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Einstein's Field Equations

Our goal is to present a brief motivation for Einstein's field equations of gravity (hereafter "Einstein's equation"). This is an equation that relates the metric of spacetime to a source consisting of, among other things, mass and energy. The philosophy of General Relativity is that "mass tells spacetime how to be curved, and this curvature of spacetime tells matter how to move". This is similiar to Newtonian ideas about gravity, in that the gravitational field is the direct result of its source mass. A key difference is that Einstein's gravity is a relativistic theory and Newton's is not.

We will first briefly discuss the metric, tensors, and index notation. Index notation is an alternative way of writing vectors. Moreover, index notation allows us to write down objects called tensors, of which vectors are a special case. As a simple example of index notation, consider the 4-momentum vector $[E/c, p_x, p_y, p_z]$; we denote such an object as p_ν, where the index used, ν, is a Greek letter. The Greek index is a variable which denotes the components of the four vector. Therefore, p_ν represents four numbers, one number for each possibility of ν $\{t, x, y, z\} \equiv \{0, 1, 2, 3\}$. For example, if $\nu = 0$, then $p_\nu = p_0 = E/c$ denotes the 0 component of the four-vector. In order to reproduce 4-vector dot products we introduce superscripts which are also used for indices. For example, p^ν is another way of denoting the four-momentum, but it will have a slighty different meaning. We define,

$$p^0 = p_0 \qquad p^x = -p_x \qquad p^y = -p_y \qquad p^z = -p_z \; .$$

These definitions apply only in the case of special relativity; they are modified in the case of curved spacetimes in general relativity. Now, we adopt the following convention whenever an upper index is repeated with a lower index in a product. We implicity assume that the index is to be summed over. For example,

$$\begin{aligned}
p^\nu p_\nu &= \sum_\nu p^\nu p_\nu \\
&= p^0 p_0 + p^x p_x + p^y p_y + p^z p_z \\
&= p_0 p_0 - p_x p_x - p_y p_y - p_z p_z \\
&= p_0^2 - p_x^2 - p_y^2 - p_z^2 = E^2/c^2 - \vec{p} \cdot \vec{p}
\end{aligned}$$

This is referred to as the "Einstein sum convention". Summing over the indices here resulted in a Lorentz invariant!

We have discussed index notation with regard to vectors, and now suppose we have an object with more than one index, such as $T_{\mu\nu}$. This actually represents 16 numbers, because there are (4 possibilities for μ) $\times$ (4 possibilities for ν). An object like this is called a "second rank tensor" because it has two indices. Likewise, vectors are referred to as "first rank tensors", because they have only one index. (Scalars are "zero rank tensors"). So $T_{\mu\nu}$ can be viewed as a matrix

$$T_{\mu\nu} = \begin{pmatrix} T_{tt} & T_{tx} & T_{ty} & T_{tz} \\ T_{xt} & T_{xx} & T_{xy} & T_{xz} \\ T_{yt} & T_{yx} & T_{yy} & T_{yz} \\ T_{zt} & T_{zx} & T_{zy} & T_{zz} \end{pmatrix} .$$

We will not discuss tensors in any detail. However, we simply note that, just as four-vectors have a transformation law under Lorentz boosts, tensors also have a transformation law (involving a factor of the Lorentz matrix for each rank). Just as we defined p^μ in terms of p_μ, we can similarly define objects such as, $T_\mu{}^\nu$, $T^\mu{}_\nu$, and $T^{\mu\nu}$ in terms of $T_{\mu\nu}$.

A second rank tensor of particular importance is the metric. The metric is an object which tells us how to measure intervals. For example, in three dimensional Euclidean space, how do we calculate the distance between two nearby points? If we work in Cartesian coordinates, then the distance is given by

$$ds^2 = dx^2 + dy^2 + dz^2 = \begin{pmatrix} dx & dy & dz \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

The right hand side is written in terms of matrix multiplication, with a row vector times a square matrix times a column vector. This is convenient, because we can interpret the square matrix as the metric. Now consider computing this same distance in spherical coordinates.

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 = \begin{pmatrix} dr & d\theta & d\phi \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix}$$

Notice, that in two different coordinate systems, the general form of the equation that gives ds^2 is the same, i.e., they both can be written as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu ,$$

where μ and ν are each summed over $\{0, 1, 2, 3\}$. In effect, the metric $g_{\mu\nu}$ determines how to measure intervals. We have also learned that the spacetime interval, $c^2 d\tau^2$, is independent of Lorentz frame. Thus, one can use a metric to determine the invariant spacetime interval:

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = \begin{pmatrix} cdt & dx & dy & dz \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} cdt \\ dx \\ dy \\ dz \end{pmatrix}$$

The metric $\mathrm{diag}(1, -1, -1, -1)$ is called the "Minkowski metric", to distinguish it from other metrics like $\mathrm{diag}(1, -1, -r^2, -r^2\sin^2\theta)$ (which is also the metric of flat spacetime, expressed in spherical polar coordinates).

The spacetime ordinarily used in special relativity is the Minkowski metric and is, by definition, flat. By contrast, spacetimes in the presence of matter are referred to as "curved" and are described by metric tensors with quite different properties. In curved spacetimes Euclid's fifth postulate is violated, i.e., initially straight parallel lines *can* intersect.

Finally, we note that $g^{\mu\nu}$ with raised indices is just the matrix inverse of $g_{\mu\nu}$. In any spacetime, the metric is used to lower indices and the inverse matrix to raise them. Thus, $p_\mu = g_{\mu\alpha}p^\alpha$ and $p^\mu = g^{\mu\alpha}p_\alpha$ where the index α in each case is summed over $\{0, 1, 2, 3\}$.

The theory of relativity was partially motivated by the frame-invariant form of Maxwell's equations under a Lorentz transformation. For this reason, we say that electromagnetism is a relativistic theory. What is the relativistic theory of gravity? Newton's theory is determined by an equation similar to Gauss' law in E&M. One sees the similarity between Newton's theory and Gauss's Law by noting that both involve fields that that fall off as r^{-2}. Newton's theory can be expressed as

$$\nabla \cdot \mathbf{g} = -4\pi G\rho_m$$

where $\mathbf{g}$ is the gravitational field and ρ_m is the mass density. Equivalently, Newton's law of gravity can be expressed in terms of the gravitational potential in the form of Poisson's equation

$$\nabla^{\mathbf{2}}\phi = 4\pi G\rho_m \,,$$

where ϕ is the gravitational potential defined so that $\mathbf{g} = -\nabla\phi$.

Unfortunately, Newton's theory is not relativistic. Among other things, the fact that the gravitational field is completely determined by a single equation that has no time derivatives implies "action at a distance", i.e., instantaneous knowledge by a test mass of the location of the source mass, without regard to propagation time delays at the speed of light. Of course, Maxwell's equations also include Poisson's equation, and yet they transform correctly in the context of special relativity. However, Newton's theory of gravity has no other equations analogous to the time-dependent Ampere's law and Faraday's law. This might motivate a simple suggestion to generate a relativistic theory of gravity. For example, we might try to add three more equations to give the same form as Maxwell's equation's, with the following substitutions in Maxwell's equations

$$\mathbf{E} \rightarrow \mathbf{g} \qquad \mathbf{B} \rightarrow \mathbf{B_m} \qquad \rho \rightarrow \rho_m \qquad \mathbf{j} \rightarrow \mathbf{j_m} \qquad \epsilon_0 \rightarrow -\frac{1}{4\pi G}$$

However, this does not actually work since charge density and mass density do not have the same Lorentz transformation properties. Remember, mass which is equivalent to energy is not a Lorentz scalar, but charge *is* a Lorentz scalar.

Then, how does one construct a relativistic theory of gravity? A helpful starting point is Einstein's Principle of Equivalence (1911) which states that it is not possible via local observations to distinguish between a uniform gravitational field $\mathbf{g}$, and a frame undergoing uniform acceleration $\mathbf{a} = -\mathbf{g}$. Thus, the apparent gravity seems to depend on the choice of reference frame. For example, if you are in an elevator in free fall or orbiting the Earth in the Space Shuttle, you observe everything to be weightless. It is as if the gravitational field is zero. Likewise, if one were in a totally isolated rocket that was undergoing a constant acceleration of $9.8\,\mathrm{m\,s^{-2}}$, then inside the rocket it would seem as if there were a gravitational field present of $g = 9.8\,\mathrm{m\,s^{-2}}$. In both cases, we see that the inferred field depends on the choice of frame. In fact, it seems as though one can always choose a frame in which the gravitational field vanishes. Note, however, that there are peculiarities with these frames. The freely-falling elevator frame, if extended a great distance (e.g., to the other side of the earth), would not be everywhere free-fall. In the case of the accelerating rocket, the frame is non-inertial.

In General Relativity, we go beyond considering only inertial frames. So this suggests we want a theory, in which the equations take on the same form after any coordinate transformation. Equations that have this property are called "generally covariant". It follows from the equivalence principle that if a material object has no forces other than gravity acting on it, the path the object follows is independent of the nature of the object. So Einstein's conjecture was that gravity is the result of spacetime curvature. In turn, the curvature and geometry of spacetime are all determined by the metric of spacetime. Hence, we want to postulate an equation that relates the metric to a source of mass/energy.

We will no longer talk about gravitational fields; however, the metric plays a role analogous to that of the gravitational potential. Recall in electromagnetism, Maxwell's equations relate the electric and magnetic fields to source charge and currents. Likewise, Einstein's equation will relate the metric to its matter/energy source. So there is some similarity between Maxwell's equations and Einstein's equation though, as we shall see, the latter is inherently a tensor equation rather than a vector equation.

We want to exploit the similarities between electromagnetism and gravity just a bit more. In particular, we note that charge conservation can be directly extracted from Maxwell's equations. To show this, start by taking the time derivative, $\frac{\partial}{\partial t}$, of Gauss's Law

$$\begin{aligned}\frac{\partial}{\partial t}(\nabla \cdot \mathbf{E}) &= \frac{\partial}{\partial t}(4\pi\rho) \\ \nabla\cdot\frac{\partial \mathbf{E}}{\partial t} &= 4\pi\frac{\partial \rho}{\partial t}\end{aligned}$$

and take the divergence, $\nabla\cdot$, of Ampere's Law

$$\begin{aligned}\nabla\cdot(\nabla \times \mathbf{B}) &= \nabla\cdot(\frac{4\pi}{c}j + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}) \\ 0 &= \frac{4\pi}{c}\nabla \cdot \mathbf{j} + \frac{1}{c}\nabla\cdot\frac{\partial \mathbf{E}}{\partial t}\end{aligned}$$

If we now combine these two results we find

$$0 = \frac{4\pi}{c}\left[\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t}\right]$$

$$\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0$$

This final equation expresses conservation of charge. In index notation, it takes the following simple form $\partial j^\mu/\partial x^\mu = 0$, where the 4-vector current density is defined as $\{c\rho; j_x; j_y; j_z\}$. This conservation equation manifestly takes on the same form in all Lorentz frames since both j^μ and x^μ transform as 4-vectors.

We want Einstein's equation to have an analogous feature. What acts as a source in Einstein's equation? A first guess might be the mass density ρ_m. However, as mentioned above, mass is not a relativistic invariant, as is charge. How does mass density fit into a relativistic structure? We know from special relativity that mass density should be replaced with mass/energy density. However, we also learned from special relativity that pure energy in one frame transforms to energy and momentum in another frame. Consideration of energy and momentum densities as well as energy and momentum fluxes leads naturally to the energy-momentum tensor, $T^{\mu\nu}$. The tensor nature arises due to the fact that flows of vector quantities require two directions for complete specification. For example, there can be a flow of the $\hat{x}$ component of momentum in the $\hat{y}$ direction, corresponding to T^{xy}. In all, the components of the stress-energy-momentum tensor contain T^{00} = energy density; cT^{0i} = energy flux across a surface with a normal in the $\hat{i}$ direction; T^{i0}/c = momentum density; and T^{ij} = momentum $\hat{i}$ flux across a surface whose normal is in the $\hat{j}$ direction. (Here the indices i and j run over only the 3 spatial components.)

Let's think about $T^{\mu\nu}$ a bit more. Since it contains various densities and "currents" of momentum and energy, we expect that it probably obeys an equation analogous to that for charge conservation. This suggests a somewhat unconventional notation for the stress-energy tensor $T^{\nu\mu} = [T^{\nu 0}, T^{\nu x}, T^{\nu y}, T^{\nu z}] = [c\rho^{\nu}, j^{\nu x}, j^{\nu y}, j^{\nu z}]$. Here we have identified the energy and momentum fluxes as "current densities", $\mathbf{j}^{\nu} \equiv T^{\nu i}$ and energy and momentum densities as $\rho_{\nu} \equiv T^{\nu 0}$. Since each component is separately conserved, this implies a grand conservation law:

$$\frac{\partial T^{\nu 0}}{\partial t} + \nabla \cdot \mathbf{j}^{\nu} = \mathbf{0}$$
$$\frac{\partial T^{\mu\nu}}{\partial x^{\mu}} \equiv \partial_{\mu} T^{\mu\nu} = 0$$

For Einstein's equation to have a similar feature, it must be constructed such that $\nabla_{\mu} T^{\mu\nu}$ automatically vanishes, where ∇_{μ} is called the covariant derivative. For our purposes, all we need to know is that the covariant derivative is General Relativity's generalization of the partial derivative ∂_{μ}.

As implied above, we expect Einstein's equation to relate the metric tensor to $T^{\mu\nu}$. We anticipate an equation of the form

$$\mathcal{O}[g_{\mu\nu}] = \kappa T_{\mu\nu}$$

where the operator, $\mathcal{O}$ that we chose operates on $g_{\mu\nu}$ to yield a constant κ, and the metric itself has been used to lower the indices on the stress-energy tensor: $T_{\mu\nu} = g_{\alpha\mu} g_{\beta\nu} T^{\alpha\beta}$. $\mathcal{O}$ will then automatically satisfy $\nabla_{\mu}\mathcal{O}^{\mu\nu} = 0$. Choosing the operator to satisfy that condition insures that energy and momentum are conserved. Furthermore, κ is a constant to be determined by the condition that Einstein's theory in the limit of small mass densities must agree with Newton's law of gravity. It turns out, we can make the following choices

$$\mathcal{O}[g_{\mu\nu}] = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \qquad\qquad \kappa = \frac{8\pi G}{c^4}$$

where $R_{\mu\nu}$ is the Ricci tensor defined in detail below. These results lead to Einstein's field equations which are summarized next.

Einstein Field Equations Summary of Definitions

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \frac{8\pi G}{c^4} T_{\alpha\beta} \ ,$$

or, alternatively

$$G_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta} \ ,$$

where

- $G_{\alpha\beta}$ is the "Einstein tensor" (not to be confused with G, the universal gravitational constant),
- $g_{\alpha\beta}$ is the "metric tensor",
- $T_{\alpha\beta}$ is the "stress-energy" or "energy-momentum tensor" involving terms in rest mass and kinetic energy, as well as pressure,
- R, the "curvature scalar" of spacetime, is given by $R = g^{\alpha\beta} R_{\alpha\beta}$, a contraction of the "Ricci tensor" with the metric,
- $R_{\alpha\gamma}$, in turn, is the Ricci tensor, defined by $R_{\alpha\gamma} = R^{\lambda}{}_{\alpha\beta\gamma} \delta^{\beta}_{\lambda} = R^{\lambda}{}_{\alpha\lambda\gamma}$, a contraction of the "Riemann curvature tensor" with a 4-dimensional Kronecker delta function (with only $1'$s along the diagonal).

Riemann Curvature Tensor

$$R^{\omega}{}_{\beta\gamma\lambda} = \frac{\partial \Gamma^{\omega}_{\beta\lambda}}{\partial x^{\gamma}} - \frac{\partial \Gamma^{\omega}_{\beta\gamma}}{\partial x^{\lambda}} + \Gamma^{\omega}_{\gamma\sigma}\Gamma^{\sigma}_{\beta\lambda} - \Gamma^{\omega}_{\lambda\sigma}\Gamma^{\sigma}_{\beta\gamma} \ ,$$

where the $\Gamma^{\alpha}_{\beta\gamma}$ are the "Christoffel symbols" (also known as the connection coefficients), defined as

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{g^{\alpha\sigma}}{2} \left[\frac{\partial g_{\sigma\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\sigma\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\sigma}} \right] \ .$$

Notes
1. All indices run over the 4 spacetime coordinates (0 = time, 1, 2, 3 = space).
2. Repeated upper and lower indices imply a summation over that index.
3. $R_{\alpha\beta}$ and $G_{\alpha\beta}$ contain 16 elements, 10 of which are independent; $R_{\alpha\beta\gamma\lambda}$ contains 256 elements, only 20 of which are independent.
4. The dimensions of $R_{\alpha\beta}$ and $G_{\alpha\beta}$ are generically (length)$^{-2}$.

Thus, Einstein's equation actually represents a set of 10 ($4 \times 4 - 6$) coupled, second order, partial, non-linear differential equations that relate $g_{\mu\nu}$ to the stress-energy-momentum tensor (i.e., the "source" term). This is analogous to Poisson's equation in which a second order differential equation relates the gravitational potential to mass density.

The Schwarzschild Metric Solution

After taking into account the various symmetries associated with a static, non-rotating, spherically symmetric mass distribution, we can write the metric in the region exterior to the mass as

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = e^{2\Phi}(cdt)^2 - e^{2\Lambda}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2$$

where Φ and Λ are functions of r only, and are written in this exponential form for convenience. Here, the coordinates are $\{x^0, x^1, x^2, x^3\} \equiv \{ct, r, \theta, \phi\}$. The metric tensor, $g_{\alpha\beta}$, is then given by

$$\begin{pmatrix} e^{2\Phi} & 0 & 0 & 0 \\ 0 & -e^{2\Lambda} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix},$$

Steps to determine $e^{2\Phi}$ and $e^{2\Lambda}$

1. Plug $g_{\alpha\beta}$ into the equations for the Christoffel symbols, $\Gamma^\alpha_{\beta\gamma}$. (These involve first-order derivatives). For the Schwarzschild metric, there are only 9 non-zero Christoffel symbols .
2. The resulting $\Gamma^\alpha_{\beta\gamma}$ are then inserted into the expression for the Riemann curvature tensor (this operation produces second derivatives). The Riemann tensor contains only 6 non-vanishing independent elements for this particular problem.
3. The Riemann curvature tensor, $R^\lambda{}_{\alpha\beta\gamma}$, is then contracted with a 4-dimensional Kronecker delta function δ^β_λ to yield the Ricci tensor $R\alpha\gamma$.
4. After computing the curvature scalar and forming the complete Einstein tensor, only diagonal elements remain; two of these provide the information needed to find $e^{2\Phi}$ and $e^{2\Lambda}$

$$\begin{aligned} G_{00} &= \frac{1}{r^2}e^{2\Phi}\frac{d}{dr}[r(1-e^{-2\Lambda})] , \\ G_{11} &= -\frac{1}{r^2}e^{2\Lambda}(1-e^{-2\Lambda}) + \frac{2}{r}\frac{d\Phi}{dr} . \end{aligned}$$

5. Since the energy-momentum tensor is zero outside the mass distribution, $G_{00} = 0$ and $G_{11} = 0$, are simple differential equations from which we find (see Problem Set 10)

$$e^{2\Phi} = e^{-2\Lambda} = \left(1 - \frac{2GM}{c^2 r}\right) .$$

Now, we actually carry out the steps of plugging in our inferred form for the metric into Einstein's equation to derive the differential equations for Φ and Λ. Recall

$$ds^2 = e^{2\Phi}dt^2 - e^{2\Lambda}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2 .$$

Step one

Start by computing the Christoffel symbols. We'll make a slight change in notation for brevity, from now on when a comma appears next to an index in a tensor it indicates a derivative. For example, $g_{\alpha\beta,\gamma} \equiv \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}$, and likewise $g_{\alpha\beta,\gamma\nu} \equiv \frac{\partial^2 g_{\alpha\beta}}{\partial x^\gamma \partial x^\nu}$.

$$\Gamma^\alpha_{\mu\nu} = \frac{g^{\alpha\sigma}}{2}\left[\frac{\partial g_{\sigma\mu}}{\partial x^\nu} + \frac{\partial g_{\sigma\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right]$$

In all there are $4 \times 4 \times 4/2 = 32$ independent elements of the Christoffel symbol. We'll work out the values of some of the elements of the Christoffel symbol as examples, and then just list the results for the rest of them. One more bit of notation: whenever we use an index such as $\{t, r, \phi, \theta\}$, we mean one of those 4 specific components; however, if we use any other Greek letter we are referring to an arbitrary component that is possibly being summed over.

Let's start by working out Γ^t_{tr}

$$\Gamma^t_{tr} = \frac{g^{t\sigma}}{2}\left[\frac{\partial g_{\sigma t}}{\partial r} + \frac{\partial g_{\sigma r}}{\partial t} - \frac{\partial g_{tr}}{\partial x^\sigma}\right]$$

First, we notice that none of the metric depends on time, t, nor on the angle ϕ. Hence, terms like, $\frac{\partial g_{\sigma r}}{\partial t}$, vanish. So consider only derivatives with respect to r or θ. Next, note that the metric is diagonal, so elements off the diagonal are all zero, such as $g_{tr} = g_{t\theta} = g_{r\phi} = 0$, etc. So looking at our expression for Γ^t_{tr}, we see that only one of the three derivatives might be non-zero, and that is $\frac{\partial g_{\sigma t}}{\partial r}$. This derivative is only non-zero when σ is t, otherwise it is an off-diagonal element of the metric. Also

$$\frac{\partial g_{tt}}{\partial r} = \frac{\partial(e^{2\Phi})}{\partial r} = 2e^{2\Phi}\frac{\partial \Phi}{\partial r}$$

So now, we need to know the value of g^{tt}. How do we find the value of the metric when the indices are raised? In other words, we want to know $g^{\alpha\beta}$, but this is just the inverse of $g_{\alpha\beta}$. Hence, $g^{\alpha\beta}$ is given by

$$\begin{pmatrix} e^{-2\Phi} & 0 & 0 & 0 \\ 0 & -e^{-2\Lambda} & 0 & 0 \\ 0 & 0 & -r^{-2} & 0 \\ 0 & 0 & 0 & -r^{-2}\sin^{-2}\theta \end{pmatrix},$$

So $g^{tt} = e^{-2\Phi}$, and hence $\Gamma^t_{tr} = \frac{\partial \Phi}{\partial r} = \partial_r \Phi$

Next, let's work out a trickier example, $\Gamma^\theta_{\phi\phi}$

$$\Gamma^\theta_{\phi\phi} = \frac{g^{\theta\sigma}}{2}\left[\frac{\partial g_{\sigma\phi}}{\partial \phi} + \frac{\partial g_{\sigma\phi}}{\partial \phi} - \frac{\partial g_{\phi\phi}}{\partial x^\sigma}\right] = \frac{g^{\theta\sigma}}{2}\left[\frac{2\partial g_{\sigma\phi}}{\partial \phi} - \frac{\partial g_{\phi\phi}}{\partial x^\sigma}\right]$$

Now remember the metric is diagonal, and notice the prefactor of $g^{\theta\sigma}$, so we only get a non-zero result when σ is θ, so we have

$$\Gamma^\theta_{\phi\phi} = -\frac{g^{\theta\theta}}{2}\frac{\partial g_{\phi\phi}}{\partial \theta} = -\frac{r^{-2}}{2}(2r^2 \sin\theta\cos\theta) = -\sin\theta\cos\theta$$

Many of the other elements of the Christoffel symbol work out to be zero. We'll show one example of this. Consider Γ^θ_{tt}

$$\Gamma^\theta_{tt} = \frac{g^{\theta\sigma}}{2}\left[\frac{\partial g_{\sigma t}}{\partial t} + \frac{\partial g_{\sigma t}}{\partial t} - \frac{\partial g_{tt}}{\partial x^\sigma}\right] = -\frac{g^{\theta\theta}}{2}\frac{\partial g_{tt}}{\partial \theta} = 0$$

Also, recall that Φ and Λ depend only on r and not on t, θ, and ϕ. Thus $\frac{\partial g_{tt}}{\partial \theta} = 0$, which implies $\Gamma^{\theta}_{tt} = 0$. Working out all the possible elements of the Christoffel symbol is quite a tedious exercise, and thus we simply state the results after all the algebra is done. For brevity, we will use the notation of ∂_r, which stands for $\frac{\partial}{\partial r}$. Only 9 of the elements work out to be non-zero and they are

$$
\begin{array}{lll}
\Gamma^t_{tr} = \partial_r \Phi & \Gamma^r_{tt} = e^{2(\Phi-\Lambda)}\partial_r \Phi & \Gamma^r_{rr} = \partial_r \Lambda \\
\Gamma^r_{\theta\theta} = -re^{-2\Lambda} & \Gamma^r_{\phi\phi} = -r\sin^2\theta e^{-2\Lambda} & \Gamma^\theta_{r\theta} = \frac{1}{r} \\
\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta & \Gamma^\phi_{r\phi} = \frac{1}{r} & \Gamma^\phi_{\theta\phi} = \cot\theta
\end{array}
$$

Step two

Now we need to compute the Riemann curvature tensor, which is given in terms of the Christoffel symbols

$$R^{\sigma}{}_{\mu\beta\nu} = \Gamma^{\sigma}_{\mu\nu,\beta} - \Gamma^{\sigma}_{\mu\beta,\nu} + \Gamma^{\sigma}_{\beta\tau}\Gamma^{\tau}_{\mu\nu} - \Gamma^{\sigma}_{\nu\tau}\Gamma^{\tau}_{\mu\beta} \ ,$$

(note the commas, indicating derivatives, in the first two terms). Now, if one works out what $R^{\sigma}{}_{\mu\beta\nu}$ is in terms of the metric, it can be shown to have a number of symmetries. These symmetries can be used to show there are only 20 independent components. In the end, we actually want to find the Ricci tensor $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$. Now using the definition of the Riemann curvature tensor and the results for the Christoffel symbol, we carry out the tedious computations similiar to computing the Christoffel symbol. Here we simply list the independent non-vanishing components

$$
\begin{aligned}
R^t_{rtr} &= \partial_r\Phi\partial_r\Lambda - \partial_r{}^2\Phi - (\partial_r\Phi)^2; \\
R^t_{\theta t\theta} &= -re^{-2\Lambda}\partial_r\Phi; \\
R^t_{\phi t\phi} &= -re^{-2\Lambda}\sin^2\theta\partial_r\Phi; \\
R^r_{\theta r\theta} &= re^{-2\Lambda}\partial_r\Lambda; \\
R^r_{\phi r\phi} &= re^{-2\Lambda}(\partial_r\Lambda)\sin^2\theta; \\
R^\theta_{\phi\theta\phi} &= (1-e^{-2\Lambda})\sin^2\theta
\end{aligned}
$$

This means we'll only need to work out a subset of the Riemann curvature elements. The subset of $\{R^t_{\mu t\nu}, R^r_{\mu r\nu}, R^\theta_{\mu\theta\nu}, R^\phi_{\mu\phi\nu}\}$.

Now, we compute the non-vanishing components of the Ricci tensor. The first example is

$$
\begin{aligned}
R_{tt} &= R^t_{ttt} + R^r_{trt} + R^\theta_{t\theta t} + R^\phi_{t\phi t} \\
&= 0 + g^{rr}g_{tt}R^t_{rtr} + g^{\theta\theta}g_{tt}R^t_{\theta t\theta} + g^{\phi\phi}g_{tt}R^t_{\phi t\phi} \\
&= e^{2(\Phi-\Lambda)}(-\partial_r\Phi\,\partial_r\Lambda + \partial_r^2\phi + (\partial_r\Phi)^2 + 2r^{-1}\partial_r\Phi)
\end{aligned}
$$

A similar calculation for the other elements of $R_{\alpha\beta}$, shows that all the off-diagonal elements are zero. The complete set of diagonal Ricci tensor elements for the Schwarzschild metric works out to be

$$
\begin{aligned}
R_{tt} &= e^{2(\Phi-\Lambda)}(-\partial_r\Phi\,\partial_r\Lambda + \partial_r^2\phi + (\partial_r\Phi)^2 + 2r^{-1}\partial_r\Phi) \\
R_{rr} &= \partial\Phi\,\partial_r\Lambda - \partial_r^2\Phi - (\partial_r\Phi)^2 + 2r^{-1}\partial_r\Lambda; \\
R_{\theta\theta} &= -e^{-2\Lambda}\left[1 + r(\partial_r\Phi - \partial_r\Lambda)\right] + 1 \\
R_{\phi\phi} &= \sin^2\theta\{-e^{2\Lambda}\left[1 + r(\partial_r\Phi - \partial_r\Lambda)\right] + 1\}
\end{aligned}
$$

Lastly, we need to calculate the scalar curvature $R = g^{\mu\nu}R_{\mu\nu}$

$$
\begin{aligned}
R &= g^{tt}R_{tt} + g^{rr}R_{rr} + g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} \\
R &= 2e^{-2\Lambda}\left\{-\partial_r\Phi\,\partial_r\Lambda + \partial_r^2\Phi + (\partial_r\phi)^2 + \frac{2}{r}(\partial_r\Phi - \partial_r\Lambda) + \frac{1}{r^2}\right\} + \frac{2}{r^2}
\end{aligned}
$$

Finally, we calculate the Einstein tensor, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$

$$
\begin{aligned}
G_{tt} &= \frac{2}{r}e^{2(\Phi-\Lambda)}\partial_r\Lambda - \frac{1}{r^2}e^{2(\Phi-\Lambda)} + \frac{1}{r^2}e^{2\Phi} \\
G_{rr} &= \frac{2}{r}\partial_r\Phi - \frac{1}{r^2}e^{2\Lambda} + \frac{1}{r^2} \\
G_{\theta\theta} &= r^2e^{-2\Lambda}\left[\partial_r^2\Phi + (\partial_r\Phi)^2 + \frac{1}{r}(\partial_r\Phi - \partial_r\Lambda) - \partial_r\Phi\partial_r\Lambda\right] \\
G_{\phi\phi} &= \sin^2\theta r^2e^{-2\Lambda}\left[\partial_r^2\Phi + (\partial_r\Phi)^2 + \frac{1}{r}(\partial_r\Phi - \partial_r\Lambda) - \partial_r\Phi\partial_r\Lambda\right]
\end{aligned}
$$

For the case of the Schwarzschild metric, in the space outside the spherically symmetric mass distribution the stress-energy-momentum tensor $T_{\mu\nu}$ vanishes, and therefore the Einstein equation reduces to $G_{\mu\nu} = 0$. Hence, each of the four elements listed above vanishes. The first two of these simple differential equations (see bottom of page 7) are sufficient to find Λ and Φ, and hence complete the Schwarzschild metric:

$$
c^2d\tau^2 = \left(1 - \frac{2GM}{c^2r}\right)c^2dt^2 - \frac{dr^2}{(1 - \frac{2GM}{c^2r})} - r^2d\theta^2 - r^2\sin^2\theta d\phi^2
$$

From this metric we will derive a number of the famous effects of General Relativity that you may have heard about. These include the gravitational red shift, bending of light, the advance of the perihelion of Mercury's orbit, and the Shapiro time delay.

For a more serious introduction to General Relativity, the interested reader is directed to such texts as:

"Gravitation and Cosmology", by Steven Weinberg (Wiley)
"Gravitation", by Charles Misner, Kip Thorne, and John Archibald Wheeler (Freeman)
"A First Course in General Relativity", by Bernard Schutz