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THE INF–SUP TEST

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Abstract—We briefly review the inf–sup condition for the finite element solution of problems in incompressible elasticity, and then propose a numerical test on whether the inf–sup condition is passed. The evaluation of elements with this test is simple, and various results are presented. This inf–sup test will prove useful for many discretizations of constrained variational problems.

1. INTRODUCTION

It has been amply recognized that many problems in solid and fluid mechanics cannot be solved efficiently using finite element discretization with only one unknown field variable. For example, the solution of almost incompressible elasticity problems with the standard displacement method and reasonable meshes can yield solutions that are grossly in error, and it is much more effective to use displacements and pressure as unknown field variables [1].

The fundamental difficulties in such problems frequently arise because the solution variables are subject to some constraints. In the case of the almost incompressible elasticity problem, the volumetric strains must be very small (and approach zero as the condition of total incompressibility is approached) while the pressure is of the order of the boundary tractions. For the analysis of such problems, it is necessary to use a mixed formulation.

In principle, finite element mixed formulations can be quite easily designed, and many different approaches and formulations have been proposed. However, the key to whether a mixed formulation is actually valuable lies, of course, in the convergence properties of the formulation. These properties are governed by the stability considerations as expressed in the ellipticity requirement and the inf–sup condition of Brezzi and Babuška [2].

At present, we find that in the engineering literature, mixed formulations are proposed and 'so-called' tested for stability by 'solvability tests' and 'counting rules' (counts and comparisons of degrees of freedom). While these rules are quite easy to use, we have pointed out earlier that they are deficient in predicting whether a mixed formulation is stable and indeed these rules can be misleading [3].

On the other hand, the inf–sup condition can be a difficult criterion to apply to new formulations, because an analytical expression must be evaluated which deals with an infinity of problems and solutions. In engineering practice, a numerical test that with relatively little effort indicates whether the inf–sup condition is passed would be very valuable. Such a test could be used much in the same way as the patch test is currently used to test incompatible displacement-based formulations.

Our objective in this paper is to propose a numerical test on whether the inf–sup condition is passed. We consider in detail the problem of incompressible elasticity but the basic steps used are also applicable to other problem formulations [1]. First we briefly review the inf–sup condition. We then present our numerical test and we apply it to various finite element discretization schemes.

2. THE INF–SUP CONDITION: CONTEXT, FORMS AND EVALUATION

2.1. *The context: constrained problems*

In the following developments, for simplicity, both the notation and the numerical results refer entirely to incompressible elasticity. This particular case, however, is sufficiently characteristic to also shed light upon similar problems. In particular, the subsequent derivations are immediately applicable to the field of analysis of incompressible fluids.

In incompressible elasticity, we want to minimize a potential of the form

$$\Pi(\mathbf{v}) = G\int_{\Omega}\sum_{i,j=1}^{3}(\epsilon_{ij}^{D}(\mathbf{v}))^2\,d\Omega + \kappa/2\int_{\Omega}(\operatorname{div}\mathbf{v})^2\,d\Omega - \int_{\Omega}\mathbf{f}\cdot\mathbf{v}\,d\Omega, \quad (1)$$

where Ω is the volume corresponding to the system considered, ϵ_{ij}^{D} are the deviatoric strains, G and κ are the shear and bulk moduli, respectively, and $\mathbf{f}$ stands for the external force field. κ is of course increasing as the material becomes more incompressible. For full incompressibility, κ is infinite.

We are seeking the displacement field $\mathbf{u}$ which minimizes the potential over a vectorial space V of the type

$$V = \left\{\mathbf{v}\in\mathscr{B}(\Gamma)\,\middle/\,\frac{\partial v_i}{\partial x_j}\in L^2(\Omega),\quad \forall i,j=1,\ldots,3\right\},$$

where $\mathscr{B}(\Gamma)$ is the space of displacements satisfying some homogeneous boundary conditions on the boundary Γ.

In finite element analysis, we approximate the continuous solution $\mathbf{u}$ by a finite element solution $\mathbf{u}_h$, belonging to V_h, a finite-dimensional subspace of V. Hence, taking $\mathbf{u}_h$ as the solution of the minimization of Π over V_h seems natural. Unfortunately, experience shows that such a scheme almost always encounters convergence difficulties when κ is very large.

What happens can be described as follows. If we define

$$D_h = \{\text{div } \mathbf{v}_h / \mathbf{v}_h \in V_h\}$$

and for all elements q_h in D_h

$$K_h(q_h) = \{\mathbf{v}_h \in V_h / \text{div } \mathbf{v}_h = q_h\}$$

minimizing Π will yield solutions, whether continuous or discretized, which are closer to incompressibility as κ becomes larger. In the limit when κ is infinite, the minimizing solutions satisfy the incompressibility condition exactly. In particular, the discretized solution $\mathbf{u}_h$ is then constrained to lie in $K_h(0)$. Of course, what we reasonably expect from our finite element analysis is *optimal convergence*. This means that, as the mesh is refined, the distance between $\mathbf{u}$ and $\mathbf{u}_h$ must remain of the same order of magnitude as the quantity $\text{d}(\mathbf{u}, V_h)$, the distance between $\mathbf{u}$ and its best possible approximation in V_h. Clearly, with $\mathbf{u}_h$ in $K_h(0)$, optimal convergence cannot be guaranteed unless $K_h(0)$ is somehow rich enough compared to V_h. More precisely, what we need to enforce is

$$\text{d}(\mathbf{u}, K_h(0)) \leqslant c \, \text{d}(\mathbf{u}, V_h) \tag{2}$$

with c independent of h. Whenever this requirement is not met, we say that we have *locking*. Conversely, when condition (2) is satisfied, other considerations show that optimal convergence is ensured for any value of κ [4].

In practice, finite elements exhibit locking when used directly with the potential Π. To circumvent this phenomenon, a convenient method consists of weakening the constraint applied on $\mathbf{u}_h$, using the modified potential

$$\Pi'_h(\mathbf{v}_h) = G \int_\Omega \sum_{i,j=1}^{3} (\epsilon_{ij}^D(\mathbf{v}_h))^2 \, \text{d}\Omega + \kappa/2 \int_\Omega (P_h(\text{div } \mathbf{v}_h))^2 \, \text{d}\Omega - \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \, \text{d}\Omega, \tag{3}$$

where P_h is an L^2-projector onto an auxiliary space Q_h. This space can be interpreted as a pressure space when considering the equivalent mixed formulation obtained by invoking the stationarity of Π'_h

$$2G \int_\Omega \sum_{i,j=1}^{3} \epsilon_{ij}^D(\mathbf{u}_h)\epsilon_{ij}^D(\mathbf{v}_h) \, \text{d}\Omega - \int_\Omega p_h \, \text{div } \mathbf{v}_h \, \text{d}\Omega = \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \, \text{d}\Omega \quad \forall \mathbf{v}_h \in V_h \tag{4}$$

$$\int_\Omega (\text{div } \mathbf{u}_h + p_h/\kappa) q_h \, \text{d}\Omega = 0 \quad \forall q_h \in Q_h, \tag{5}$$

where we recognize in (5) the projection definition in the form: $p_h = -\kappa P_h(\text{div } \mathbf{u}_h)$. The matrix problem corresponding to this variational formulation is then of the following type

$$\begin{pmatrix} \mathbf{A}_h & \mathbf{B}_h^t \\ \mathbf{B}_h & -\frac{1}{\kappa}\mathbf{T}_h \end{pmatrix} \begin{pmatrix} \mathbf{U}_h \\ \mathbf{P}_h \end{pmatrix} = \begin{pmatrix} \mathbf{F}_h \\ \mathbf{0} \end{pmatrix}, \tag{6}$$

where all the matrices are naturally characterized by the variational formulation (4), (5). Incidentally, comparing the two relations $p_h = -\kappa P_h(\text{div } \mathbf{u}_h)$ and $\mathbf{B}_h\mathbf{U}_h - (1/\kappa)\mathbf{T}_h\mathbf{P}_h = \mathbf{0}$, we note that $-\mathbf{T}_h^{-1}\mathbf{B}_h$ is the matrix form associated with the linear operator $P_h(\text{div})$.

Going back to (3) and defining, for every q_h in $P_h(D_h)$

$$K'_h(q_h) = \{\mathbf{v}_h \in V_h / P_h(\text{div } \mathbf{v}_h) = q_h\},$$

the limit constraint enforced by κ infinite is $\mathbf{u}_h \in K'_h(0)$ and the non-locking condition becomes, from (2)

$$\text{d}(\mathbf{u}, K'_h(0)) \leqslant c \, \text{d}(\mathbf{u}, V_h). \tag{7}$$

Choosing Q_h smaller makes $K'_h(0)$ larger and thus renders (7) easier to satisfy. Yet, trying to avoid locking by taking the pressure space arbitrarily small is clearly not desirable—for $Q_h = \{0\}$, $K'_h(0) = V_h$ and nothing remains from the incompressibility constraint! We understand that some level of 'constraint accuracy' must be preserved. Therefore, our aim shall be to reduce the size of Q_h sufficiently to avoid locking, but also to keep it as large as possible for reasons of accuracy. With this in mind, a reliable test to detect locking is very useful. This is precisely the purpose of the inf–sup condition.

Before discussing this condition we should recall an important general observation. Namely, when examining the stability of a mixed formulation, we require that both the inf–sup and the ellipticity conditions be satisfied [1–3]. Hence, the ellipticity condition also requires special attention in general. However, our objective in this paper is to concentrate on the inf–sup condition, and, for the problem area considered herein, the ellipticity condition is immediate.

2.2. *Inf–sup condition: the natural form*

The natural condition arising in the context presented above is the condition

$$\inf_{q_h \in P_h(D_h)} \sup_{\mathbf{v}_h \in V_h} \frac{\int_\Omega q_h \operatorname{div} \mathbf{v}_h \, d\Omega}{\| q_h \| \, \| \mathbf{v}_h \|} \geqslant \alpha > 0, \tag{8}$$

where the norm symbol in Q_h stands for the L^2-norm, while V_h is measured by the standard norm

$$\| \mathbf{v} \|_V^2 = \int_\Omega \sum_{i,j=1}^{3} \left(\frac{\partial v_i}{\partial x_j} \right)^2 d\Omega.$$

The satisfaction of condition (8) is of several important consequences. First, it strongly excludes locking by implying that inequality (7) holds for any $\mathbf{u}$ with the same constant c. Hence, an optimal error bound can be derived for the limit problem (κ infinite) [2, 3]

$$\| \mathbf{u} - \mathbf{u}_h \| \leqslant c_1 \, d(\mathbf{u}, V_h) + c_2 \, d(p, Q_h), \tag{9}$$

where p denotes the continuous solution for pressure. Here, the last term $d(p, Q_h)$ displays the meaning of what we earlier called 'constraint accuracy'. In order to guarantee optimal convergence, Q_h ought to be chosen such that, not only the inf–sup condition (8) is satisfied, but also $d(p, Q_h)$ is at least of the same order of magnitude as $d(\mathbf{u}_h, V_h)$. Since the norm of $\mathbf{u}$ involves first-order derivatives, (9) tells that we should aim to use an interpolation for p no more than one degree lower than for $\mathbf{u}$.

Another direct implication of condition (8) is the convergence of the pressures in the mixed formulation (4), (5). No other assumption is indeed necessary to derive pressure error estimates as good as those for the displacements [4, 5].

In order to evaluate the inf–sup expression contained in condition (8), some preliminary transformations are necessary. First, we note that, with $q_h \in P_h(D_h)$, we can always find $\mathbf{w}_h$ such that $q_h = P_h(\operatorname{div} \mathbf{w}_h)$. Thus, an all-displacement form of the expression can be obtained, which we write equivalently in terms of the nodal-displacement vectors instead of the fields

$$\inf_{\mathbf{W}_h} \sup_{\mathbf{V}_h} \frac{\mathbf{W}_h^t \mathbf{G}_h \mathbf{V}_h}{\sqrt{\mathbf{W}_h^t \mathbf{G}_h \mathbf{W}_h} \cdot \sqrt{\mathbf{V}_h^t \mathbf{S}_h \mathbf{V}_h}} \geqslant \alpha > 0, \tag{10}$$

where $\mathbf{S}_h$ is the norm-matrix and $\mathbf{G}_h$ is defined by

$$\begin{aligned} \mathbf{W}_h^t \mathbf{G}_h \mathbf{V}_h &= \int_\Omega P_h(\operatorname{div} \mathbf{w}_h) \operatorname{div} \mathbf{v}_h \, d\Omega \\ &= \int_\Omega P_h(\operatorname{div} \mathbf{w}_h) P_h(\operatorname{div} \mathbf{v}_h) \, d\Omega \\ &= \int_\Omega \operatorname{div} \mathbf{w}_h P_h(\operatorname{div} \mathbf{v}_h) \, d\Omega. \end{aligned}$$

The matrix $\mathbf{G}_h$ is symmetric positive semi-definite, whereas $\mathbf{S}_h$ is of course symmetric positive definite. Moreover, $\mathbf{S}_h$ can be directly assembled at the element level and is banded. The same holds true for $\mathbf{G}_h$, provided that the projector P_h is element-internal, or in other words, that the pressure field is discontinuous between the elements.

Assuming that the pressure field is continuous, $\mathbf{G}_h$ can still be derived from the general relation $\mathbf{G}_h = \mathbf{B}_h^t \mathbf{T}_h^{-1} \mathbf{B}_h$, directly deduced from the matrix form of $P_h(\operatorname{div})$. In this case, more computations are involved to evaluate $\mathbf{G}_h$; also the resulting matrix is full.

2.3. *The inf–sup and spurious pressure modes*

Before we proceed to our computational considerations, we need to reflect on the form itself of our original condition (8). That it involves $P_h(D_h)$ instead of Q_h should not be a surprise—it is $P_h(D_h)$ and not Q_h that matters in (3). Yet, considering the stability of system (6) from the algebraic point of view [3], with κ infinite, leads to the slightly different form

$$\inf_{q_h \in Q_h} \sup_{\mathbf{v}_h \in V_h} \frac{\int_\Omega q_h \operatorname{div} \mathbf{v}_h \, d\Omega}{\| q_h \| \, \| \mathbf{v}_h \|} \geqslant \alpha > 0. \tag{11}$$

Fortunately, a straightforward relation between (8) and (11) can readily be established. Of course, for a given h, if $P_h(D_h) = Q_h$, the two expressions yield the same result. Also, whenever $P_h(D_h)$ is strictly included in Q_h, if we denote by $P_h^\perp(D_h)$ the orthogonal subspace to $P_h(D_h)$ in Q_h, we have the following property

$$\forall q_h \in P_h^\perp(D_h), \quad \forall \mathbf{v}_h \in V_h, \quad \int_\Omega q_h \operatorname{div} \mathbf{v}_h \, d\Omega = 0. \tag{12}$$

Hence, the inf–sup expression in (11) yields exactly zero. The elements of $P_h^\perp(D_h)$ are usually called *spurious pressure modes*. They correspond to particular pressure fields which do not interact with the displacements in the u/p formulation. Hence, what condition (11) does is two-fold:

1. Condition (11) tests whether any spurious pressure mode is present.
2. When no spurious pressure mode is detected, condition (11) tests whether condition (8) itself is satisfied.

Noting from (12) that the spurious modes are in fact the elements of $\operatorname{Ker}(\mathbf{B}_h^t)$, we understand why their existence has to be tested in (11)—for κ infinite, they render system (6) singular [3]. By contrast, as long as κ is finite, formulation (4), (5) is clearly regular and system (6) is invertible with or without spurious modes. Furthermore, for finite values of κ (partially incompressible or penalized problems) and homogeneous boundary conditions, pressure modes do not affect the numerical conditioning and are automatically filtered out of the results, as shown by Malkus [6, App. 4-II], see also [3].

Yet, spurious modes should be avoided and can lead to large solution errors in totally incompressible problems or in cases with prescribed non-zero displacements. Nevertheless, it is important to realize that locking and spurious modes correspond to two very different issues. Locking—pertaining to the question of stability—refers to the convergence behavior intrinsic to a given element. On the other hand, spurious modes—which can only affect solvability—occur with certain elements in certain meshes and depend on the boundary conditions.

Thus, studying spurious modes through (11), which is an asymptotic condition, does not seem very appropriate. In order to detect them, it is certainly more natural to consider the matrix $\mathbf{B}_h^t$, checking that it has full column-rank. Assuming for instance that the basic properties of the matrix $\mathbf{G}_h$ are known from the analysis of condition (10), the rank of $\mathbf{B}_h^t$ can be directly inferred since, with $\mathbf{G}_h = \mathbf{B}_h^t \mathbf{T}_h^{-1} \mathbf{B}_h$, we have: $\mathrm{rank}(\mathbf{B}_h^t) = \mathrm{rank}(\mathbf{B}_h) = \mathrm{rank}(\mathbf{G}_h)$. Incidentally, we note that the test advocated by Zienkiewicz *et al.* [7, 8] (practiced by counting and comparing DOFs of displacement and pressure) is a weaker form of this rank verification. Therefore, this test only addresses the issue of solvability for which, as pointed out by Zienkiewicz *et al.*, it constitutes a necessary but not a sufficient condition.

Globally, we have every reason to be satisfied with condition (8). It focuses indeed on locking, our primary concern, yet still allows the detection of spurious modes. We may now concentrate on a very interesting feature of formula (8)—its numerical applicability.

2.4. *Numerical evaluation: the equivalent eigenproblems*

In so far as it involves an infinite number of meshes, verifying a condition like (8), strictly speaking, cannot be performed numerically. What we can do, however, is obtain a numerical value of the expression

$$\alpha_h = \inf_{q_h \in P_h(D_h)} \sup_{v_h \in V_h} \frac{\int_\Omega q_h \operatorname{div} v_h \, d\Omega}{\| q_h \| \, \| v_h \|} = \inf_{\mathbf{W}_h} \sup_{\mathbf{V}_h} \frac{\mathbf{W}_h^t \mathbf{G}_h \mathbf{V}_h}{\sqrt{\mathbf{W}_h^t \mathbf{G}_h \mathbf{W}_h} \cdot \sqrt{\mathbf{V}_h^t \mathbf{S}_h \mathbf{V}_h}}$$

for a well-chosen set of meshes, and attempt to draw a prediction on whether the inf–sup condition is satisfied. The numerical evaluation is made possible by the following result (proven in [2], see also [1, 9]).

Proposition 2.1. Consider the following eigenproblem

$$\mathbf{G}_h \mathbf{V}_h = \lambda \mathbf{S}_h \mathbf{V}_h \tag{13}$$

and call λ_p the first non-zero eigenvalue. Then, the value of α_h is simply $\sqrt{\lambda_p}$.

This property is crucial, for it renders an inf–sup value as easy to evaluate as a modal frequency in a dynamic anaysis, at least in the case of the discontinuous pressure field. Also the number of zero eigenvalues immediately tells whether spurious modes are present [1]. For continuous pressures, a second result is still useful.

Proposition 2.2. Consider the second eigenproblem

$$\mathbf{G}_h' \mathbf{Q}_h = \lambda' \mathbf{T}_h \mathbf{Q}_h, \tag{14}$$

where $\mathbf{G}_h' = \mathbf{B}_h \mathbf{S}_h^{-1} \mathbf{B}_h^t$. Call now λ_r' the first non-zero eigenvalue. Then λ_r' is equal to λ_p in the first eigenproblem (13).

In the continuous case, $\mathbf{G}_h'$ is not easier to calculate than $\mathbf{G}_h$, but at least $\mathbf{G}_h'$ has the dimension of the pressure space instead of the displacement space, and the eigenproblem is simplified accordingly.

2.5. *The inf–sup in practice*

We suggest that a particular element be tested by calculating α_h using meshes of increasing refinement. On the basis of three results, we should be able to predict whether the inf–sup value is probably bounded from underneath or, on the contrary, goes down to zero when the mesh is refined.

It is evident that such a prediction, as valuable as it is, can never be as reliable as a definitive analytical proof. Nevertheless, in engineering practice where elements of various formulations are proposed and used in distorted meshes, it may well remain the only available tool to test elements. And even for theoretical purposes, we believe that such a numerical scheme can be useful as a quick test to give guidance for mathematical analysis.

In the following section, we demonstrate the use of our numerical inf–sup test on several examples of elements for the incompressible elasticity problem.

3. THE INF–SUP AT WORK: NUMERICAL EXPERIMENTS

3.1. *Testing conditions and result interpretation*

The numerical scheme described in Sec. 2.5 was used to test several elements, with continuous as well as discontinuous pressure fields, in two-dimensional plane strain analysis. Throughout the tests, the same mechanical system is used: a simple cantilevered square block, shown in Fig. 1. In every instance, a sequence of three of four meshes is considered—successive refinements are obtained by dividing the characteristic dimension of the element by a factor 2. The results are plotted in the form $\log(IS) = f(\log(1/N))$, where IS stands for the calculated value of the inf–sup expression, and N is the square-root of the total number of elements (i.e. the number of elements per side for square elements). In order to satisfy the inf–sup condition (8), an element must have its inf–sup values bounded away from zero

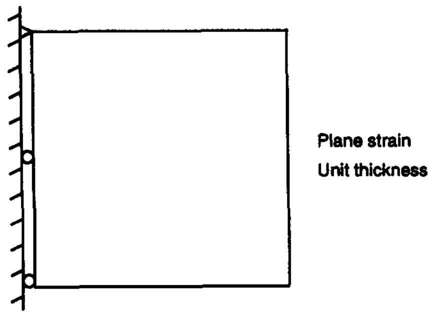

Fig. 1. Inf–sup experiments, problem considered.

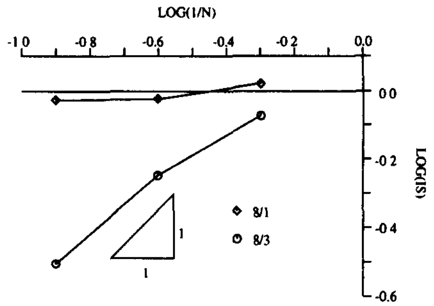

Fig. 3. Inf–sup results, eight-node elements.

when N increases. Therefore, when a steady decrease of log(IS) is observed on the graph, the element is predicted to violate the inf–sup condition and said to fail the numerical test. By contrast, if the inf–sup value stops decreasing as the number of elements increases, the test is passed.

Figures 2–7 show the numerical results. The two expected types of inf–sup behavior are encountered. When the inf–sup is not bounded from underneath, its rate of decrease with the element size always appears clearly on the log/log graph: 1 for all elements tested, except for the 9/5c element in Fig. 7 for which it approaches 0.5. In all cases, the predictions are readily drawn. They are summarized in Table 1, together with the corresponding theoretical results whenever an analytical proof is known to exist. The exact correspondence between theoretical and numerical results testifies to the excellent reliability of this numerical test. It also makes us confident about the possibility to predict the behavior of an element for which no prior theoretical result exists. One such element proposed by Gresho *et al.* [10], the 9/(4c + 1) element, is featured in the table and is predicted to satisfy the inf–sup condition.

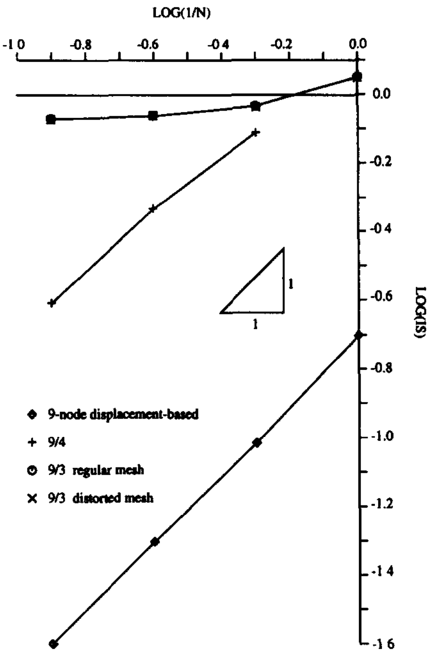

Fig. 4. Inf–sup results, nine-node elements.

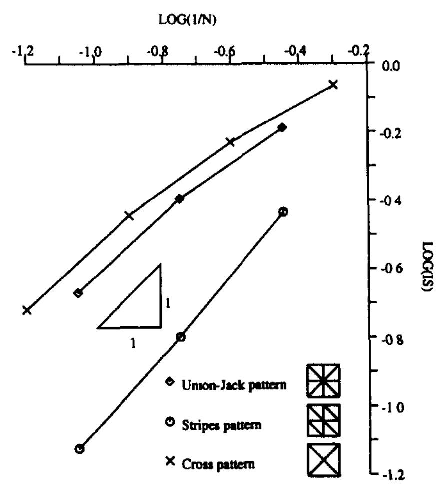

Fig. 5. Inf–sup results, 3/1 triangular element.

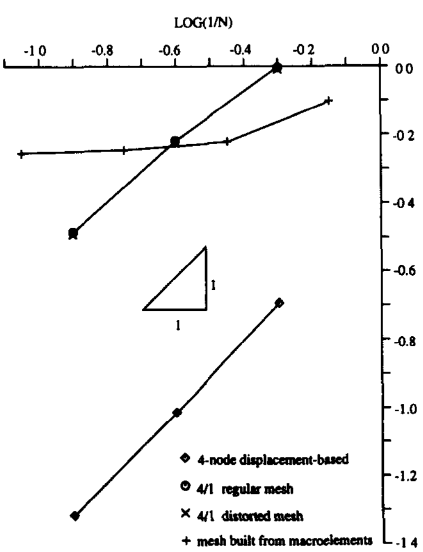

Fig. 2. Inf–sup results, four-node elements.

3.2. *Distorted meshes. Macroelements*

Distorting meshes have sometimes been thought of as a remedy to improve an element behavior when the element does not satisfy the inf–sup con-

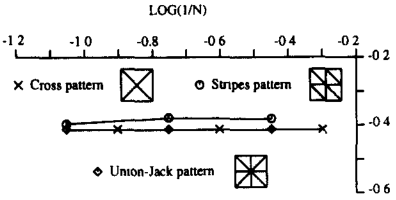

Fig. 6. Inf–sup results, MINI element.

the 4/1 element does not satisfy the inf–sup condition in a regular mesh while the 9/3 element does. When distorted meshes are employed instead of regular ones, the same figures display no significant difference in the inf–sup values. In particular, the inf–sup behavior of the 4/1 element is not improved by the new mesh. Thus, in this case the mesh distortion does not help in satisfying the inf–sup condition.

Our inf–sup test can also be used to study special meshes built from macroelements. As an example, we tested the macroelement shown in Fig. 9—a particular assemblage of five 4/1 elements. The results are plotted in Fig. 2. These results show that the numerical inf–sup test is passed for this mesh and in fact, this behavior was proven analytically [2], see also [11]. Note that our numerical test can be similarly employed, with ease, to investigate any kind of different element and mesh patterns.

dition in regular meshes. Whereas distorted meshes are difficult to analyze theoretically, they do not involve any particular complication for numerical testing.

Two elements, the 4/1 (also called Q1/P0) and the 9/3 (or Q2/P1) elements, were therefore tested in distorted meshes generated from the pattern shown in Fig. 8. As proven analytically and confirmed by the numerical tests (see Figs 2 and 4),

Table 1. Inf–sup numerical predictions

ELEMENT	INF-SUP CONDITION		CONSTRAINT RATIO
	ANALYTICAL PROOF	NUMERICAL PREDICTION	
3/1 *	FAIL	FAIL	1
4/1 *	FAIL	FAIL	2
8/3	FAIL	FAIL	2
8/1	PASS	PASS	6
9/4	FAIL	FAIL	2
9/3	PASS	PASS	8/3
MINI	PASS	PASS	6
9/9c	FAIL	FAIL	2
9/8c	FAIL	FAIL	8/3
9/5c	?	FAIL	4
9/4c	PASS	PASS	8
9/(4c+1)	?	PASS	4

O continuous pressure dof

X and - discontinuous pressure dof

* 3/1 and 4/1 element discretizations can contain spurious pressure modes [1]

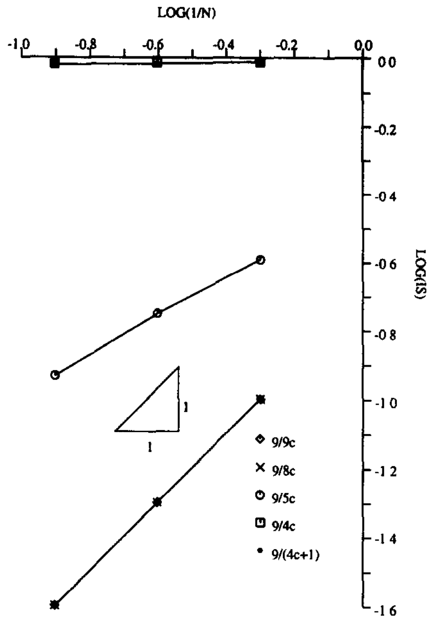

Fig. 7. Inf–sup results, continuous pressure elements.

3.3. *Inf–sup condition: a reliable and discriminate criterion*

Mathematically speaking, the inf–sup condition is a *sufficient* condition which guarantees that, for both displacements and pressures, optimal error estimates like (9) can be derived from coefficients independent of κ. This is a very strong result, and therefore any element which satisfies the condition may be considered as very robust in incompressible analysis. On the other hand, one might hestitate to disqualify elements, like the Q1/P0 element, which do not satisfy the inf–sup condition, on the argument that the condition may be too strong.

These doubts are not confirmed by experience. Among the elements presented in Table 1, all those which violate the inf–sup condition were tested in the nearly incompressible problem defined in Fig. 10. Using the Sussman–Bathe pressure band plots [12], in every instance without exception, the calculated pressures yielded very unsatisfactory band-plots. Two

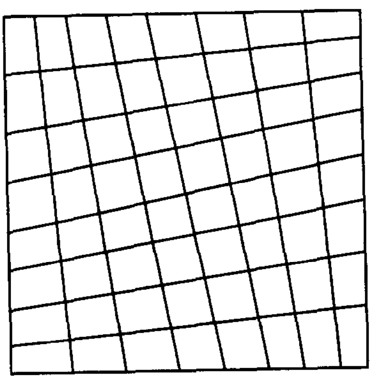

Fig. 8. Distorted mesh pattern.

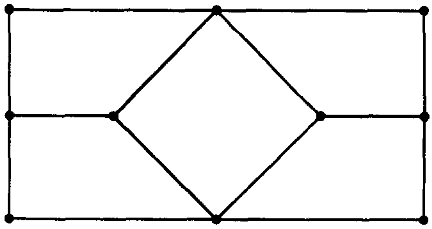

Fig. 9. Macroelement.

examples, the results of the 4/1 and 9/4 elements, are displayed in Figs 11 and 12. They contrast with the regular plot obtained for the 9/3 element, shown in Fig. 13.

The inf–sup condition is therefore confirmed as a very discriminate test, capable of detecting phenomena which escape simpler analyses. For instance, the constraint ratio defined in [6] for a given element as the limit quotient, when the mesh is refined, of the number of displacement DOFs by the number of pressure DOFs, is certainly a useful quick indicator to calculate. Of course, the more constraints there are in proportion to the number of displacement DOFs, the higher the chance that locking will occur. But what is more relevant is the exact nature of the constraints and not only their number. Looking at Table 1, the impeccable 9/3 element has the same constraint ratio as the 9/8c element which does not satisfy the inf–sup condition. Even more strikingly, the 9/5c element, very similar apparently to the 9/(4c + 1) element, behaves totally differently. Obviously, to estimate an element's robustness in incompressible conditions requires more insight than what a mere constraint count provides. A careful investigation of the limit influence of the particular constraints is necessary, and this is achieved by the inf–sup condition.

4. CONCLUDING REMARKS

Our objective in this paper was to introduce a numerical method designed to test whether an element satisfies the inf–sup condition. We also reported upon the results that were obtained by submitting a series of two-dimensional elements to our

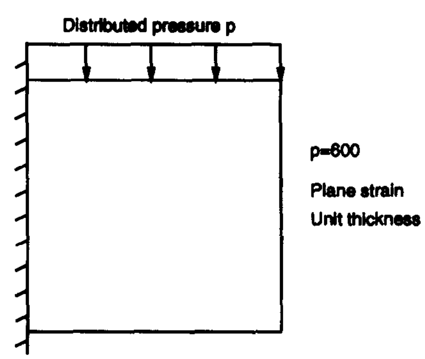

Fig. 10. Loading case, problem definition.

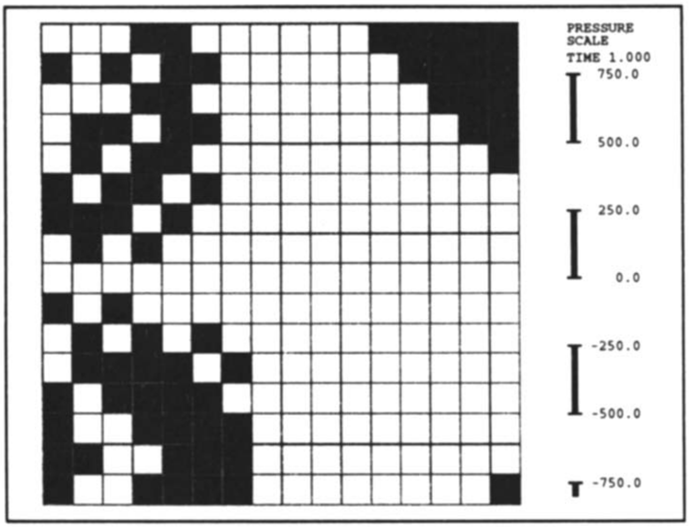

Fig. 11. Pressure band-plot, 4/1 element, $\nu = 0.499$.

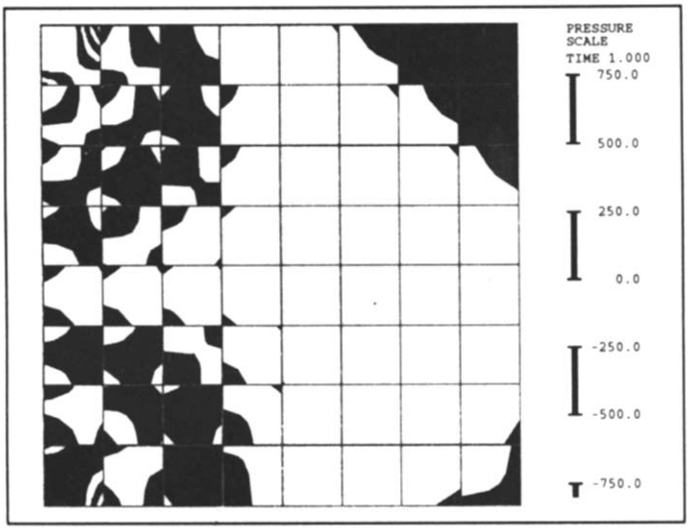

Fig. 12. Pressure band-plot, 9/4 element, $\nu = 0.499$.

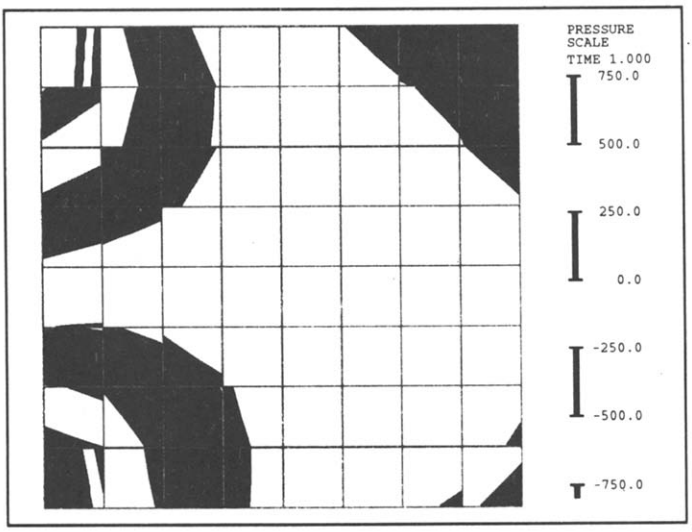

Fig. 13. Pressure band-plot, 9/3 element, $\nu = 0.499$.

numerical test, and showed that all the predictions drawn from these experiments were in perfect agreement with existing theoretical results.

Supported by these very encouraging results, we believe that our technique can be used to investigate the inf–sup behavior of new elements, like the 9/4c + 1 element, for which no analytical result is available. Of course—very much like the numerical evaluation of the patch test—our numerical inf–sup test considers only a discrete set of examples instead of the infinity involved in the corresponding theoretical condition. Thus, our test's ambition is not to supersede the analytical proof whose absolute degree of certainty it will never attain. Instead, we want to provide engineers with a simple computational tool to assess whether the inf–sup condition is passed.

Although all experiments reported herein were carried out in two-dimensional plane strain analysis, the numerical scheme that we have presented can be applied to any constraints problem in which an inf–sup condition appears. In particular, all our results immediately carry over to the two-dimensional Stokes problem. As a natural extension, three-dimensional incompressible media can be studied with the same method. But the inf–sup numerical test can also be used in problems where a constraint of a different type is acting, like in the formulation of plate and shell elements resulting in shear and membrane locking. In this field where mathematical results are still sparse, our test should prove particularly useful. Finally, while we have used the test in finite elment analysis, it would also be valuable to apply the test to finite difference/control volume schemes and to control volume/finite element methods.

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