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STUDIES OF FINITE ELEMENT PROCEDURES—STRESS SOLUTION OF A CLOSED ELASTIC STRAIN PATH WITH STRETCHING AND SHEARING USING THE UPDATED LAGRANGIAN JAUMANN FORMULATION

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Abstract—The updated Lagrangian Jaumann formulation is used for the analysis of an elastic closed strain path involving stretching and shearing. The analytical stress solution for each phase of the strain path is given and the final stresses at zero strain are calculated.

We do not present any new concept or procedures but study an interesting example solution that gives insight into the use of the updated Lagrangian Jaumann formulation.

1. INTRODUCTION

Much research effort has during the recent years been spent on the development of physically realistic and computationally efficient large strain finite element formulations. This development is particularly difficult in inelastic analysis, where the material characterizations are very complex and the computations may not be tractable.

A formulation that has found wide acceptance for large strain inelastic analysis is the updated Lagrangian Jaumann (U.L.J.) formulation [1, 2]. Here the Jaumann stress rate is used to express the stress–strain material behavior,

$$ {}^t\overset{\nabla}{\tau}_{ij} = C_{ijrs}\,{}^t\dot{e}_{rs} \tag{1}$$

and then the Cauchy stress is calculated from the relation

$$ {}^t\dot{\tau}_{ij} = {}^t\overset{\nabla}{\tau}_{ij} + {}^t\tau_{im}\,{}^t\Omega_{mj} + {}^t\tau_{jm}\,{}^t\Omega_{mi}, \tag{2}$$

where the left superscript t denotes 'at time t' and

${}^t\overset{\nabla}{\tau}_{ij}$ = components of Jaumann stress rate tensor,

C_{ijrs} = components of constitutive tensor,

${}^t\dot{e}_{ij}$ = components of velocity strain tensor

$$= \frac{1}{2}\left(\frac{\partial^t\dot{u}_i}{\partial^t x_j} + \frac{\partial^t\dot{u}_j}{\partial^t x_i}\right),$$

${}^t\Omega_{ij}$ = components of spin tensor

$$= \frac{1}{2}\left(\frac{\partial^t\dot{u}_j}{\partial^t x_i} - \frac{\partial^t\dot{u}_i}{\partial^t x_j}\right),$$

${}^t\dot{u}_i$ = velocity of material particle into direction i,

${}^t x_i$ = coordinate of material particle, direction i.

A main disadvantage of the U.L.J. formulation is that it is a *rate* formulation, which necessitates the integration of the stress as given by eqn (2). Hence, in analysis with large rigid body rotations, the magnitude of rotation—even when the material is elastic—that can be analyzed in one step is limited by the integration formula used. This is different in the updated Lagrangian Hencky formulation which is a *total* formulation and can accommodate large rigid body rotations in single solution steps [3, 4].

In addition to the shortcoming of the U.L.J. formulation mentioned above, there is the concern that, since it is a rate formulation, the stress solution is path-dependent, and hence for a closed strain path of an elastic material strain energy may be stored. Since the U.L.J. formulation is primarily employed in inelastic analysis, the importance of this point can be argued. However, the observation is at least interesting and we may note that the updated Lagrangian Hencky formulation does not accumulate strain energy under these conditions.

The objective of this short paper is to study, using the U.L.J. formulation, the residual stresses in a relatively simple case of a closed elastic strain path. We do not present any new concepts or procedures but our purpose is to study an example that gives some insight into what we consider to be a shortcoming of the updated Lagrangian Jaumann formulation.

2. SOLUTION OF THE CLOSED PATH STRAIN PROBLEM

Figure 1 shows the problem considered. A square piece of material—in effect a four-node finite element—of sides h is stretched, sheared and compressed until the fibers have returned to their orig-

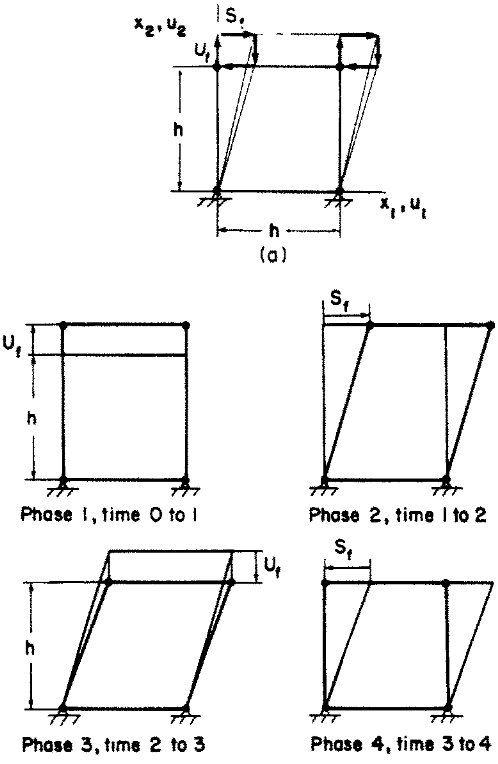

Fig. 1. Problem analyzed. (a) Strain path considered. (b) Individual strain phases.

inal positions. As indicated in Fig. 1, the complete strain path can be thought of to consist of four individual phases.

The stress–strain law for the material is given by the matrix **C**:

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix}, \tag{3}$$

where for plane stress analysis

$$C_{11} = C_{22} = \frac{E}{1-\nu^2}$$

$$C_{12} = C_{21} = \frac{\nu E}{1-\nu^2} \tag{4}$$

$$C_{33} = \frac{E}{1+\nu},$$

and for plane strain analysis

$$C_{11} = C_{22} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$$

$$C_{12} = C_{21} = \frac{E\nu}{(1+\nu)(1-2\nu)} \tag{5}$$

$$C_{33} = \frac{E}{1+\nu}.$$

Here E and ν are the Young's modulus and Poisson's ratio, respectively.

Solution of phase 1

Let U be the vertical upward displacement of the upper edge; U increases monotonically in time, t, to U_f, where $0 \leqslant t \leqslant 1$. Then we have in the first phase of the strain path, using the notation of [1, 5],

$$\begin{aligned} {}^t x_1 &= {}^0 x \\ {}^t x_2 &= {}^0 x_2 \left(1 + \frac{U}{h}\right), \end{aligned} \tag{6}$$

where the ${}^0 x_i$ are the original coordinates of a material particle and the ${}^t x_i$ are the coordinates at time t.

Hence the velocity gradient ${}^t\mathbf{L} = [\partial {}^t \dot{u}_i / \partial {}^t x_j]$ is given by

$${}^t\mathbf{L} = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{\dot{U}}{h+U} \end{bmatrix}, \tag{7}$$

and since

$${}^t\dot{\mathbf{e}} = \tfrac{1}{2}({}^t\mathbf{L} + {}^t\mathbf{L}^T)$$

$${}^t\mathbf{\Omega} = \tfrac{1}{2}({}^t\mathbf{L}^T - {}^t\mathbf{L}),$$

we have

$$\begin{aligned} {}^t\dot{\mathbf{e}} &= \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{\dot{U}}{h+U} \end{bmatrix} \\ {}^t\mathbf{\Omega} &= \mathbf{0}. \end{aligned} \tag{8}$$

Using next eqns (1) and (2) we obtain

$$\begin{aligned} {}^1\tau_{11} &= C_{12} \ln\left(1 + \frac{U_f}{h}\right) \\ {}^1\tau_{22} &= C_{22} \ln\left(1 + \frac{U_f}{h}\right) \\ {}^1\tau_{12} &= 0, \end{aligned} \tag{9}$$

where the superscript 1 denotes time 1, i.e. the end of phase 1.

Solution of phase 2

Let S be the horizontal movement to the right of the upper edge which increases from 0 to S_f monotonically in time t, $1 \leqslant t \leqslant 2$. Now we have

$$\begin{aligned} {}^t x_1 &= {}^0 x_1 + \frac{{}^0 x_2}{h} S \\ {}^t x_2 &= {}^0 x_2 \left(1 + \frac{U_f}{h}\right). \end{aligned} \tag{10}$$

We note that S is the only variable in these relations. Hence

$$ {}^{t}\mathbf{L} = \begin{bmatrix} 0 & \dfrac{\dot{S}}{h+U_f} \\ 0 & 0 \end{bmatrix} \tag{11} $$

$$ {}^{t}\dot{\mathbf{e}} = \frac{1}{2}\begin{bmatrix} 0 & \dfrac{\dot{S}}{h+U_f} \\ \dfrac{\dot{S}}{h+U_f} & 0 \end{bmatrix} $$

$$ {}^{t}\boldsymbol{\Omega} = \frac{1}{2}\begin{bmatrix} 0 & -\dfrac{\dot{S}}{h+U_f} \\ \dfrac{\dot{S}}{h+U_f} & 0 \end{bmatrix} \tag{12} $$

The stresses are obtained using eqns (1) and (2):

$$ {}^{t}\dot{\tau}_{11} = {}^{t}\overset{\nabla}{\tau}_{11} + {}^{t}\tau_{12}\frac{\dot{S}}{h+U_f} $$

$$ {}^{t}\dot{\tau}_{22} = {}^{t}\overset{\nabla}{\tau}_{22} - {}^{t}\tau_{12}\frac{\dot{S}}{h+U_f} \tag{13} $$

$$ {}^{t}\dot{\tau}_{12} = {}^{t}\overset{\nabla}{\tau}_{12} - \frac{\dot{S}}{2(h+U_f)}({}^{t}\tau_{11} - {}^{t}\tau_{22}) $$

and

$$ \begin{bmatrix} {}^{t}\overset{\nabla}{\tau}_{11} \\ {}^{t}\overset{\nabla}{\tau}_{22} \\ {}^{t}\overset{\nabla}{\tau}_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ C_{33}\dfrac{\dot{S}}{2(h+U_f)} \end{bmatrix}. \tag{14} $$

Using as the initial conditions the stress state at time 1, we obtain for time 2

$$ {}^{2}\tau_{11} = C_{12}\ln\left(1+\frac{U_f}{h}\right) + A\left(1-\cos\frac{S_f}{h+U_f}\right) $$

$$ {}^{2}\tau_{22} = C_{22}\ln\left(1+\frac{U_f}{h}\right) - A\left(1-\cos\frac{S_f}{h+U_f}\right) \tag{15} $$

$$ {}^{2}\tau_{12} = A\sin\frac{S_f}{h+U_f}, $$

where, in plane stress and plane strain conditions,

$$ A = \frac{E}{2(1+\nu)}\left(1+\ln\left(1+\frac{U_f}{h}\right)\right). $$

Solution of phase 3

Let U now be the vertical downward displacement of the upper edge, with U increasing from 0 to U_f in time 2 to time 3. We now have

$$ {}^{t}x_1 = {}^{0}x_1 + \frac{{}^{0}x_2}{h}S_f $$

$$ {}^{t}x_2 = {}^{0}x_2\left(1+\frac{U_f-U}{h}\right), \tag{16} $$

and hence

$$ {}^{t}\mathbf{L} = \begin{bmatrix} 0 & 0 \\ 0 & -\dfrac{\dot{U}}{h+U_f-U} \end{bmatrix} \tag{17} $$

$$ {}^{t}\dot{\mathbf{e}} = \begin{bmatrix} 0 & 0 \\ 0 & -\dfrac{\dot{U}}{h+U_f-U} \end{bmatrix} \tag{18} $$

$$ {}^{t}\boldsymbol{\Omega} = \mathbf{0}. $$

Further,

$$ {}^{t}\dot{\tau}_{11} = -C_{12}\frac{\dot{U}}{h+U_f-U} $$

$$ {}^{t}\dot{\tau}_{22} = -C_{22}\frac{\dot{U}}{h+U_f-U} \tag{19} $$

$$ {}^{t}\dot{\tau}_{12} = 0, $$

and with the initial conditions at time 2 given in eqn (15) we obtain:

$$ {}^{3}\tau_{11} = A\left(1-\cos\frac{S_f}{h+U_f}\right) $$

$$ {}^{3}\tau_{22} = -A\left(1-\cos\frac{S_f}{h+U_f}\right) \tag{20} $$

$$ {}^{3}\tau_{12} = A\sin\frac{S_f}{h+U_f}. $$

When comparing the solutions in eqns (9), (15) and (20) we note that the response at time 3 is obtained by subtracting the stresses corresponding to time 1 from the stresses at time 2, and that at time 3 the plane stress and plane strain solutions are identical.

Solution of phase 4

Let S be now the horizontal movement of the upper edge to the left, with S increasing from 0 to S_f in time 3 to time 4. Thus we have

$$ {}^{t}x_1 = {}^{0}x_1 + \frac{{}^{0}x_2}{h}(S_f - S) $$

$$ {}^{t}x_2 = {}^{0}x_2. \tag{21} $$

Hence

$$
{}^{t}\mathbf{L} = \begin{bmatrix} 0 & -\dfrac{\dot{S}}{h} \\ 0 & 0 \end{bmatrix} \tag{22}
$$

$$
{}^{t}\dot{\mathbf{e}} = \frac{1}{2}\begin{bmatrix} 0 & -\dfrac{\dot{S}}{h} \\ -\dfrac{\dot{S}}{h} & 0 \end{bmatrix}
$$

$$
{}^{t}\mathbf{\Omega} = \frac{1}{2}\begin{bmatrix} 0 & \dfrac{\dot{S}}{h} \\ -\dfrac{\dot{S}}{h} & 0 \end{bmatrix}, \tag{23}
$$

and the stresses are calculated from

$$
\begin{aligned}
{}^{t}\dot{\tau}_{11} &= -{}^{t}\tau_{12}\frac{\dot{S}}{h} \\
{}^{t}\dot{\tau}_{22} &= {}^{t}\tau_{12}\frac{\dot{S}}{h} \\
{}^{t}\dot{\tau}_{12} &= -C_{33}\frac{\dot{S}}{2h} + ({}^{t}\tau_{11} - {}^{t}\tau_{22})\frac{\dot{S}}{2h},
\end{aligned} \tag{24}
$$

with the initial conditions in eqn (20). Then we obtain for plane stress and plane strain conditions

$$
\begin{aligned}
\frac{{}^{4}\tau_{11}}{G} &= 1 - \cos\bar{S} + [1 + \ln(1+\bar{U})] \\
&\times \left[\left(1 - \cos\frac{\bar{S}}{1+\bar{U}}\right)\cos\bar{S} - \sin\frac{\bar{S}}{1+\bar{U}}\sin\bar{S}\right] \\
{}^{4}\tau_{22} &= -{}^{4}\tau_{11} \\
\frac{{}^{4}\tau_{12}}{G} &= -\sin\bar{S} + [1 + \ln(1+\bar{U})] \\
&\times \left[\sin\frac{\bar{S}}{1+\bar{U}}\cos\bar{S} + \left(1 - \cos\frac{\bar{S}}{1+\bar{U}}\right)\sin\bar{S}\right],
\end{aligned} \tag{25}
$$

where we have used $\bar{S} = S_f/h$ and $\bar{U} = U_f/h$, and $G = E/2(1+\nu)$ (the shear modulus).

A graphical representation of these results is given in Fig. 2. We note that when either $\bar{U} = 0$ or $\bar{S} = 0$, the final stresses are zero, but the stresses at time 4 do not vanish when values of both $\bar{U}$ and $\bar{S}$ are nonzero.

3. CONCLUSIONS

The objective of this short paper was to present an interesting example for the updated Lagrangian Jaumann formulation. The solution of the problem considered, which has been calculated analytically, shows a fundamental property of the U.L.J. formulation: residual stresses may be measured at the end of an elastic closed strain path.

There is no surprise in this observation; however, the possibility of strain energy accumulation during an elastic closed strain path is surely disturbing. The updated Lagrangian Hencky formulation, being a total formulation, does not have this undesirable property [3, 4].

REFERENCES

1. K. J. Bathe, *Finite Element Procedures in Engineering Analysis*. Prentice-Hall, Englewood Cliffs, NJ (1982).
2. K. Willam (ed.), Constitutive equations: macro and computational aspects. Proceedings ASME, WAM (1984).
3. K. J. Bathe, R. Slavković and M. Kojić. On large strain elasto-plastic and creep analysis. In *Finite Element Methods for Nonlinear Problems* (Edited by P. G. Bergan, K. J. Bathe and W. Wunderlich), pp. 175–190. Springer, New York (1986).

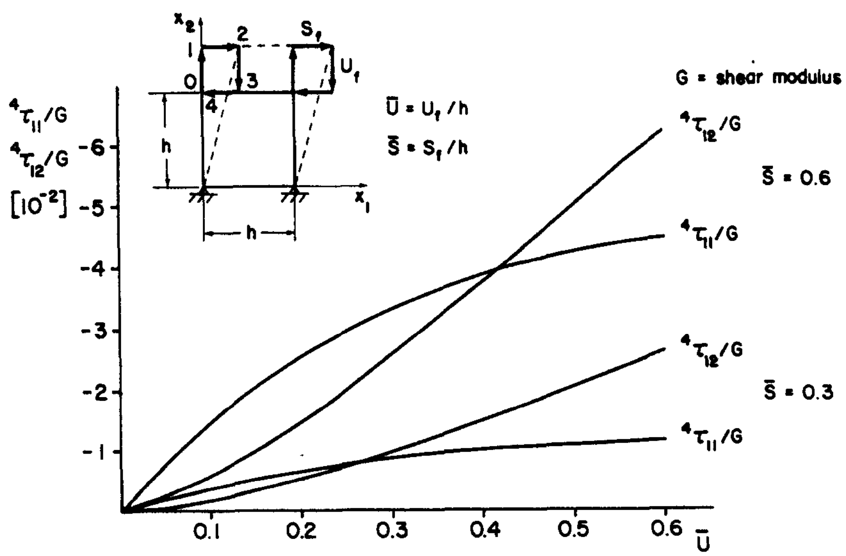

Fig. 2. Residual stresses as functions of elongation and shear measures $\bar{U}$ and $\bar{S}$ (U.L.J. analytical solution).

4. K. J. Bathe, M. Kojić and J. Walczak, Large strain finite element solutions. In *Computational Plasticity* (Edited by E. Oñate *et al.*). Pineridge Press, Swansea (1987).

5. K. J. Bathe, Finite element procedures for solids and structures—nonlinear analysis. Video-Course Study Guide, Centre for Advanced Engineering Study, M.I.T. (1986).