ii} = \Delta t^i$. As we would expect the spectral radii of the approximation operators therefore depend on $\Delta t/T$, ξ, θ, α and δ but are independent of Δt.

The unconditional stability of the Newmark and the Houbolt method was discussed in References 1, 4 and 7.

Consider the stability of the Wilson operator. Figure 1 shows $\rho(\mathbf{A})$ as a function of θ for different values of $\Delta t/T$ and ξ. We note that the curves for $\Delta t/T = 0$ and $\Delta t/T = \infty$ are independent of ξ, and that the method is unconditionally stable, i.e. $\rho(\mathbf{A}) \leqslant 1$ for any $\Delta t/T$ ratio, provided $\theta \geqslant 1{\cdot}37$. For $\theta < 1{\cdot}37$. Where the method is only conditionally stable, the stability limit depends on the physical damping in the system.

There are therefore many different operators which can be used in a practical analysis. In the discussion to follow we consider the Houbolt method and two typical operators each of the Newmark and the Wilson method. In the Newmark method we let $\delta = \frac{1}{2}$ with $\alpha = \frac{1}{4}$ and $\delta = \frac{11}{20}$ with $\alpha = \frac{3}{10}$. In the Wilson method we consider the cases $\theta = 1{\cdot}4$ and $\theta = 2{\cdot}0$. The spectral radii of the corresponding operators as a function of $\Delta t/T$ are shown in Figure 2, where the unconditional stability of these integration schemes can be noted.

INTEGRATION ACCURACY

The accuracy of a numerical integration depends, in general, on the loading, the physical parameters of the system and the time step size. To obtain an idea of the integration accuracy using the five schemes mentioned above the response of a system with no physical damping is evaluated for two different initial conditions and no loading:

(1) $x_0 = 1{\cdot}0$, $\dot{x}_0 = 0{\cdot}0$ and $\ddot{x}_0 = -\omega^2$ for which the exact solution is $x = \cos \omega t$;

(2) $x_0 = 0{\cdot}0$, $\dot{x}_0 = \omega$ and $\ddot{x}_0 = 0{\cdot}0$ for which the exact solution is $x = \sin \omega t$.

In the Wilson and Newmark methods, equation (14) can be used directly with these initial conditions. In the Houbolt scheme the exact displacement values for $x_{\Delta t}$ and $x_{2\Delta t}$ have been used.

The solutions show that the errors in the numerical integration can be measured in terms of period elongation and amplitude decay. Figures 3–6 show the percentage period elongations and amplitude decays in the integration schemes as a function of $\Delta t/T$. In general, the integration is accurate when $\Delta t/T$ is smaller than about 0·01, but as $\Delta t/T$ increases the numerical integration results in large period elongations and amplitude decays. The Newmark method with $\delta = \frac{1}{2}$ and $\alpha = \frac{1}{4}$ is most accurate and only gives period

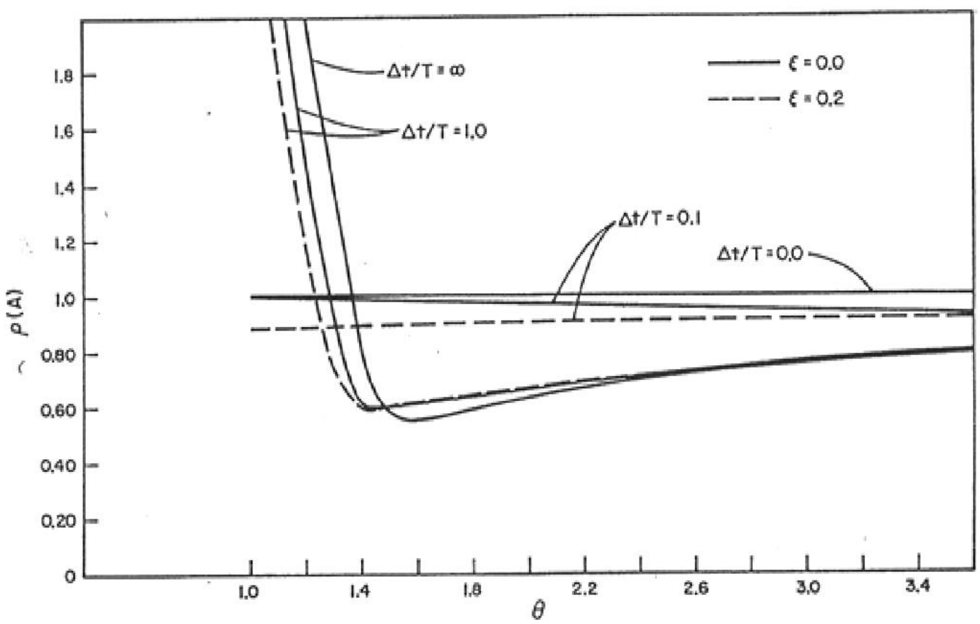

Figure 1. Spectral radius $\rho(A)$ as a function of θ in the Wilson method

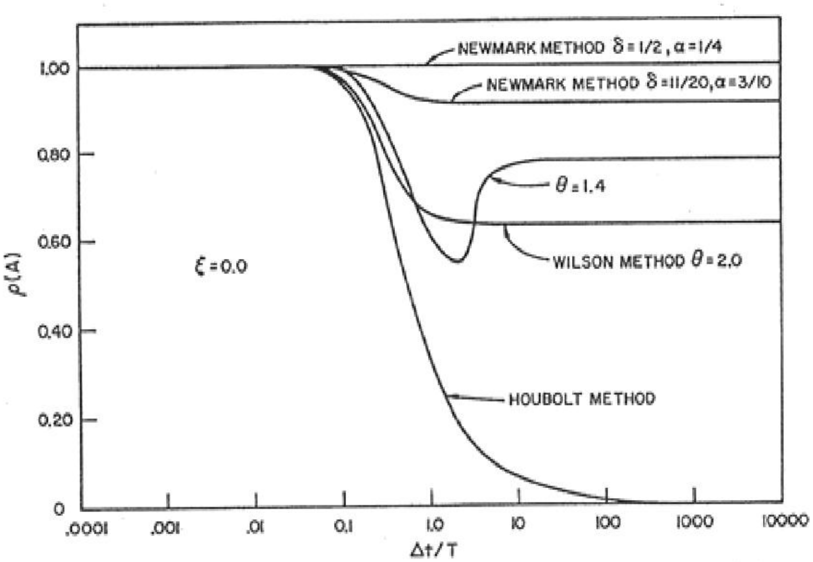

Figure 2. Spectral radii of approximation operators as a function of $\Delta t/T$

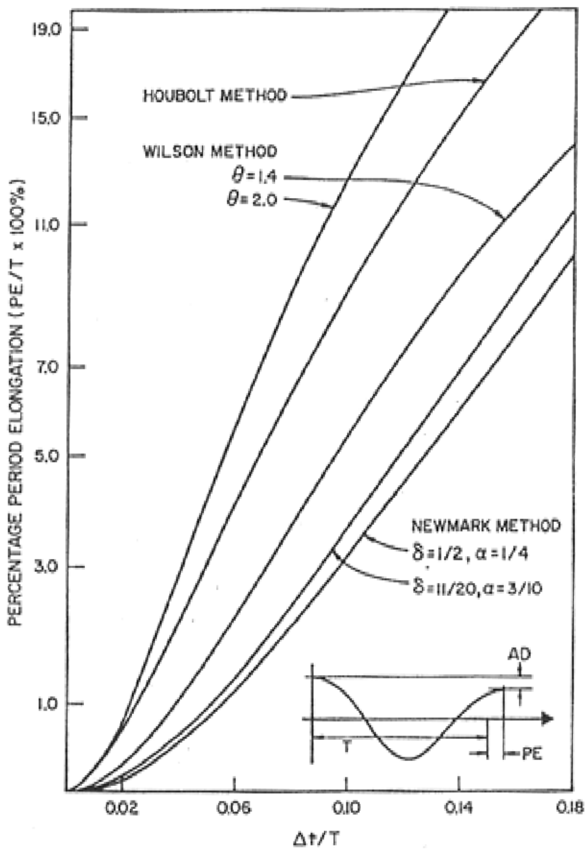

Figure 3. Percentage period elongations

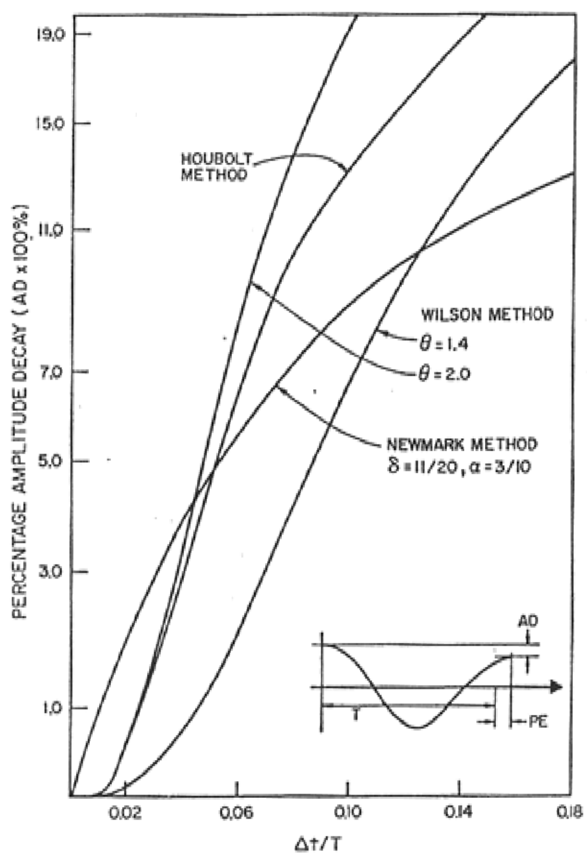

Figure 4. Percentage amplitude decays

elongations. A small negligible variation of the amplitude results from using equations (15) and (16) to find the maximum displacement at times between the discrete time points. Note the improvement in the accuracy of the Wilson method if $\theta = 1{\cdot}4$ is used.

From the discussion in the previous section the amplitude decay in an integration is directly related to the spectral radius $\rho(A)$ of the approximation operator. Figure 2 shows that for the Newmark method with $\delta = \frac{1}{2}$ and $\alpha = \frac{1}{4}$ the spectral radius is unity for any value $\Delta t/T$, but that for the other integration schemes $\rho(A)$ is smaller than unity for $\Delta t/T$ larger than about 0·01.

Consider now the simultaneous integration of all rows in equation (5), i.e. the direct integration of equation (1). We may choose a time step to obtain accuracy in the low mode responses in which we are interested. The question is then, what results are obtained with the same time step in the integration of the response in the higher modes. For illustration, assume that using Wilson's method with $\theta = 1{\cdot}4$ a time step is selected which gives $\Delta t/T_1 = 0{\cdot}01$, where T_1 is the fundamental period of the system. Let the initial conditions in each mode be those given in (1) above and let the integration be performed over 100 time steps. Figure 7 indicates the response in the higher modes. We observe that the amplitude decay caused by the numerical integration errors effectively 'filters' the high mode response out of the solution. The same effect is observed using the other integration schemes except when Newmark's method with $\delta = \frac{1}{2}$ and $\alpha = \frac{1}{4}$ is used. In this case the response in the high frequency components is retained in the solution with large errors in the periods.

The effective filtering of the high frequency response from the solution appears to be beneficial. Integration accuracy cannot be obtained in the response of the modes for which $\Delta t/T$ is large. But the filtering process allows one to obtain, with a relatively large time step, a total system solution in which the low mode response is accurately observed. Naturally, in this integration a scheme should be used which has minimum integration error when $\Delta t/T$ is small.

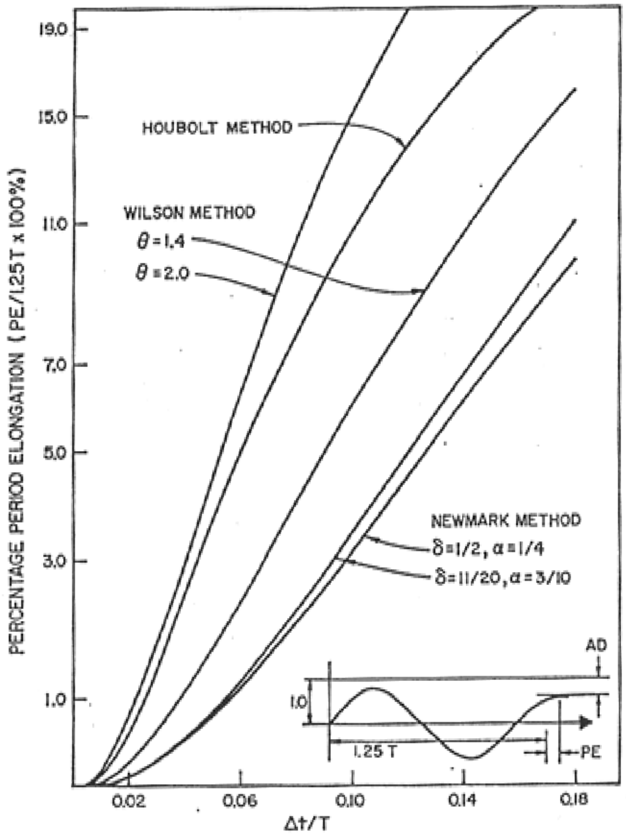

Figure 5. Percentage period elongations

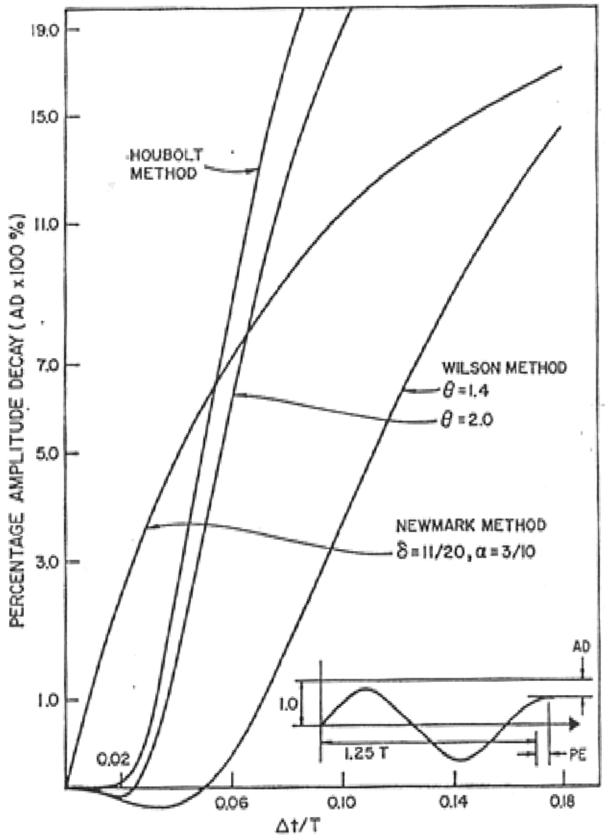

Figure 6. Percentage amplitude decays

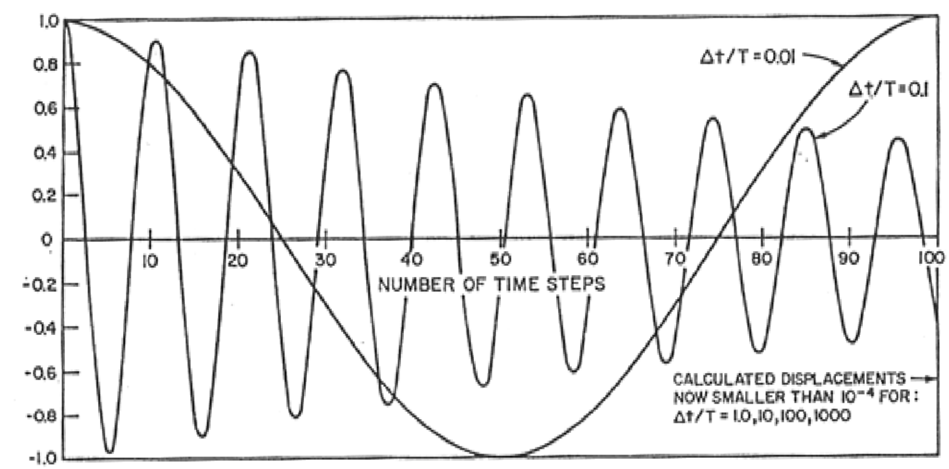

Figure 7. Displacement response in 100 time steps (Wilson method, $\theta = 1{\cdot}4$)

DIRECT INTEGRATION VERSUS MODE SUPERPOSITION

The observations made in the previous section may be used to draw an instructive comparison between a mode superposition analysis and the direct integration of equation (1) using an integration scheme in which the high frequency response is filtered out of the solution. As discussed, the direct integration is equivalent to a mode superposition analysis in which all eigenvalues and vectors have been calculated and the uncoupled equations in equation (5) are integrated with a common time step Δt. The integration is accurate for those modes for which $\Delta t/T$ is small, but the response in the modes for which $\Delta t/T$ is large is eliminated by the artificial damping. Therefore, the direct integration is quite equivalent to a mode superposition analysis in which only the lowest modes of the system are considered. The exact number of modes effectively included in the analysis depends on the time step Δt and the distribution of the periods. Clearly, the direct integration is most efficient when all important periods of the system are clustered together. In this case a time step which is chosen using the smallest of those periods is not unnecessarily small with respect to the largest period.

The comparison of the direct integration with mode superposition analysis also indicates on which basis the time step size should be chosen. The most important modes are those for which r in equation (6) is non-zero. Thus, the load distribution and frequency content of the loading largely determine which modes need to be integrated accurately and hence what size of time step should be used. In practice, the mode shapes and frequencies of the system are not known, and it is best to select a time step increment which will accurately represent all of the frequency content of the load.

CONCLUSIONS

A systematic and fundamental procedure for the stability and accuracy analysis of direct integration methods has been presented. The procedure was applied to the Newmark generalized acceleration method, the Houbolt method and the Wilson θ-method, which was optimized for integration accuracy. It is concluded that all of these methods will yield accurate results for certain types of problems. In addition, the relationship between direct integration and mode superposition was discussed. Both methods of analysis can be used to truncate the frequency domain.

It should be emphasized that the discussion of direct integration methods presented here has been limited to linear problems. Additional difficulties arise with the stability and accuracy of numerical methods applied to non-linear systems, and further research is needed concerning such cases. It is believed that the approach to stability and accuracy analysis described in this paper would be of value in these investigations.

REFERENCES

1. N. M. Newmark, 'A method of computation for structural dynamics', *Proc. Am. Soc. Civ. Engs*, 85, EM3, 67–94 (1959).
2. J. C. Houbolt, 'A recurrence matrix solution for the dynamic response of elastic aircraft', *J. Aeronaut. Sci.* 17, 540–550 (1950).
3. E. L. Wilson, 'A computer program for the dynamic stress analysis of underground structures', *SESM Report* 68–1, Department of Civil Engineering, University of California, Berkeley, 1968.
4. D. E. Johnson, 'A proof of the stability of the Houbolt method', *AIAA J.* 4, 1450–1451 (1966).
5. C. W. Gear, 'Numerical integration of stiff ordinary differential equations', *Report* No. 221, Department of Computer Science, University of Illinois, Urbana, 1967.
6. P. D. Lax and R. D. Richtmyer, 'Survey of the stability of finite difference equations', *Communs. Pure Appl. Math.* 9, 267–293 (1956).
7. R. E. Nickell, 'On the stability of approximation operators in problems of structural dynamics', *Int. J. Solids Structures*, 7, 301–319 (1971).
8. R. W. Clough, 'Analysis of structural vibrations and dynamic response', Proceedings, U.S.–Japan Symposium, *Recent Advances in Matrix Methods of Struct. Analysis and Design*, University of Alabama Press, University of Alabama, 1971.
9. O. C. Zienkiewicz and Y. K. Cheung, *The Finite Element Method in Structural and Continuum Mechanics*, McGraw-Hill, 1967.
10. E. L. Wilson and J. Penzien, 'Evaluation of orthogonal damping matrices', *Int. J. num. Meth Engng*, 4, 5–10 (1972).
11. K. J. Bathe, 'Solution methods for large generalized eigenvalue problems in structural engineering', *SESM Report* 71–20, Department of Civil Engineering, University of California, Berkeley, 1971.
12. J. H. Wilkinson, *The Algebraic Eigenvalue Problem*, Clarendon Press, Oxford, 1965.