The internal fractional function revisited
An uncommon approximation for nongray radiation exchange

John H. Lienhard V
Rohsenow Kendall Heat Transfer Lab
Massachusetts Institute of Technology
Cambridge MA 02139-4307 USA

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Net radiation exchange
Small object in large isothermal surrounds

The net radiation leaving this surface is

\[ q_{\text{net}} = \sigma \varepsilon(T_1)T_1^4 - \sigma \alpha(T_1, T_2)T_2^4 \]  \hspace{1cm} (1)

Total hemispherical emissivity and absorptivity

\[ \varepsilon(T_1) = \frac{1}{\sigma T_1^4} \int_0^{\infty} \alpha(\lambda, T_1)e_{\lambda, b}(T_1) \, d\lambda \]

\[ \alpha(T_1, T_2) = \frac{1}{\sigma T_2^4} \int_0^{\infty} \alpha(\lambda, T_1)e_{\lambda, b}(T_2) \, d\lambda \]
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If \( T_2 \to T_1 \), then \( \alpha(T_1, T_2) \to \varepsilon(T_1) \), but ...
Non-gray error

Linearization about $T_1$ for small temperature differences

The slope as $T_2 \to T_1$ is different when $d\alpha/dT_2 \neq 0$.

$$\alpha(T_1, T_2) T_2^4 \approx \alpha(T_1, T_1) T_1^4 + \frac{d}{dT_2} (\alpha(T_1, T_2) T_2^4) \bigg|_{T_1} (T_2 - T_1)$$

$$= \varepsilon(T_1) T_1^4 + 4T_1^3 \left[ \varepsilon(T_1) + \frac{T_1}{4} \frac{d\alpha}{dT_2} \bigg|_{T_1} \right] (T_2 - T_1)$$
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Thus,

$$q_{net} \approx 4\sigma T_1^3 \left[ \varepsilon(T_1) + \frac{T_1}{4} \frac{d\alpha}{dT_2} \bigg|_{T_1} \right] (T_1 - T_2) \tag{2}$$
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For a gray (or black) surface, $d\alpha/dT_2 = 0$, so: $q_{\text{net}} \approx 4\sigma \varepsilon(T_1) T_1^3 \Delta T$. 
Background
External and internal emissivities

DK Edwards (1932–2009)
Heat Transfer Memorial Award (1973)

In his work on radiative property measurements, he studied the failure of gray-body approximations at even small $\Delta T$

- Edwards suggested the *internal radiation fractional function* for linearizing net heat flux between surfaces at small $\Delta T$. Appears in several textbooks by Edwards and his coworkers.

- Internal to a spacecraft: small $\Delta T$
- External to a spacecraft: large $\Delta T$
Edwards defined the *internal* total hemispherical emissivity as

\[
\varepsilon^i(T_1) \equiv \lim_{T_2 \to T_1} \frac{\varepsilon(T_1)\sigma T_1^4 - \alpha(T_1, T_2)\sigma T_2^4}{\sigma T_1^4 - \sigma T_2^4} = \lim_{T_2 \to T_1} \frac{\int_0^\infty \alpha(\lambda, T_1) \frac{\partial}{\partial T_2} e_{\lambda, b}(T_2) d\lambda}{4\sigma T_2^3}
\]

Thus, when \( T_2 \) is not too much different from \( T_1 \)

\[
q_{\text{net}} \approx \varepsilon^i(T_1) 4\sigma T_1^3 (T_1 - T_2)
\]

with

\[
\varepsilon^i(T) = 1 - 4\sigma T^3 \int_0^\infty \frac{\partial}{\partial T} e_{\lambda, b}(T) d\lambda = \int_0^1 \alpha(\lambda, T) d\lambda^i(\lambda T)
\]

where the internal fractional function is

\[
f^i(\lambda T) \equiv \frac{1}{4\sigma T^3} \int_0^\lambda \frac{\partial}{\partial T} e_{\lambda, b}(T) d\lambda
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\]

(3)

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(4)
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\]

where the *internal fractional function* is

\[
f_i(\lambda T) \equiv \frac{1}{4\sigma T^3} \int_0^\lambda \frac{\partial e_{\lambda, b}}{\partial T} \, d\lambda
\]
External Fractional Function

What we usually called the radiation fractional function

The fraction of blackbody radiation between wavelengths of 0 and $\lambda$ is

$$f(\lambda T) = \frac{1}{\sigma T^4} \int_0^\lambda e_{\lambda, b} \, d\lambda$$

$$= 1 - \frac{90}{\pi^4} \zeta\left(c_2/\lambda T, 4\right)$$

(7)

where $\zeta(X, s)$ is the incomplete zeta function. (Details in paper.)
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The external total hemispherical emissivity is

$$\varepsilon(T) = \int_0^1 \alpha(\lambda, T) \, df(\lambda T)$$

From these relationships, one can show that

$$f_i(\lambda T) - f(\lambda T) = F(X) = \frac{15}{4\pi^4} \frac{X^4}{e^X - 1}$$

(8)

where $X \equiv c_2/\lambda T$. 

\[\text{John Lienhard (MIT)}\]

\[\text{The internal fractional function revisited} \]

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\[ f_i(\lambda T) - f(\lambda T) = F(X) \]
The fraction of blackbody radiation between wavelengths of $0$ and $\lambda$ is
\[ f_1(\lambda T) - f(\lambda T) = F(X) \]
where
\[ X = \frac{c_2}{\lambda T} \]
(see Appendix C). The function $dF/dX$ is positive for $X < X_z$.

Thus, when $T$ is not too much different from 1, the internal radiation fractional function is:

\[ f_1(\lambda T) = \frac{1}{e^{X z} - 1} \]

Because the numerator and denominator both go to zero in the limit, L'Hôpital's rule (or Taylor expansion) is required:

\[ \lim_{X \to 0} \frac{f_1(X) - f(0)}{X} = \frac{1}{e^{X z} - 1} \]

Edwards [1, 4, 5] refers to internal emissivities.

John Lienhard (MIT)
The fraction of blackbody radiation between wavelengths of 0 and $\lambda T$ is

$$f_i(\lambda T) - f(\lambda T) = F(X)$$  \hspace{1cm} X = c_2/\lambda T$$

where the internal radiation fractional function is

$$f_i(\lambda T) = \frac{dF}{dX}$$  \hspace{1cm} X = \zeta$$

The function $dF/dX$ is positive for $X < \zeta$. From these relationships, one can show that $f_i(\lambda T) - f(\lambda T)$ is the incomplete zeta function (see derivation in Appendix C). The function $dF/dX$ is not too much different from $f(\lambda T)$; therefore, Eqns. (7) and (14) are identical. Thus, when $T \rightarrow 0$, $f(\lambda T)$ goes to 0 and denominator in the limit, so Eqns. (7) and (14) are identical.

We now derive an upper bound on the difference of these two functions, as a change of variables and integration by parts of

$$f(\lambda T) = \int_{X_z}^{\infty} \frac{1}{x^{1/2} \xi(x)} \xi(x) dx$$

and

$$dF/dX = \int_{X_z}^{\infty} \frac{1}{x^{1/2} \xi(x)} \xi(x) dx$$

where $\xi(x)$ is the incomplete zeta function (see derivation in Appendix C). The function $dF/dX$ is positive for $X < \zeta$.

FIGURE 1: Fractional function $f_i$, $f$, and $F$, with $X_z = 3.92 \ldots$
Difference between external and internal emissivities

\[ \varepsilon - \varepsilon^i = \int_0^1 \alpha(\lambda, T) \, df(\lambda T) - \int_0^1 \alpha(\lambda, T) \, df_i(\lambda T) = \int_0^\infty \alpha(\lambda, T) \frac{dF}{dX} \, dX \]

where \( \frac{dF}{dX} = 0 \) at \( X = \frac{z}{3.92069} \). Because \( \frac{dF}{dX} > 0 \) for \( X < \frac{z}{3.92069} \) and \( < 0 \) for \( X > \frac{z}{3.92069} \):

\[ \varepsilon - \varepsilon^i \leq \int_0^{\frac{z}{3.92069}} dF \, dX = F(X^{\frac{z}{3.92069}}) \quad \text{if} \quad \varepsilon - \varepsilon^i > 0, \]

and

\[ \varepsilon^i - \varepsilon \leq \int_{\frac{z}{3.92069}}^\infty dF \, dX = F(X^{\frac{z}{3.92069}}) \quad \text{if} \quad \varepsilon^i - \varepsilon > 0. \]
Difference between external and internal emissivities

\[ \varepsilon - \varepsilon^i = \int_0^1 \alpha(\lambda, T) \, df(\lambda T) - \int_0^1 \alpha(\lambda, T) \, df_i(\lambda T) = \int_0^\infty \alpha(\lambda, T) \frac{dF}{dX} \, dX \]

\[ = \int_0^{X_z} \alpha(\lambda, T) \frac{dF}{dX} \, dX + \int_{X_z}^\infty \alpha(\lambda, T) \frac{dF}{dX} \, dX \]

where \( \frac{dF}{dX} = 0 \) at \( X_z = 3.92069 \).
Difference between external and internal emissivities

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where \(dF/dX = 0\) at \(X_z = 3.92069\). Because \(dF/dX > 0\) for \(X < X_z\) and \(< 0\) for \(X > X_z\):

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Difference between external and internal emissivities

\[
\epsilon - \epsilon^i = \int_0^1 \alpha(\lambda, T) f(\lambda T) \, d\lambda - \int_0^1 \alpha(\lambda, T) f_i(\lambda T) \, d\lambda = \int_0^\infty \alpha(\lambda, T) \frac{dF}{dX} \, dX
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\[
\epsilon - \epsilon^i \leq \int_0^{X_z} \frac{dF}{dX} \, dX = F(X_z) \quad \text{if } \epsilon - \epsilon^i > 0, \text{ and}
\]

\[
\epsilon^i - \epsilon \leq \int_{X_z}^\infty \frac{dF}{dX} \, dX = F(X_z) \quad \text{if } \epsilon^i - \epsilon > 0
\]

Evaluating

\[
|\epsilon - \epsilon^i| \leq 0.18400
\]
Model surfaces: Switch between $\alpha(\lambda) = 0$ and $\alpha(\lambda) = 1$ at

$$X_z = \frac{c_2}{\lambda_z T} = 3.92069$$

Emissivities evaluated numerically

**Case 1:** 300 K surface, black for $\lambda_z \leq 12.23$ µm, but reflective on other wavelengths.

$$\varepsilon = 0.4177, \quad \varepsilon_i = 0.6017, \quad \text{and} \quad \varepsilon^i - \varepsilon = 0.1840 \quad (10)$$

**Case 2:** 300 K surface, black for $12.23$ µm $\leq \lambda_z$, but reflective on other wavelengths:

$$\varepsilon = 0.5823, \quad \varepsilon_i = 0.3983, \quad \text{and} \quad \varepsilon - \varepsilon^i = 0.1840 \quad (11)$$

In both cases $\alpha(T_1, T_2)$ is a strong function of $T_2$. 
Linearization of $q_{\text{net}}$ about $T_1$ is less accurate than for $T_m$

Consider $q_{\text{net}}$ for a black surface: $T_1$, eqn. (32); $T_m$, eqn. (33). $T_m = (T_1 + T_2)/2$
Linearization with internal emissivity

Linearize about $T_m = (T_1 + T_2)/2$

Linearization accuracy is also greater for a non-gray surface when using $T_m$, but must include temperature dependence of $\alpha(T_1, T_2)$.
Linearization with internal emissivity

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- Linearization about $T_1$ is just Edward’s definition: $q_{\text{net}} \approx \varepsilon^i(T_1) \cdot 4\sigma T_1^3 \Delta T$
  It is a first-order, single-step, Euler approximation.

\[ q_{\text{net}} \approx 4\varepsilon^i(T_m) \cdot \sigma T_m^3 \Delta T \]
Linearization with internal emissivity

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  It is a first-order, single-step, Euler approximation.

- Linearization about $T_m$ is a second-order, single-step Runge-Kutta approximation. Calculation gives (details in paper)

\[
q_{\text{net}} \approx 4\varepsilon^i(T_m) \cdot \sigma T_m^3 \Delta T
\]  \hspace{1cm} (12)

to an accuracy of $O(\Delta T^3)$. 
FIGURE 4. COMPARISON OF MODELS FOR $q_{\text{net}}$ (300 K SURFACE, BLACK BELOW 12.23 $\mu$m)
The ratio of gray slope to exact slope is \( \frac{1}{1.46} \) for Figs. 4 and 5, respectively. For the surface black on short wavelengths, with \( T = 300 \) K, the compute heat flux evaluated, to an accuracy of \( O(1) \). At seen in Fig. 3, the linearization around \( T = 1 \) is equivalent to a second-order term and has errors on the order of \( 1 \). The reason for the greater severity and has errors on the order of \( T = 1 \). The charts show: 1. The gray surface approximation, \( f(T_1) \) has the correct slope as \( f(T_m) \); 2. The approximation using \( f(T_m) \), has the wrong slope as \( f(T_1) \) and \( T \) is fixed. The trend seen will be seen for all \( T \), although \( f(T_m) \) will also have a poor accuracy for even the smallest temperature differences. Although Figs. 4 and 5 use \( 300 \) K as a reference point, Eqns. (14) and (32) are both very specific in recommending to evaluate \( f(T) \) at \( T = 1 \). Edwards et al. [6] did not Copyright © 2018 by ASME.
Polycrystalline alumina, normal emissivity

99.5% Al$_2$O$_3$, 6 mm thick, 1 µm roughness, $T_1 = 823$ K (Teodorescu and Jones, 2008)
Polycrystalline alumina, normal emissivity
99.5% Al$_2$O$_3$, 6 mm thick, 1 µm roughness, $T_1 = 823$ K (Teodorescu and Jones, 2008)

Total, normal
\[ \varepsilon_n = 0.506 \]
\[ \varepsilon_i = 0.404 \]
Polycrystalline alumina at $T_1 = 823$ K

$\varepsilon^i(T_m)$ provides much wider accuracy than $\varepsilon^i(T_1)$
Platinum, $T_1 = 373$ K

Drude/Hagen-Rubens model for spectral hemispherical emissivity (Baehr & Stephan, 1998)

\[
e(\lambda, T) = 48.70\sqrt{\frac{r_e}{\lambda}} \left\{ 1 + \left[ 31.62 + 6.849 \ln\left(\frac{r_e}{\lambda}\right) \right] \sqrt{\frac{r_e}{\lambda}} - 166.78 \frac{r_e}{\lambda} + \cdots \right\}
\]

\[\text{FIGURE 8. COMPARISON OF MODELS FOR PLATINUM SURFACE: SOFT ANODIZED ALUMINUM AT T = 373 K.}
\]

\[\text{TABLE 1: SELECTIVE SOLAR REFLECTOR: SOFT ANODIZED ALUMINUM.}
\]

\[\begin{array}{c|cc}
\text{Selectivity} & \text{Exact Model} & \text{Eqn. (14)} \\
0 & 0 & 462.1 \\
10 & 0.6237 & 371.0 \\
20 & 0.7094 & 371.0 \\
30 & 0.9206 & 371.0 \\
40 & 0.9806 & 371.0 \\
50 & 0.9957 & 371.0 \\
\end{array}
\]
Model of spectrally selective surface

Similar to data for soft-anodized aluminum in Edwards’ *Radiation Heat Transfer Notes*

\[
\alpha(\lambda) = \begin{cases} 
\alpha_{sw} & \text{for } \lambda \leq \lambda_c \\
\alpha_{lw} & \text{for } \lambda > \lambda_c 
\end{cases}
\]
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\end{cases}
\]

Can write

\[
\varepsilon(T) = \alpha_{\text{sw}} f(\lambda_c T) + \alpha_{\text{lw}} [1 - f(\lambda_c T)] = \alpha_{\text{sw}} + \frac{90}{\pi^4} \Delta\alpha \zeta(X_c, 4)
\]

where \(X_c = c_2/\lambda_c T\) and \(\Delta\alpha = \alpha_{\text{lw}} - \alpha_{\text{sw}}\).
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\]

where \(X_c = c_2/\lambda_c T\) and \(\Delta \alpha = \alpha_{lw} - \alpha_{sw}\). Further,

\[
\varepsilon^i(T_m) = \alpha_{sw} + \Delta \alpha \left[ \frac{90}{\pi^4} \zeta(X_{c,m}, 4) - F(X_{c,m}) \right]
\]

where \(X_{c,m} = c_2/\lambda_c T_m\).
Model of spectrally selective surface

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\alpha(\lambda) = \begin{cases} 
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\]

where \( X_c = c_2/\lambda_c T \) and \( \Delta\alpha = \alpha_{lw} - \alpha_{sw} \). Further,

\[
\varepsilon^i(T_m) = \alpha_{sw} + \Delta\alpha \left[ \frac{90}{\pi^4} \zeta(X_{c,m}, 4) - F(X_{c,m}) \right]
\]

where \( X_{c,m} = c_2/\lambda_c T_m \). Finally,

\[
\alpha(T_1, T_2) = \alpha_{sw} + \frac{90}{\pi^4} \Delta\alpha \zeta(X_{c,2}, 4)
\]

with \( X_{c,2} = c_2/\lambda_c T_2 \). Impact of selectivity greatest when \( X_c \) and \( X_z \) are close.
Soft anodized aluminum at $T_1 = 360$ K with $T_2 = 290$ K
Selective solar reflector: $\alpha_{sw} = 0.1$, $\alpha_{lw} = 0.85$, and $\lambda_c = 7 \, \mu$m. Heat flux in W/m$^2$.

<table>
<thead>
<tr>
<th>$\varepsilon(T_1)$</th>
<th>$\varepsilon^i(T_1)$</th>
<th>$\varepsilon^i(T_m)$</th>
<th>$\alpha(T_1, T_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7258</td>
<td>0.6237</td>
<td>0.6807</td>
<td>0.7964</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$q_{\text{gray}}$</th>
<th>$q_{\text{int}, T_1}$</th>
<th>$q_{\text{int}, T_m}$</th>
<th>$q_{\text{exact}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>400.2</td>
<td>462.1</td>
<td>371.0</td>
<td>371.8</td>
</tr>
</tbody>
</table>

\[\lambda_z = 10.2\]
Radiation thermal resistance

$\varepsilon^i(T_m)$ should be used for this linearization

$$R_{t_{\text{rad}}} = \frac{1}{h_{\text{rad}}A}$$

$$= \frac{1}{4\varepsilon \sigma T_m^3 A}$$

$$= \frac{1}{4 \varepsilon^i(T_m) \sigma T_m^3 A}$$
Summary

ε^(i)(T_m) is useful for radiation thermal resistance

Edwards and others have suggested ε^(i)(T_1) for non-gray exchange in enclosures with modest ΔT, to provide a correct linearization of q_{net}.
Summary

$\epsilon^i(T_m)$ is useful for radiation thermal resistance

Edwards and others have suggested $\epsilon^i(T_1)$ for non-gray exchange in enclosures with modest $\Delta T$, to provide a correct linearization of $q_{net}$.

1. Theory and examples for several non-gray materials show that the gray-body approximation gives the wrong slope for heat flux as $T_2 \to T_1$.

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The internal fractional function revisited

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Supplementary slides
Second-order, single-step, Runge-Kutta approximation

\[ q_{\text{net}} = Y(T_2) = \sigma \varepsilon(T_1) T_1^4 - \sigma \alpha(T_1, T_2) T_2^4 \]
Second-order, single-step, Runge-Kutta approximation

\[ q_{\text{net}} = Y(T_2) = \sigma \varepsilon(T_1)T_1^4 - \sigma \alpha(T_1, T_2)T_2^4 \]

A second-order Runge-Kutta method works from \( T_m \) with expansions toward both \( T_1 \) and \( T_2 \), subtracting the former from the latter:

\[
Y(T_2) = Y(T_m) + Y'(T_m) \frac{\delta T}{2} + Y''(T_m) \frac{\delta T^2}{8} + O(\delta T^3)
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Y(T_1) = Y(T_m) - Y'(T_m) \frac{\delta T}{2} + Y''(T_m) \frac{\delta T^2}{8} - O(\delta T^3)
\]

Subtract

\[
Y(T_2) = Y(T_1) + Y'(T_m) \cdot \delta T + O(\delta T^3)
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Y(T_2) \approx Y'(T_m) \cdot \delta T
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\[
Y'(T_m) = - \frac{d}{dT} \left( \sigma T_1^4 \alpha(T_1, T) \right) \bigg|_{T_m} = \cdots = -4 \sigma T_m^3 \cdot \epsilon_i(T_m)
\]
The internal fractional function revisited

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Incomplete zeta function and $f(\lambda T)$

\[
f(\lambda T) = \frac{1}{\sigma T^4} \int_0^\lambda \frac{2\pi h c_o^2}{\lambda^5} \left[ \exp\left(\frac{hc_o}{k_B T \lambda}\right) - 1 \right] d\lambda = \frac{1}{\sigma T^4} \frac{2\pi k_B^4 T^4}{h^3 c_o^2} \int_{c_o^2/\lambda T}^\infty \frac{t^3}{e^t - 1} dt
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When $\lambda T \to \infty$, $f = 1$ and so

$$\sigma T^4 = \frac{2\pi k_B^4 T^4}{h^3 c_o^2} \int_0^\infty \frac{t^3}{e^t - 1} \, dt \equiv \zeta(4) \Gamma(4)$$

where $\Gamma(4) = 3!$ and $\zeta(4)$ is the Riemann zeta function (Euler: $\zeta(4) = \pi^4 / 90$).
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f(\lambda T) = \frac{15}{\pi^4} \int_0^\infty \frac{t^3}{e^t - 1} \, dt - \frac{15}{\pi^4} \int_{c_2 / \lambda T}^\infty \frac{t^3}{e^t - 1} \, dt
\]

\[
= 1 - \frac{15}{\pi^4} \Gamma(4) \zeta(X, 4) = 1 - \frac{90}{\pi^4} \zeta(X, 4)
\]

where \( X = c_2 / \lambda T \), and \( \zeta(X, s) \) is the incomplete zeta function.
Integration of directional emissivity for alumina

\[ \varepsilon(\lambda, T) = \int_0^{\pi/2} \varepsilon'(\theta, \lambda, T) \sin(2\theta) \, d\theta \]

Data in 12° increments over 0° ≤ θ ≤ 72°. Essentially constant from 0 to 36°; this range was integrated analytically. From 36° to 84° a five-point trapezoidal rule was used, and the integral from 84° to 90° was approximated as a trapezoid. The value at 90° was set to zero, in line with theory. Numerical truncation error is 1.0% for a gray surface.

The data showed angular behavior consistent with a dielectric. On this basis, interpolated using a value representative of large angle for a dielectric:

\[ \varepsilon(84°, \lambda) \approx 0.75 \varepsilon(72°, \lambda) \]

Without more data, cannot exclude peak emissivity above 80° predicted by Drude's model for metals; but sensitivity analysis letting \( \varepsilon(84°, \lambda) \approx 2.5 \varepsilon(72°, \lambda) \) increases the hemispherical emissivity by only about 5% of the previous estimate.
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Nondimensional results for model surfaces

\( \varepsilon'(T_m) \) excellent for \( T_2/T_1 = 1 \pm 30\% \) or more
Model surfaces: \( \alpha(T_1, T_2) \) has strong dependence on \( T_2 \)

\[
\begin{align*}
\text{Total absorptivity, } \alpha & \quad \text{Black for } \lambda > 12.23 \text{ \( \mu \)m} \\
\text{\( T_2 \) [K]} & \\
200 & \quad 0.8 \\
250 & \quad 0.6 \\
300 & \quad 0.4 \\
350 & \quad 0.2 \\
400 & \quad 0.0 \\
\end{align*}
\]
The constant $X_z$, the finite solution of $dF/dX = 0$

$$4 \left(1 - e^{-X_z}\right) = X_z$$

In terms of the Lambert $W$ function

$$X_z = 4 - W(4e^{-4}) = 3.92069 \ldots$$

$X_z$ is irrational. Diophantine approximation by continued fractions:

$$X_z = 3.92069 \ldots = 3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{11 + \ddots}}}$$

Successive convergents give rational approximations:

$$X_z \approx \left\{4, \frac{47}{12}, \ldots, \frac{149}{38}, \frac{247}{63}, \ldots, \frac{1137}{290}, \ldots\right\}$$

$2^{nd}$ one is within $0.1\%$