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Experimental investigation of the transverse vibrational frequency ratio for identical loaded and unloaded rubber strings

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The elementary problems of small transverse vibrations of loaded and unloaded strings are briefly reviewed, and an example that shows the effect on the vibrational frequency of the amount of stretch of different kinds of materials is presented. A detailed review of the problem of the transverse vibration of a loaded string when the inertia of string is not neglected is presented, and certain exact and approximate results are examined graphically. A universal formula for the ratio of the fundamental transverse vibrational frequencies of identical loaded and unloaded strings is derived; and new experimental results describing the response of loaded rubber cords for a wide range of attached loads are compared with the universal formula.

I. INTRODUCTION

Investigation of the problem of the small-amplitude motion of a vibrating string usually is included in a beginning course in mechanical vibrations or acoustics. It is learned, though seldom emphasized, that the transverse vibrational frequency derived is a universal result in the sense that it is independent of the elastic behavior of the material. On the other hand, the manner in which the frequency may vary with the amount of stretch, which all too often and unnecessarily is assumed infinitesimal, will depend on the elastic response of the material. The same may be said of the simpler problem of the small-amplitude vibration of a mass attached to an elastic string. In this case, the influence of the inertia of the string on the motion of the load is commonly neglected. In fact, this effect is not discussed in most textbooks; consequently, we lose the opportunity for the teacher to explore with the student the degree to which the easy approximate solution is valid and for the student to learn method from the analysis of the problem when the inertia of the string is included. We hope that this didactic note will help to remedy the situation and at the same time provide some fresh thoughts together with some new experimental results related to the transverse vibration of identical loaded and unloaded rubber strings.

With these objectives in mind, we shall review briefly in Sec. II the problems of small transverse vibration of loaded and unloaded strings. We present an illustrative example that reveals an interesting effect on the frequency due to variation in material response for two different materials, a highly elastic rubber cord and a typical metal wire. Then, we turn to a careful review of the transverse vibration of the loaded string in the case when the inertia of the string is included. The first- and second-order fundamental-mode approximations, and the higher-mode and nodal solutions are compared graphically with the exact solution; the results are interpreted physically and the extent to which the approximate solutions are meaningful is discussed. It is shown that regardless of the amount of stretch of the string and the nature of its material the ratio of the fundamental transverse vibrational frequencies of identical loaded and unloaded strings is a constant that depends only on the ratio of their masses, a simple result that seems to have passed unnoticed in the usual sources. We also show easily that the frequency of vibration of the load always is less in the case when the inertia of the string is included.

An easy experiment that compares the approximate and exact expressions for the aforementioned transverse frequency ratio to its experimentally determined values for identical loaded and unloaded rubber strings is described in Sec. III. Three kinds of rubber cords subjected to stretches ranging up to nearly 400% elongation were studied for a wide range of centrally attached loads, the period of vibration being measured with a novel laser beam device. The experimental results, presented in Sec. IV, show remarkably good agreement with the theoretical predictions derived in Sec. II. We find no significant variation in the data among the materials used nor for the full range of stretches studied. Therefore, our experiment, extending over a wide range of loads, confirms that the ratio of the fundamental transverse vibrational frequencies of identical loaded and unloaded rubber strings that may be subjected to finite stretch is a constant that depends only upon the ratio of their masses.

II. TRANSVERSE VIBRATION OF LOADED AND UNLOADED STRINGS REVISITED

We consider an elastic, perfectly flexible string, wire, or fiber of undeformed length l_0 and total mass m stretched by

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a force $T(\lambda)$ to length $l = \lambda l_0$ and clamped in rigid supports at both ends. We shall neglect gravity, momentarily ignore the inertia of the string itself, and focus our attention on a centrally attached load of mass M that is given a small displacement normal to the string and released to perform small oscillations of circular frequency ω and amplitude $A \ll l$. The tension is assumed constant at all times t during the motion $Y(M, t)$ of M, which is determined by the differential equation

$$\frac{d^2Y}{dt^2} + \omega^2 Y = 0 \tag{1}$$

with

$$\omega = 2\sqrt{\frac{T}{Ml}} = 2\sqrt{\frac{T/\lambda}{Ml_0}}. \tag{2}$$

The solution of (1) yields the small-amplitude, simple harmonic motion of M:

$$Y(M, t) = A\cos(\omega t - \phi). \tag{3}$$

The amplitude A and phase angle ϕ are constants determined by specification of the initial data as usual.[1]

If the load is attached to the string at distance $a \equiv \alpha l$ from one end, the circular frequency for the small amplitude motion (3) is given by

$$\omega = \sqrt{\frac{T/\lambda}{Ml_0\alpha(1-\alpha)}}. \tag{4}$$

This reduces to (2) when $\alpha = \frac{1}{2}$. The derivation of (4) provides an easy exercise for the student.

When the load is removed, but the aforementioned circumstances are otherwise the same, the equation for the motion $y(x, t)$ of a particle of the string distant x from one end is given by

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \tag{5}$$

with $c = \sqrt{(T/\rho)}$, where $\rho \equiv m/l$ denotes the mass per unit length of the stretched string.[1] In this case, the general solution of (5) that meets the fixed end conditions $y(0, t) = 0$, $y(l, t) = 0$ for all times t delivers the motion

$$y(x, t) = \sum_{n=1}^{\infty} A_n \cos(\Omega_n t - \phi_n)\sin(n\pi x/l), \tag{6}$$

wherein the normal-mode circular frequencies are defined by $\Omega_n = n\pi c/l$, $n = 1,2,3,\ldots$, and the constants A_n and ϕ_n are determined from assigned initial conditions.[1] With the definition of c in (5), the circular frequency $\Omega \equiv \Omega_1$ of the fundamental mode is given by

$$\Omega = \frac{\pi}{l}\sqrt{\frac{T}{\rho}} = \pi\sqrt{\frac{T/\lambda}{ml_0}}. \tag{7}$$

The relations (2) and (7) show that in either case the transverse vibrational frequency depends on the length of the string, the amount of stretch and the elastic response of the material. In particular, a rubber cord having an undeformed cross sectional area A_0 and elastic modulus E may be characterized by the constitutive equation

$$T(\lambda) = \tfrac{1}{3}A_0E(\lambda - \lambda^{-2}). \tag{8}$$

Of course, in this case the stretch λ may by fairly large.[2] On the other hand, for a linearly elastic, metal wire characterized by

$$T(\lambda) = A_0E(\lambda - 1) \equiv A_0E\epsilon, \tag{9}$$

the strain ϵ is infinitesimal so that terms of the order ϵ^2 are negligible. Therefore, in (2) and (7) the length $l = \lambda l_0$ and the mass density $\rho = \lambda\rho_0$ in the deformed state may be confounded with their values l_0, ρ_0 in the natural state; that is, for a metal wire, $T(\lambda)/\lambda = T(\lambda) + O(\epsilon^2)$, very nearly. It is clear that the exact expressions derived above must be employed when the strain is finite; consequently, in both the loaded and unloaded cases, the transverse frequency for these materials certainly will differ significantly. We shall

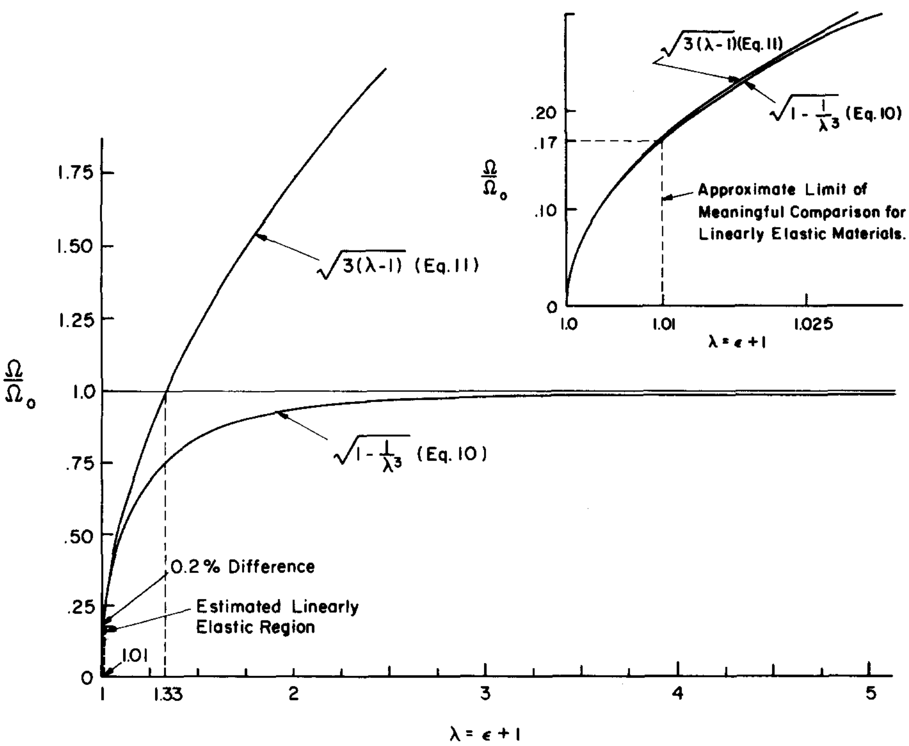

Fig. 1. Normalized transverse vibrational frequency of a linearly elastic string and a rubberlike, nonlinearly elastic cord as a function of the stretch. Upper right-hand corner: Magnification of the linearly elastic region for which (11) holds.

see that the rubber cord exhibits especially interesting behavior.

Substitution of (8) into (7) yields

$$\Omega = \Omega_0\sqrt{(1-\lambda^{-3})} \quad \text{with} \quad \Omega_0 \equiv \pi\sqrt{(A_0E/3ml_0)}, \tag{10}$$

whereas use of (9) gives

$$\Omega = \Omega_0\sqrt{(3\epsilon)} \tag{11}$$

for the unloaded case. Similar equations can be written for the loaded case (2). It is easy to see that (10) reduces to (11) when the deformation of the rubber cord is infinitesimal; so, according to (11), for sufficiently small deformation, the squared frequency increases linearly with the stretch. But (10) reveals that as the stretch of the rubber cord increases, the transverse frequency approaches a constant value. This behavior is illustrated in Fig. 1. The graph of (11), though drawn for comparison with the expression (10) valid for large deformations of certain rubberlike materials, actually applies only over a very small range of deformation, which is shown with slight exaggeration in the upper right-hand corner of Fig. 1. The difference between the actual values on these curves at 1% strain is roughly 0.2%.

Thus we reach the interesting result derived by Beatty and Chow[3] that when a rubber band characterized by (8) is stretched sufficiently between the fingers of both hands and plucked, the pitch will vary only slightly as the tension is increased. In fact, the first experimental investigations of this phenomenon were reported independently at the turn of this century by von Lang[4] and Baker,[5] although neither author was able to relate his data to a theoretical model as we have done here. Of course, the foregoing analytical prediction that so far as variation in frequency is concerned the sound power radiated by a rubber string is bounded is predicated upon a specific and simple constitutive model for rubber, commonly called the neo-Hookean model; other acceptable models of rubber elasticity,[2] of which the neo-Hookean theory is a special case, may yield significantly different results, as shown by Beatty and Chow.[3] However, we wish to focus on another issue.

Let us observe that regardless of the length, the stretch or the elastic stiffness of the string the ratio R of the transverse vibrational frequencies (2) and (7) of identical loaded and unloaded strings is a constant that depends solely upon the ratio of the string mass to the load mass:

$$R = \omega/\Omega = (2/\pi)\sqrt{(m/M)}. \tag{12}$$

Since, in effect, we have assumed that the mass of the string is small compared with that of the load, it is evident from (12) that the vibrational frequency of the load always is less than the fundamental frequency of the string. Let us now review the more difficult case when the inertia of the string is included and all other assumptions are the same as before.

The load is identified as a particle of mass M attached to the string distant a from one end so that its motion $Y(M,t)$ coincides with the motion $y_r(a,t)$ of the point at $x=a$ on the parts $r = 1, 2$ of the cord. Then, for continuity at $x=a$, for all times t

$$Y(M,t) = y_1(a,t) = y_2(a,t). \tag{13}$$

The motion $y_r(x,t)$ of each part of the string is determined by (5):

$$\frac{\partial^2 y_1}{\partial t^2} = c^2\frac{\partial^2 y_1}{\partial x^2}$$

on $0<x<a$ with $y_1(0,t)=0$, (14)

$$\frac{\partial^2 y_2}{\partial t^2} = c^2\frac{\partial^2 y_2}{\partial x^2}$$

on $a<x<l$ with $y_2(l,t)=0$. (15)

Further, since the string tension has been approximated for all times by its uniform equilibrium value, the tension must be the same throughout both parts of the string during the small amplitude oscillation of the system. It therefore, follows that the motion of M is governed by the differential equation

$$\frac{d^2Y}{dt^2} = \frac{T}{M}\left(\frac{\partial y_2}{\partial x} - \frac{\partial y_1}{\partial x}\right)\bigg|_{x=a}. \tag{16}$$

We seek solutions of the form $y_r(x,t) = A_r(x)e^{i\omega t}$, which will satisfy (14) and (15), and thus obtain

$$y_1(x,t) = Ce^{i\omega t}\sin\mu x \quad \text{for} \quad x\epsilon[0,a], \tag{17a}$$

$$y_2(x,t) = De^{i\omega t}\sin\mu(l-x) \quad \text{for} \quad x\epsilon[a,l]. \tag{17b}$$

Here C and D are certain constants and $\mu\equiv\omega/c$. Equation (13) requires

$$Y(M,t) = Ce^{i\omega t}\sin\mu a = De^{i\omega t}\sin\mu b,$$

where $b\equiv l-a$. Therefore, the motion of M is described by

$$Y(M,t) = Ae^{i\omega t} \tag{18}$$

with $A = C\sin\mu a = D\sin\mu b$. Consequently, if the point $x=a$ is not a nodal point for which $Y(M,t)=0$, that is, $A\neq 0$, (17) delivers

$$y_1(x,t) = Y(M,t)\frac{\sin\mu x}{\sin\mu a} \quad \text{for} \quad x\epsilon[0,a], \tag{19a}$$

$$y_2(x,t) = Y(M,t)\frac{\sin\mu(l-x)}{\sin\mu b} \quad \text{for} \quad x\epsilon[a,l]. \tag{19b}$$

However, if the point $x=a$ is nodal, (17) must satisfy $y_1(a,t) = y_2(a,t) = 0$. Then M remains at rest as the two parts of the cord oscillate with frequency determined by $\mu a = s\pi$, and b is a suitable multiple of a such that $\mu b = q\pi$, where $s, q = 1,2,3,\ldots$. Otherwise, substitution of (19) into (16) yields (1) with

$$\omega^2 = \mu^2c^2 = (T\mu/M)(\cot\mu a + \cot\mu b);$$

that is, with $c^2 = T/\rho$ and $m=\rho l$,

$$\mu l = \gamma(\cot\mu a + \cot\mu b), \tag{20}$$

where $\gamma\equiv m/M$.

This transcendental equation determines $\mu = \omega/c$ when the other parameters are assigned. In particular, when $a=b=l/2$, we obtain the simpler formulas

$$\theta\tan\theta = \gamma \quad \text{with} \quad \theta\equiv\mu l/2 = \omega\pi/2\Omega, \tag{21}$$

wherein we recall (7) and the previous relation for c in (5). Equation (21) determines $\theta = \theta(\gamma)$ for all values of the mass ratio γ. This result demonstrates that when the inertia of the string is taken into account, the ratio R of the fundamental transverse vibrational frequencies of a loaded and an otherwise identical unloaded string is a constant that depends solely on the ratio of their masses, that is,

$$R\equiv\omega/\Omega = (2/\pi)\theta(\gamma) \tag{22}$$

is independent of the amount of stretch and the nature of the material.

When only terms of second order in θ are retained in the series expansion of (21), we have $\theta(\gamma) = \sqrt{\gamma}$. Hence, if γ is considerably smaller than one, use of this approximation in (22) yields our earlier result (12) for which case the mass of

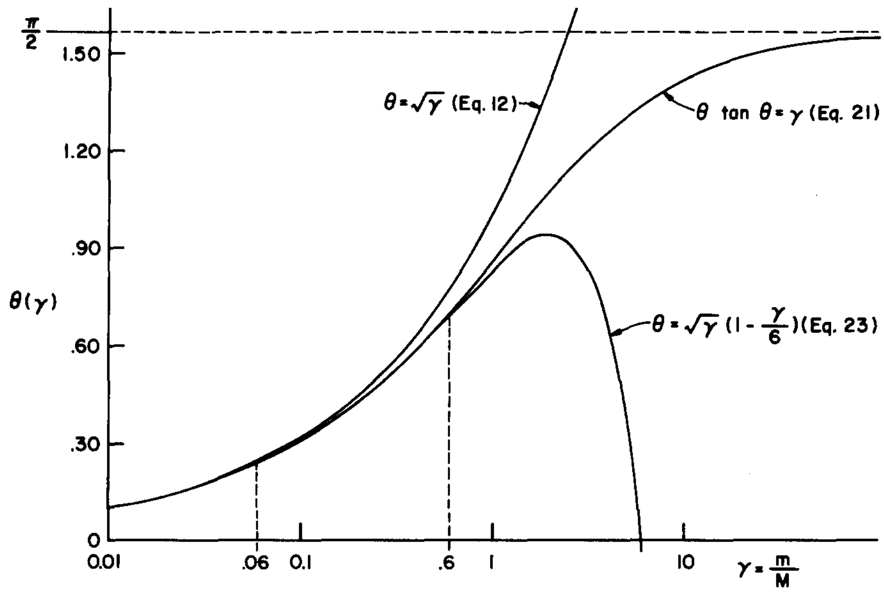

Fig. 2. Comparison of first- and second-order approximate solutions with the exact solution for the transverse oscillation of a centrally loaded string in the fundamental mode (semilog graph).

the string is negligible compared to that of the load. To obtain the lowest-order correction that accounts for the inertia of the string, terms of the fourth order in θ are retained in (21); we find

$$\theta(\gamma) = (1 - \gamma/6)\sqrt{\gamma}. \tag{23}$$

Here γ must be small enough that terms of second order in γ are negligible. For the lowest mode, these approximate solutions are compared graphically with the exact solution (21) in Fig. 2 for $\theta\epsilon(0,\pi/2)$. The simple solution (12) approximates the exact curve (21) to only 1% error at $\gamma = 0.06$, so (12) is an accurate estimate for heavily loaded strings. The second-order solution (23) extends this range to almost $\gamma = 0.60$ where the simple solution errs by nearly 10%. In fact, we see that (23) approximates (21) nicely for $\gamma < 1$ at which value the error is roughly 3%. Nevertheless, the experiments to be described later on for lightly to heavily loaded rubber strings will extend well beyond the range of applicability of even the second-order solution.

More generally, the positive solutions θ_n, $n = 0,1,2,\ldots$, of (21) and the aforementioned nodal solutions $\theta_s = s\pi$, $s = 1,2,3\ldots$(which are independent of M) also may be graphically illustrated. The θ_n are the points of intersection of the graphs of the trigonometric function $z = \tan\theta$ and the rectangular hyperbola $z = \gamma/\theta$ shown in Fig. 3. The unique solution $\theta = \theta_0\epsilon(0, \pi/2)$ is the exact value for the dominant mode. Since all higher modes tend to and are preceded by nodal values, we may write for the nth root of (21)

$$\theta_n = \theta'_n + \delta_n = n\pi(1 + \delta_n/n\pi) \quad \text{for} \quad n = 1,2,3,\ldots.$$

To the first order in δ_n, (21) yields $\delta_n = \gamma/n\pi$; hence, the roots of (21) are approximated by

$$\theta_n = n\pi(1 + \gamma/n^2\pi^2) \quad \text{for} \quad \text{n} = 1,2,3,\ldots \tag{24}$$

when γ is sufficiently small. Of course, the general relation (21) is valid for all γ and always may be used to find the effect on the frequency of the various component vibra-

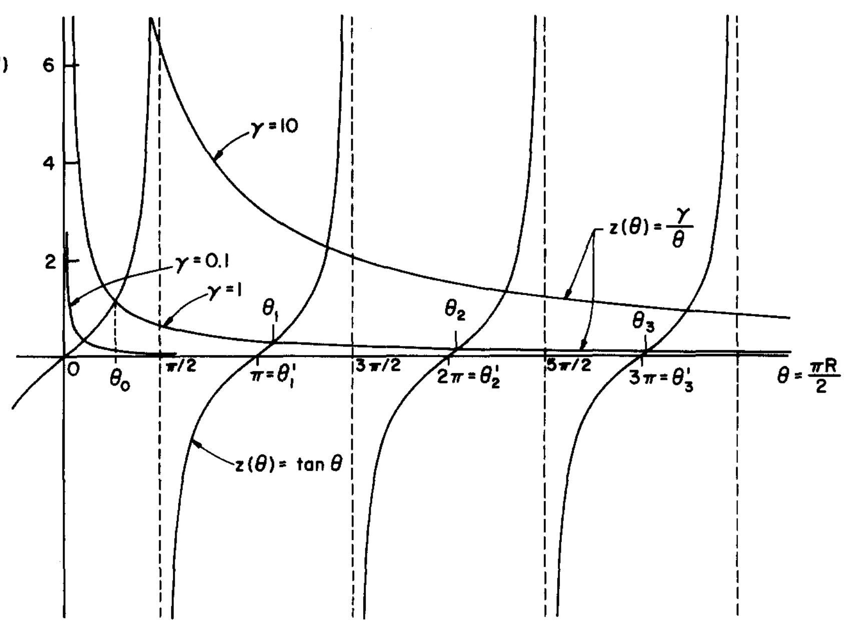

Fig. 3. Graphical illustration of several higher-mode solutions and neighboring nodal solutions for selected values of γ; θ_0 indicates a typical dominant fundamental mode of principal interest.

tions due to even a small mass M attached to a considerably more massive string.

Finally, let us write ω_0 for the circular frequency (2) of the load when the inertia of the string is neglected. Then use of (7) in (22) shows that $\omega = \omega_0 \gamma^{-1/2} \theta(\gamma)$, and application of the first-order correction for the inertia of the string in (23) reveals that

$$\omega = \omega_0(1 - \gamma/6) < \omega_0. \tag{25}$$

Indeed, the graph in Fig. 2 shows clearly that the exact solution (21) for $\theta(\gamma)$ always lies below the simple solution (12). Thus the circular frequency ω of the load always will be less in the case when the inertia of the string is taken into account. Equation (25) also shows that, to the first order in γ, the error in frequency due to neglecting the mass of the string itself is

$$(\omega_0 - \omega)/\omega = \gamma/6. \tag{26}$$

Here we are able to see clearly that this error will be small only when γ is small, i.e., only when $M \gg m$.

III. EXPERIMENTAL APPARATUS AND PROCEDURE

A simple laboratory experiment that serves to reinforce these rudimentary physical results has been developed. This experiment compares the simple and exact predictions of R, (12), (23), and (21), respectively, to an experimentally determined R. The apparatus, which was developed by Beatty and Chow[3] to measure the transverse vibrational frequency of rubber strings, is shown in Fig. 4; it consists of a bridge carriage to support the cord, a laser light source and photoelectric cell receiver whose output is fed into an electronic digital counter that records the average over several cycles of either the frequency or the period of the oscillation. However, the period could be measured more accurately. The laser beam provided a narrow, concentrated light beam that was focused normally to the midpoint of the string and directly in front of the photoelectrc cell aperture. The string was plucked at its midpoint, and its transverse motion in the plane normal to the beam interrupted the laser twice in each cycle. Therefore, the counter recorded half the actual period of the vibration.

Gum rubber, neoprene, and buna-N rubber materials were used. All of the strings were fabricated from standard 40 durometer rubber "O"-ring cord of nominal diameters 2.362 mm (0.093 in.) for the gum rubber and 2.616 mm (0.103 in) for the others. The manufacturer's standard dimensional tolerance was specified as ± 0.127 mm (± 0.005 in.) for all three varieties. Three samples of the buna-N and one sample of each of the other materials were tested. The samples were cut in nominal lengths of 50.8 cm (20 in.) from rolls of stock and weighed in order to determine their natural state mass density ρ_0. In order to test the same set of mass ratios γ for each string, using an assigned set of loads M, each string's test segment, that portion between the bridge supports, was to be of the same mass. A 10-in. segment of buna-N rubber cord was arbitrarily selected as the reference mass. Then, the gum rubber sample, which was of lower density, and the neoprene sample, which was of higher density, were marked off in the lengths l_0 shown in Table I. We note that for two measurements of the gum rubber sample, the test segment l_0 was reduced to 10 in. so that its maximum stretched length would not exceed the expanded bridge length, thus reducing the values of γ. Proper allowances were made in calculations.

The loads consisted of six small, axially symmetrical metal weights having an axial hole through which the string could be threaded. The hole was about half of the string's natural state diameter d_0, in order that, after threading the string through the hole while at high stretch, releasing it to a lower stretch or to its natural state had the effect of clamping the load tightly in place at the midpoint of the test segment. The masses of the loads, ranging from 0.38 to 13.81 g, were selected so as to provide a wide range of ratios γ. We also found that the average values of d_0 determined for several samples of the stock material did fall within the aforementioned tolerances; these are given in Table I.

Owing to relaxation effects the strings exhibited continual, though small, variation in tension. With the counter

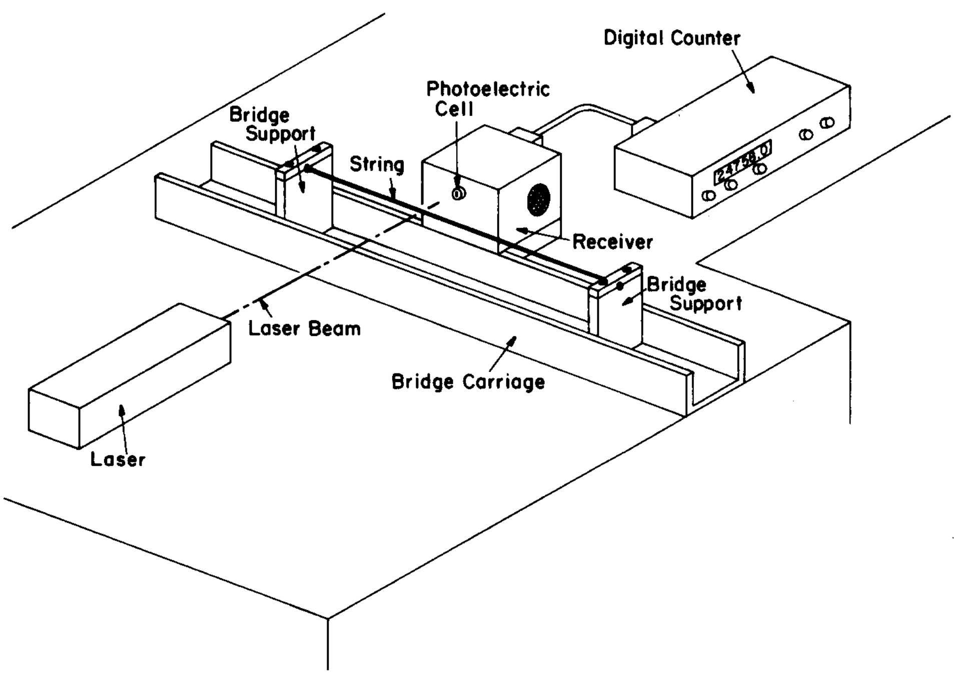

Fig. 4. Schematic of the experimental apparatus for measurement of transverse frequency of vibration of a rubber string.

Table I. Physical properties of the strings.

Material	d_0(mm)	ρ_0(kg/m)	l_0(cm, in.)	m (kg) (test segment)
Buna-N	2.602	5.689×10^{-3}	25.4, 10.0	1.445×10^{-3}
Natural gum	2.390	4.764×10^{-3}	30.3, 11.9	1.445×10^{-3}
neoprene	2.556	6.693×10^{-3}	21.6, 8.50	1.445×10^{-3}

averaging the period over 10 cycles, this led to small fluctations of the order $\pm 10^{-3}$– $\pm 10^{-4}$ sec in the period reading. In an effort to control variable tension effects, the periods for ω and Ω were read in quick succession and at the smallest possible amplitude of oscillation. The experiment proceeded as described below.

The ends of the test segment were clamped to movable bridge supports separated initially by distance l_0. The supports were then further separated symmetrically in the carriage to increase the stretch λ, which was measured to within ± 0.5 mm ($\pm 1/64$ in.) by a calibrated tape fastened to the bridge carriage. Six values of stretch covering a wide range of elongation were tested in increasing order. The unloaded string was taken to the lowest state of stretch, plucked at its midpoint, and the unloaded period measured. Then, the smallest load was quickly added at the center, the string plucked again, and the loaded period measured. After a brief pause, the loaded period was again measured, the load quickly removed, and the unloaded period measured. These four measurements were used to compute R for the given γ, λ, and modulus E (i.e., material). The load was then increased, while holding λ constant, and the process repeated. When all six loads had been tested, the stretch was increased and the entire procedure repeated. In this manner the frequency ratio R was determined using the same wide variety of γ's and λ 's for each of the three kinds of material studied.

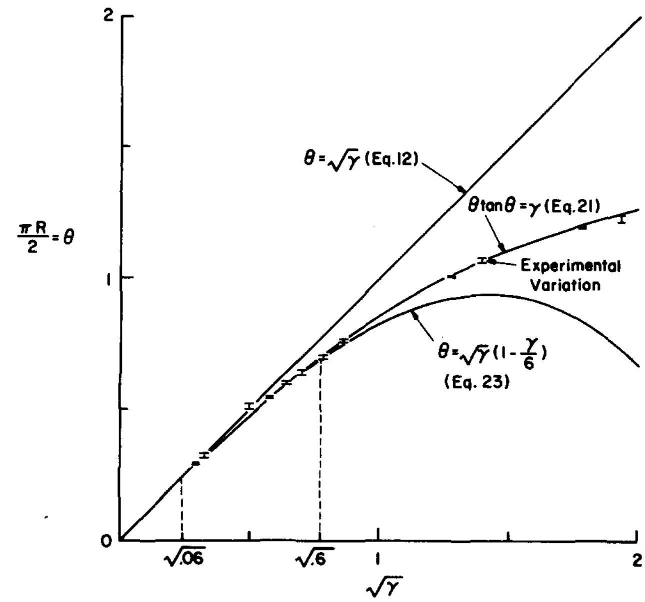

Fig. 6. Comparison of experimental and theoretical results for $\theta(\gamma)$ determined for a gum rubber string when the stretch is varied at each value of γ.

IV. SUMMARY AND DISCUSSION OF THE EXPERIMENTAL RESULTS

The test results are summarized in Figs. 5–8. The data obtained for the single neoprene and gum rubber samples are plotted in Figs. 5 and 6, respectively. The I symbols shown at various fixed values of γ denote the range of variation in the experimentally determined values of θ as the stretch was increased over its range from 1.20 to 3.70 in increments of 0.5. Similarly, the data for the values of θ averaged from the three tests on buna-N rubber cord, for the same stretches, is presented in Fig. 7. The graph in Fig. 8 shows the variation in the experimental values of θ when, for each stretch in the above range, the values of θ for all the

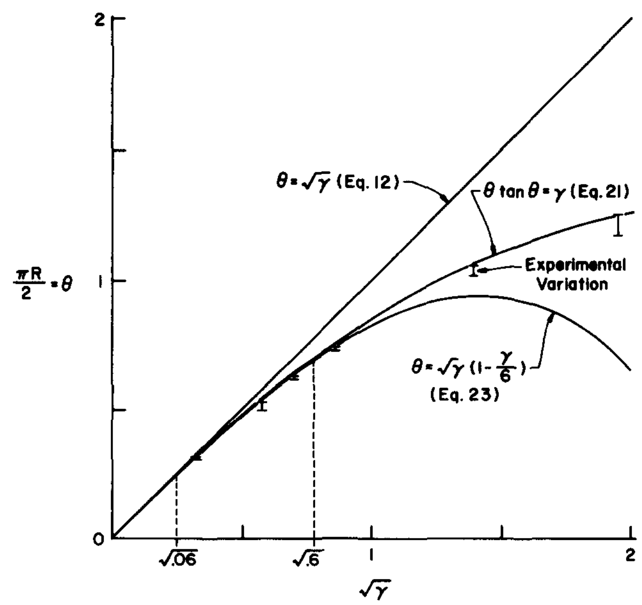

Fig. 5. Comparison of experimental and theoretical results for $\theta(\gamma)$ determined for a neoprene rubber cord when the stretch is varied at each value of γ.

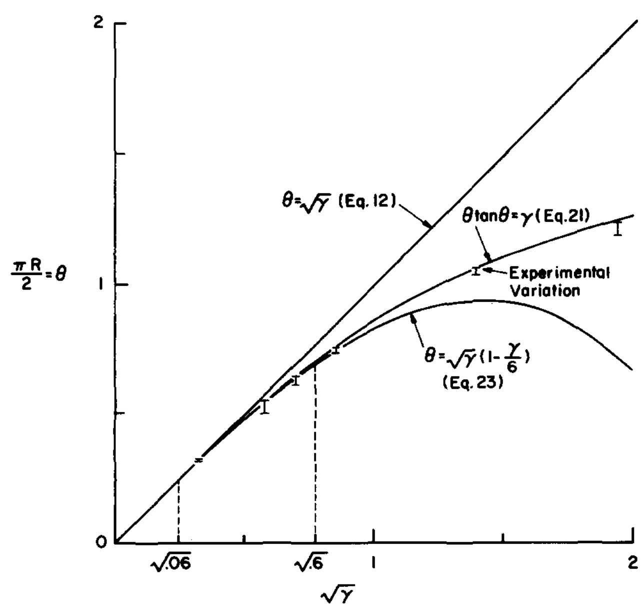

Fig. 7. Comparison of theoretical results with experimental values of θ averaged from three tests on Buna-N rubber cord as the stretch was varied over the same range at each value of γ.

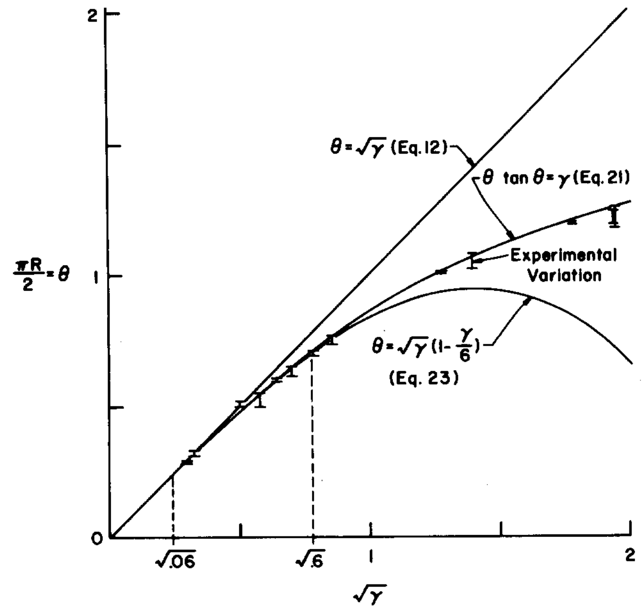

Fig. 8. Comparison of theoretical results with experimental values of θ for all materials averaged together over the same range of stretch at each value of γ.

materials are averaged together. Thus congruent with the theoretical expression derived here, the data confirms that the frequency ratio is independent of the amount of stretch and the nature of the material. Indeed, the results of the experiment correlate very well with the prediction of the exact relation (21). It is seen in Figs. 5–8 that the experimental values of R fall on or quite near the graph of the theoretical prediction over the whole range of γ's and λ 's tested and manifest no significant variation among the several materials used.

We observe some increase in the variation of R at the highest values of γ, at least in part due to experimental difficulties associated with the use of the small masses needed to produce large γ's. Irregularities in the small masses introduced slight rotational effects that tended to damp vibrations to a greater extent than for larger masses, making readings more erratic as well as decreasing the number of measurable cycles. We note, however, that irregularities in the shapes of the loads had some effect at all values of γ and that this effect has been neglected here.

For the lowest value of stretch ($\lambda = 1.20$), at which tension was least, vibrations were difficult to sustain; in fact, the buna-N strings would not vibrate in any measurable fashion at the smallest γ for this value of stretch. Deviations from the theory consistently decreased as the stretch was increased for the buna-N data set. The decrease was sharpest when the stretch was increased from 1.20 to 1.70 and slower thereafter. This behavior is consistent with the theoretical assumption of constant tension. In the gum rubber tests, which show exceptionally good agreement with (21), this trend occurred to a lesser extent; in the neoprene data it was not apparent.

For the gum rubber tests we recall that the string was shortened to accommodate two measurements at high stretch. We note that the data continued to follow the theory well, except for the two measurements at $\sqrt{\gamma} = 0.50$, shown in Fig. 6. These measurements differed fairly widely, probably due to some random effect involving the mass used for this γ.

For the buna-N and neoprene strings there existed some tendancy for the experimental R values to fall short of the theoretical values. This effect was smaller in the gum rubber tests, but in all tests was more pronounced at larger values of γ. The buna-N data indicates that this effect is largely the result of the aforementioned tension effects at low stretch. In the other data no cause seems evident. Of course, unknown effects due to damping are also present in all cases.

Excessive gravitational deflection prohibited tests in the region of very small γ values where the first approximation is valid to less than 1% error at $\gamma = 0.06$, but we note from Figs. 5–8 that the data fits the first approximation accurately at small values of $\gamma < 0.10$. On the other hand, the experiment shows excellent agreement with the second-order approximation over its full range of applicability and somewhat beyond the point of 1% error at $\gamma = 0.6$, roughly up to $\gamma = 0.75$, as shown in Figs. 5–8.

In sum, the results confirm that our theoretical equation (22), based on the constant tension assumption for small amplitude oscillations of identical loaded and unloaded rubber strings, is quite accurate for all materials tested and for all values of the stretch. Thus this experiment, extending over a wide range of mass ratios, confirms that the ratio of the fundamental transverse vibrational frequencies of identical loaded and unloaded rubber strings that can be subjected to finite stretch is a constant that depends solely on the ratio of their masses. It is our view that this example, which is a fragment extracted from a deeper research program in finite elasticity, constitutes an excellent laboratory exercise for students.

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