An Equilibrium Model of Irreversible Investment

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Abstract

This paper presents a general equilibrium model of a two-sector production economy with irreversible real investment. Irreversibility of investment is the most prominent feature of the productive sector. It restricts capital accumulation, affecting firms' investment decisions, which in turn determine properties of asset prices. Thus, this model provides a framework for connecting stock returns to firm characteristics that proxy for real economic activity. The primary focus of this paper is on the analysis of the equilibrium and the effects of irreversibility on investment.

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1. Introduction

Most asset pricing models focus on the demand side of the economy, making extremely simple assumptions about the supply side. For example, consider two of the most influential papers in this literature, Lucas (1978), and Cox, Ingersoll, and Ross (1985). Lucas assumes that the supply of risky assets in the economy is completely exogenous. Thus, the elasticity of supply is equal to zero and demand shocks are absorbed entirely by changes in asset prices. On the other hand, Cox, Ingersoll, and Ross assume the opposite extreme. In their model the supply of basic risky assets is perfectly elastic. As a result, demand shocks have no effect on the prices of these assets. In both cases the elasticity of supply is fixed, either at infinity or at zero.

The focus on the demand side of the economy proves to be fruitful by delivering tractable models. The obvious drawback is that such models do not lead to a realistic description of supply dynamics, limiting one's understanding of the interaction between real economic activity and prices of financial assets. To learn more about such interaction, the traditional paradigm must be augmented by incorporating economic activity of firms, such as their production and investment decisions.

In this paper, I develop a two-sector continuous-time general-equilibrium model of a production economy with irreversible investment. Irreversibility of investment is the most prominent feature of the productive sector. It restricts the process of capital accumulation, affecting firms' investment decisions, which in turn determine properties of asset prices. Thus, this model leads to a structural relation between stock returns and firm characteristics that proxy for real economic activity.

The main objective of this paper is to introduce and analyze a tractable equilibrium model with irreversibility, with a focus on the investment behavior. I study the impact of irreversible investment on the behavior of stock returns in Kogan (2000).

Extensive literature analyzes effects of irreversibility and adjustment costs on investment

activity. Earlier contributions to this field investigate implications of capital immobility under certainty. Examples of this line of research include Arrow (1968), Johansen (1967), Dasgupta (1969), Ryder (1969), Bose (1970), Floyd and Hynes (1979), Smith and Starnes (1979), LeRoy (1983), and others.

More recent literature has focused on the interaction between irreversibility and uncertainty. One avenue of the literature is concerned with optimal timing of irreversible investment projects, emphasizing the value of the option to delay investment that arises due to irreversibility.[1] Another strand of the literature includes models of incremental capital accumulation by a single firm facing an uncertain economic environment. In particular, some researchers focus on a competitive firm, assuming exogenous stochastic processes of the output price and factor prices. These include Majd and Pindyck (1987), Pindyck (1988), Bertola and Caballero (1994), Caballero (1991), Leahy (1993), Abel and Eberly (1994, 1997a) and Abel et al. (1996). Yet others focus on a monopolist facing an exogenously given stochastic sequence of demand curves. Models of this type are developed by Pindyck (1988), Bertola (1989), He and Pindyck (1992), Abel and Eberly (1994, 1995, 1996, 1997b), and Dixit and Pindyck (1994).

In order to characterize the impact of uncertainty and irreversibility on the investment behavior of a competitive firm, it is important to recognize that the output price is determined endogenously in equilibrium, as highlighted by Pindyck (1993). This leads to the third branch of the literature, encompassing equilibrium models of a competitive industry. Examples of such models are Lucas and Prescott (1971), Dixit (1989b, 1991, 1992), Caballero and Pindyck (1996), Leahy (1993), and Dixit and Pindyck (1994). Following Lucas

[1] This direction is explored by Henry (1974), Baldwin and Meyer (1979), Baldwin (1982), Brennan and Schwartz (1985), McDonald and Siegel (1986), Ingersoll and Ross (1987), Dixit (1989a,b, 1992) and others. Some of the models incorporate the process of firms' learning about the parameters of the model, e.g., Cukierman (1980), Bernanke (1983), and Caplin and Leahy (1994). Clearly, the very possibility of firms' learning over time creates an additional incentive to delay investment. In this paper, however, I completely ignore such informational problems, assuming that firms possess complete knowledge of their economic environment.

and Prescott (1971), most researchers determine the equilibrium allocations by maximizing the total social surplus. My analysis of the equilibrium is also based on this idea.

Sargent (1979) and Olson (1989) develop representative agent, one-sector general equilibrium growth models with irreversible investment. These models are designed to provide some insight into equilibrium dynamics of aggregate investment. However, as pointed out by Bertola and Caballero (1994), the empirical relevance of such models is questionable, since aggregate investment is not volatile enough to make aggregate irreversibility constraints binding. Instead, irreversibility is more likely important at the level of individual industries, which I capture using the two-sector structure of the economy.

With a few exceptions, models of irreversible investment cannot be solved in closed form and require complex numerical computations. A few known examples of models with closed-form solutions include Pindyck (1988), Dixit (1989a, 1991), He and Pindyck (1992), Abel and Eberly (1994, 1995, 1996, 1997a,b). In this paper, I use singular perturbation techniques to obtain accurate closed-form approximations to the exact solution of the model.

The paper is organized as follows. Section 2 formulates and analyzes the general-equilibrium model of the economy with irreversible investment. In particular, I reduce the problem to a central planner's problem, which I then solve both using perturbation methods and numerically. Section 3 is the conclusion.

2. Irreversible investment in general equilibrium

My focus on irreversibility as the main property of real investment is motivated by empirical evidence. In many (if not most) industries physical investment is to a large extent irreversible. Little value can be salvaged by selling off capital, since many production factors are industry-specific. This prompts companies to adjust their investment rules, drastically changing the capital accumulation process from what it would have been under perfectly reversible investment.

I model irreversibility at the level of one industry. Thus, the production side of the economy consists of two sectors, each using its own, sector-specific production factor (a *capital good*). One sector represents the industry under consideration, while the other sector represents the rest of the economy. The industry consists of a large number of essentially identical firms, facing the irreversibility constraint. For instance, the firms in the industry could be producing electricity (consumption good) using power plants (capital stock), or these could be semiconductor factories producing computer chips. The key feature of firms in the industry is that their capital stock is highly specialized and can only be used for production of the industry-specific consumption good. The rest of the economy is assumed to be perfectly reversible, to isolate the effects of irreversibility at the industry level. Specifically, while it is possible to transfer capital into the industry, the reverse process is assumed to be technologically infeasible. As a result, the capital stock of the industry cannot be maintained at the level that would be optimal if investment were perfectly reversible. Investment is infrequent: it takes place only when demand for the industry's assets (capital) is sufficiently high.

Alternatively, one can characterize the investment process using q. This concept was pioneered by Tobin (1969) and later refined by Abel (1979) and Hayashi (1982). Tobin defined q as the ratio of the market value of a firm to the replacement cost of its capital, which is now known as *average* q. However, incremental investment is determined by *marginal* q, defined as the marginal value of installed capital. To be precise, marginal q is defined as the market value of a marginal unit of capital installed in the firm relative to its replacement cost. Abel et al. (1996) provides further information. Marginal q coincides with Tobin's q only under certain conditions, which are satisfied by my model. Average q equals marginal q in my model, because of perfect competition, constant returns to scale and zero adjustment costs. Hayashi (1982) provides the original result in a deterministic setting while Abel and Eberly (1994) extend it to stochastic models. Investment is triggered when q reaches an

endogenously determined threshold. In particular, due to the absence of adjustment costs, firms find it profitable to invest whenever the market value of capital exceeds its replacement cost.

The rest of this section develops a general equilibrium model with irreversible investment. The equilibrium is constructed in two steps. First, I find the Pareto optimal allocation and study its properties. In particular, I characterize the optimal consumption/investment policy and establish technical conditions under which the problem is well-posed and under which endogenous variables in the economy follow a stationary process. Second, I prove that the Pareto optimum can be implemented as an outcome of a competitive equilibrium.

Section 2.1 states the assumptions about the information structure, technology, preferences, and financial markets in the economy and defines the competitive equilibrium. In section 2.2, I formulate and analyze the central planner's problem. In particular, I develop accurate closed-form approximations to the solution and compare them with the results of numerical simulations. Section 2.4 demonstrates that the Pareto optimal allocation can be supported as an outcome of a competitive equilibrium.

2.1. The economy and the competitive equilibrium

2.1.1. Information structure

I make standard technical assumptions about the information structure of the economy. I assume that there exists a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, supporting the Brownian motion W_t. $\mathcal{P}$ is the corresponding Wiener measure. The flow of information is described by a right-continuous increasing filtration $\mathcal{F}_t$, $t \in [0, \infty)$, $\mathcal{F}_t \subset \mathcal{F}$. Each $\mathcal{F}_t$, $t \in [0, \infty]$ is an augmentation of the sigma-field generated by the Brownian motion $\{W_s : s \in [0, t]\}$ and $\mathcal{F}_\infty \equiv \vee_{t \geq 0} \sigma \{W_s : s \in [0, t]\}$. Karatzas and Shreve, (1991, Sec. 2.7) define concepts related to Brownian filtrations.

2.1.2. Production technology

The productive side of the economy consists of two sectors. Sector one represents the bulk of the economy, excluding the industry being analyzed, which is modeled as sector two. There are two capital goods in the economy. Capital good of type one can be used for production within sector one and can be converted "one-to-one" into the capital good of type two or into perishable consumption good one. The second capital good is industry-specific and cannot be used for anything other than production of perishable consumption good two. The absence of storage is an important assumption. An inexpensive storage technology could mitigate the effects of irreversibility to some extent. This, however, depends on the precise combination of model parameters. Under certain conditions, storage is not optimal in equilibrium, even under zero storage cost. Thus, the first sector has a perfectly reversible technology, while the technology of the second sector exhibits irreversibility.

The production technology of the first sector has constant returns to scale. In particular, over a time-interval of length dt, it transforms x units of capital into $x + x \times (\alpha\, dt + \sigma\, dW_t)$ units, where dW_t is an increment of a standard Brownian motion. This is the same production technology as in Cox, Ingersoll and Ross (1985). Let K_{1t} denote the total amount of physical capital of type one. If, between time t and $t + dt$, $c_{1t}\, dt$ units of capital are converted into consumption good one and dI_t units are converted into capital good two, the total change in the capital stock K_{1t} is given by

$$dK_{1t} = (\alpha K_{1t} - c_{1t})\, dt + \sigma K_{1t}\, dW_t - dI_t. \tag{1}$$

Formally, I_t is a nonnegative, nondecreasing singular process.

The production technology of the second sector is somewhat different. In particular, physical capital cannot be directly converted into the consumption good two. Instead, it can be used to produce a flow of consumption good. This production process has constant returns

to scale: each unit of physical capital produces the same amount of consumption good. Thus, the total output of consumption good two is proportional to the stock of capital two. I assume that the productivity of capital two is constant over time, therefore $c_{2t} = XK_{2t}$, where parameter X controls the productivity of capital in the industry. To simplify the analysis, I further assume that $X = 1$. This does not lead to any loss of generality, given the properties of the model. Furthermore, I assume that the stock of physical capital depreciates exponentially at a constant rate $\delta > 0$. Thus, between t and $t + dt$, δK_{2t} units of capital are lost to depreciation. Since consumption good two cannot be converted back into physical capital, capital stock two can be increased only by converting some of the capital from type one into type two. Therefore, the total change in the capital stock between t and $t + dt$ is given by

$$dK_{2t} = -\delta K_{2t}\, dt + dI_t, \tag{2}$$

where dI_t is the amount of capital converted.

2.1.3. Firms

The technology of the first sector involves no production decisions. Capital is simply transformed over time using the constant-returns-to-scale production technology. Therefore, I do not explicitly identify firms in this sector. Effectively, households own the entire stock of the first capital good and invest directly into the production technology of the first sector.

The second sector is organized differently. It consists of a large number of competitive firms, which have identical technology and may differ only in size. In the aggregate, firms own the entire stock of capital good two, use it to produce consumption good two and sell their output at the spot market. The only decision that the firms must make is when and how much to invest. The investment process amounts to purchasing good one and converting it into capital good two. Since capital markets are dynamically complete, the exact form of financing of these firms is irrelevant and I assume for simplicity that they are financed

entirely by equity. Thus, consumers can get access to the production technology of the second sector only by purchasing the equity of firms in that sector.

Even though I do not model firms in the first sector, they might very well comprise the bulk of the economy. My focus is instead on firms in the second sector, which are affected by irreversibility. Thus, one should not be left with the impression that all firms in this model economy have irreversible production technology. I simply choose to focus on a particular industry exhibiting irreversibility, modeling the rest of the firms in the economy as the first sector.

Firms in the second sector make investment decisions to maximize their stock price, which is determined by the value of their output and investment expenses. To simplify notation, one can formally replace the entire industry by a single representative firm. Specifically, the representative firm solves the following problem:

$$\max_{\{I_t\}} \mathrm{E}_0 \left[\int_0^\infty \eta_{0,t} S_t K_{2t} dt - \int_0^\infty \eta_{0,t} dI_t \right], \tag{3}$$

subject to Eq. (2) and $I_{0-} = 0$, $dI_t \geq 0$. K_{2t} here denotes the capital stock owned by a firm and I_t is its investment strategy. Firms compute the present value of future cash flows using the stochastic discount factor $\eta_{t,s} > 0$, $t \leq s$. To ensure that firms hold rational expectations, $\eta_{t,s}$ has to be consistent with observed market prices, which is formalized by conditions in Eq. (12) through Eq. (14) below. I am looking for an equilibrium with dynamically complete markets, in which case there exists a unique stochastic discount factor (see Harrison and Pliska 1983).

2.1.4. Financial markets

At all points in time, there exists a spot market, where the consumption good two is traded against the numeraire good at the prevailing spot price S_t (all financial prices are expressed in units of consumption good one). Households purchase good two for consumption

at the spot market. They also have access to two long-lived financial assets. The first asset generates the cumulative return process identical to the constant-returns-to-scale production technology of the first sector:

$$\frac{dv_{1t}}{v_{1t}} = \alpha\, dt + \sigma\, dW_t, \tag{4}$$

where v_{1t} is the amount invested in this asset at time t. The second asset, the stock, is a claim on the total stream of cash flows generated by the second sector. I assume that at any point in time there is exactly one share of equity outstanding. Hence, each share generates a flow of dividends at rate $S_t K_{2t}^*$ minus investment expenses, which total I_t^* by time t. I let P_t denote the ex-dividend stock price. The third asset, the bond, earns an instantaneously riskless rate of interest r_t. Given that the investment process is singular, one could expect returns of financial assets to have a singular component. This turns out not to be the case in equilibrium. This can be verified directly for stock returns. In general, absence of arbitrage implies that returns on all financial assets must have identical singular components in equilibrium (see Karatzas, Lehoczky, and Schreve 1991 (Section 4)). Since the return process of the first long-lived asset is a regular diffusion process, this implies that none of the other assets have singular components in returns. In particular, the risk free rate r_t is well defined.

2.1.5. Households

The economy is populated by identical competitive households, who derive utility from the two consumption goods. In particular, I assume that the households have time-separable isoelastic preferences:

$$\mathrm{E}_0 \left[\int_0^\infty e^{-\rho t} U(c_{1t}, c_{2t}) dt \right], \tag{5}$$

where

$$U(c_1, c_2) = \begin{cases} \frac{1}{1-\gamma} c_1^{1-\gamma} + \frac{b}{1-\gamma} c_2^{1-\gamma}, & \gamma > 0, \quad \gamma \neq 1, \\ \ln(c_1) + b \ln(c_2), & \gamma = 1. \end{cases} \tag{6}$$

The utility function is separable across consumption goods. While somewhat restrictive, this assumption simplifies the analysis and provides a benchmark for more general studies. Moreover, the mathematical structure of the production technology and preferences in this economy could be given an alternative economic interpretation. The capital stock of the second sector could be viewed as a durable consumption good, while the flow of perishable consumption good two could be viewed as the flow of services from the durable good. The utility function of households is then separable between consumption of good one (c_1) and consumption of services (c_2). This interpretation is pursued by Mamaysky (2001), who also generalizes the model to allow for Cobb-Douglas preferences over the consumption of the perishable good and services. The entire population of households in the economy can be modeled as a single representative consumer. In equilibrium, the representative consumer maximizes her expected utility of consumption given by Eq. (5), subject to the nonnegativity constraint $c_{1t} \geq 0$, $c_{2t} \geq 0$, and the budget constraint

$$\begin{aligned} dV_t = &- (c_{1t} + S_t c_{2t})\, dt + v_{bt} r_t dt + \pi_{Pt} (S_t dt + dP_t) \\ &+ \alpha v_{1t} dt + \sigma v_{1t} dW_t, \end{aligned} \tag{7}$$

$$V_t = v_{bt} + \pi_{Pt} P_t + v_{1t}, \tag{8}$$

$$V_t \geq 0, \tag{9}$$

where V_t is the individual wealth process, v_{bt} is the amount of wealth invested in the bond and π_{Pt} is the number of shares of the stock held at time t. The nonnegative-wealth constraint given by Eq. (9) rules out arbitrage opportunities (see Dybvig and Huang 1989). To make sure that the wealth process is well defined by Eq. (7), I assume that both the consumption policy (c_{1t}, c_{2t}) and the portfolio policy $(v_{bt}, \pi_{Pt}, v_{1t})$ are progressively measurable processes,

satisfying standard integrability conditions:

$$\int_0^{\tau_n} c_{1t} + S_t c_{2t} + |v_{bt} r_t + \pi_{Pt} (S_t + \mu_{Pt} P_t) + \alpha v_{1t}| \, dt < \infty, \tag{10}$$

$$\int_0^{\tau_n} |\pi_{Pt}|^2 \, d \langle P \rangle_t + \int_0^{\tau_n} |\sigma v_{1t}|^2 \, dt < \infty \tag{11}$$

for a sequence of stopping times $\tau_n \nearrow \infty$, where $\mu_{Pt} P_t$ and $\langle P \rangle_t$ are respectively the drift and the quadratic variation processes of P_t.

2.1.6. Competitive equilibrium

I use the following definition of a competitive equilibrium.

Definition 1. *A competitive equilibrium with dynamically complete markets is a collection of stochastic processes K_{1t}^*, K_{2t}^*, c_{1t}^*, c_{2t}^*, I_t^*, v_{bt}^*, v_{1t}^*, π_{Pt}^*, S_t, r_t, P_t and $\eta_{t,s}$, such that (i) c_{1t}^*, c_{2t}^*, v_{bt}^*, v_{1t}^*, and π_{Pt}^* solve the representative household's optimization problem, given S_t, r_t, P_t and K_{2t}^*; (ii) I_t^* is the aggregate investment process solving Eq. (3), given S_t and $\eta_{t,s}$; (iii) capital stocks K_{1t}^* and K_{2t}^* solve Eqs. (1,2), given c_{1t}^* and I_t^* and the initial stocks of capital in the economy; (iv) markets clear: $c_{2t}^* = K_{2t}^*$, $v_{bt}^* = 0$, $v_{1t}^* = K_{1t}^*$, $\pi_{Pt}^* = 1$; (v) S_t, r_t, P_t, I_t^* and K_{2t}^* are such that $\eta_{t,s}$ is the unique stochastic discount factor satisfying*

$$-\lim_{\Delta t \searrow 0} \frac{E_t \left[\eta_{t,t+\Delta t} - 1 \right]}{\Delta t} = r_t, \tag{12}$$

$$E_t \left[\int_t^T \eta_{t,s} S_s K_{2s}^* ds - \int_t^T \eta_{t,s} dI_s^* + \eta_{t,T} P_T \right] = P_t, \tag{13}$$

$$E_t \left[\eta_{t,T} \exp \left(\left(\alpha - \frac{\sigma^2}{2} \right) (T - t) + \sigma (W_T - W_t) \right) \right] = 1 \tag{14}$$

for arbitrary t and T, such that $T > t$.

2.2. *The central planner's problem*

In this section, I formulate the central planner's problem (the problem of optimal allocation of resources subject to technological constraints) and analyze it both analytically and numerically. As I demonstrate below in Proposition 3, as in other models with perfect financial markets, the solution of the central planner's problem can be used to construct a competitive equilibrium.

2.2.1. *Formulation*

In addition to the constraints imposed by the investment and production technologies Eqs. (1, 2), feasible investment policies are restricted to be right-continuous with left limits, nonnegative and nondecreasing and feasible consumption policies are restricted to be nonnegative and integrable on any finite time interval. Both are further constrained by the requirement that the stocks of capital goods must remain nonnegative at all times. Thus, the central planner's problem takes the form

$$\max_{\{c_{1t},I_t\}} \mathrm{E}_0 \left[\int_0^\infty e^{-\rho t} U(c_{1t}, c_{2t}) dt \right], \tag{15}$$

subject to

$$dK_{1t} = (\alpha K_{1t} - c_{1t})dt + \sigma K_{1t} dW_t - dI_t, \tag{16}$$

$$dK_{2t} = -\delta K_{2t} dt + dI_t, \tag{17}$$

$$c_{2t} = X K_{2t}, \tag{18}$$

$$c_{1t} \geq 0, \quad \int_0^T |c_{1t} + c_{2t}| \, dt < \infty, \quad I_{0-} = 0, \tag{19}$$

$$dI_t \geq 0, \quad K_{1t} \geq 0, \quad K_{2t} \geq 0, \quad \forall t, T \geq 0. \tag{20}$$

For this problem to be well defined, certain restrictions must be imposed on the parameters of the model. The following proposition provides a set of sufficient conditions.

Proposition 1. *If*

$$\alpha(\gamma-1)+\rho>0, \quad \gamma<1, \tag{21}$$

$$\min\left(\alpha(\gamma-1)-\frac{\sigma^2}{2}\gamma(\gamma-1)+\rho, \rho-\delta(\gamma-1)\right)>0, \quad \gamma\geq 1, \tag{22}$$

the value function of the central planner's problem is finite.

2.2.2. Characterization of the optimal consumption/investment policy

In this section, I will focus on the case $\gamma \neq 1$. The analysis for the case of $\gamma = 1$ is very similar and can be found in Kogan (1999). The problem faced by the central planner is of singular control type (for background on singular control of diffusion processes, see Fleming and Soner (1993, Ch. 8)). Similar problems arise in analysis of portfolio decisions under transactions costs. A good example of a (highly technical) rigorous analysis of a very similar mathematical problem is Shreve and Soner (1994).

Shreve and Soner (1994) present an extensive analysis of a mathematically similar problem of portfolio optimization and consumption with transaction costs. In particular, they prove that the value function of their problem is a classical solution of the corresponding system of differential inequalities (which provides an infinitesimal representation of the dynamic programming principle for problems of this type) and use this fact to establish existence of the optimal consumption/investment policy. Their results can be applied with only minor modifications to the problem at hand. In particular, the special structure of my model leads to an important simplification of the analysis. The fact that the stock of physical capital is restricted to be nonnegative, combined with an infinite marginal utility of consumption at zero, implies that the second capital stock will remain positive at all times. The same result

follows for the first stock, given the irreversibility of investment. Thus, the second-order differential operator in the dynamic programming principle becomes nondegenerate after the problem is reduced to one state variable, which ensures that the value function is twice continuously differentiable everywhere. In contrast, the value function in Shreve and Soner (1994) can loose smoothness over the set of points at which the differential operator becomes degenerate. This forces them to provide additional arguments to handle such cases.

Here I summarize only the main properties of the solution, emphasizing the economic intuition behind it. Let $J(K_1, K_2)$ denote the value function corresponding to Eq. (15). The first step in the analysis is to provide a "differential" characterization of the value function. First, suppose that investment is strictly suboptimal in a small neighborhood of (K_1, K_2) ((K_1, K_2) belongs to the interior of the zero-investment region). This would be the case if the amount of capital installed in the second sector was relatively large. This assumption implies that a small positive amount of investment would reduce the value function, i.e., $J(K_1 - dI, K_2 + dI) < J(K_1, K_2)$. Put differently, a marginal unit of capital of the first type is more valuable to the planner than a marginal unit of capital of the second type: $J_{K_1} - J_{K_2} > 0$. According to the definition of the value function and the dynamic programming principle,

$$J(K_{1t}, K_{2t}) = \sup_{c \geq 0} \left\{ e^{-\rho dt} \mathrm{E}_t \left[J(K_{1,t+dt}, K_{2,t+dt}) \right] \right\}, \tag{23}$$

where $K_{1,t+dt}$ and $K_{2,t+dt}$ are the values of capital stocks at time $t+dt$, determined according to the evolution law Eqs. (1, 2) subject to $dI_t = 0$. In differential form, this implies that at (K_1, K_2) the value function and the optimal consumption policy must satisfy the standard Hamilton-Jacoby-Bellman equation $\rho J - \mathcal{L}J = 0$, where

$$\mathcal{L}J \equiv \sup_{c \geq 0} \left\{ \frac{1}{1-\gamma} c^{1-\gamma} + \frac{b}{1-\gamma} K_2^{1-\gamma} + (\alpha K_1 - c) J_{K_1} + (-\delta K_2) J_{K_2} + \frac{1}{2} \sigma^2 K_1^2 J_{K_1 K_1} \right\}. \tag{24}$$

Next, consider the situation in which the planner does find it optimal to invest, i.e., converting a small amount of capital from type one into type two increases the value function (or at least leaves it unchanged). In this case,

$$J(K_1 - dI, K_2 + dI) \geq J(K_1, K_2) \geq \sup_{c \geq 0} \left\{ e^{-\rho dt} \mathrm{E}_t \left[J(K_{1,t+dt}, K_{2,t+dt}) \right] \right\} \tag{25}$$

The first inequality in Eq. (25) states that nonnegative investment is optimal and implies that $J_{K_1} \leq J_{K_2}$. Furthermore, under the optimal investment policy strict inequality is impossible, since the planner is free to convert capital from type one into type two instantaneously and will do so until the marginal values of the two types of capital are equalized. Thus, the first inequality is always satisfied as an equality and $J_{K_1} = J_{K_2}$. The second inequality in Eq. (25) captures the fact that the planner is free to postpone investment by dt (therefore the last expression in Eq. (25) provides a lower bound on the value function), but finds this suboptimal compared to immediate investment. In differential terms, this condition implies that $\rho J - \mathcal{L}J \geq 0$.

Finally, when the planner is indifferent between investing and waiting, $J_{K_1} - J_{K_2} = \rho J - \mathcal{L}J = 0$. All of the conditions stated above can be summarized concisely as a single differential inequality

$$\min\left(\rho J - \mathcal{L}J, J_{K_1} - J_{K_2}\right) = 0. \tag{26}$$

2.3. Solution of the central planner's problem

The first part of this section establishes several qualitative properties of the solution of the central planner's problem and simplifies the differential characterization of the value function. In addition, it contains conditions on model parameters under which the value function is finite and key economic variables in the economy follow stationary stochastic processes. In the rest of this section I derive closed-form approximations to the value function and compare them to the numerical solution. These approximations turn out to be highly accurate for a

wide range of parameter values.

2.3.1. Properties of the solution and further characterization

Due to the isoelastic form of the utility function, $J(\beta K_1, \beta K_2) = \beta^{1-\gamma} J(K_1, K_2)$. This implies that the value function has a particularly simple functional form:

$$J(K_1, K_2) = \frac{1}{1-\gamma} K_1^{1-\gamma} j\left(\frac{K_2}{K_1}\right). \tag{27}$$

Thus, the state of the economy can be characterized by a single state variable, defined as the ratio of capital stocks: $\Omega = K_2/K_1$.

The value function J is increasing in K_1 and K_2 and concave: it inherits these properties from the utility function. Concavity of the value function implies that the zero-investment region $\{J_{K_1} - J_{K_2} > 0\}$ has the form $\{K_2 > K_2^*(K_1)\}$, or equivalently $\{\Omega > \Omega^*\}$. Therefore, investment is optimal only when the capital stock of the second sector is sufficiently low compared to the capital stock of the first sector. Moreover, the optimal amount of investment is such that the maintained ratio of the two capital stocks never falls below the threshold Ω^* (a formal characterization of the investment process is provided below). These qualitative features of the optimal investment problem are familiar from the optimal portfolio-consumption choice in presence of transactions costs (e.g., Davis and Norman 1990), in which case the relative value of stock and bond holdings is maintained within the optimally chosen range.

Formally, the value function can be shown to be a classical solution of Eq. (26), in particular, it is twice continuously differentiable everywhere. This translates into the well known "value matching" and "smooth pasting" (or "high contact") conditions at the boundary of

the zero-investment region:

$$\left(J_{K_1} - J_{K_2}\right)\big|_{\frac{K_2}{K_1}=\Omega^*} = 0, \tag{28}$$

$$\left(J_{K_1K_1} - J_{K_2K_1}\right)\big|_{\frac{K_2}{K_1}=\Omega^*} = 0, \tag{29}$$

$$\left(J_{K_2K_2} - J_{K_2K_1}\right)\big|_{\frac{K_2}{K_1}=\Omega^*} = 0. \tag{30}$$

Given (27), only two of these conditions are linearly independent. Dumas (1991) provides an extensive discussion of these optimality conditions.

The problem can be simplified further by a change of variables. Let the new independent variable be $\omega = \ln(\Omega)$, and define the new unknown function $f(\omega) \equiv j\left(K_2/K_1\right)$. For the case $\gamma \neq 1$, the new unknown function $f(\omega)$ satisfies

$$p_2 f'' + p_1 f' + p_0 f + \gamma\left(f - \frac{1}{1-\gamma}f'\right)^{1-1/\gamma} = -be^{(1-\gamma)\omega}, \tag{31}$$

where

$$p_2 = \frac{\sigma^2}{2}, \tag{32}$$

$$p_1 = -\alpha - \delta + \frac{2\gamma - 1}{2}\sigma^2, \tag{33}$$

$$p_0 = (1-\gamma)\alpha - \gamma(1-\gamma)\frac{\sigma^2}{2} - \rho, \tag{34}$$

inside the zero-investment region and the boundary conditions

$$f'(\omega^*)(1+\Omega^*) = f(\omega^*)\Omega^*(1-\gamma), \tag{35}$$

$$f''(\omega^*)(1+\Omega^*) = f'(\omega^*)(1+(1-\gamma)\Omega^*). \tag{36}$$

The optimal consumption policy is given by

$$\tilde{c}^* \equiv \frac{c_1^*}{K_1} = \left(f - \frac{1}{1-\gamma}f'\right)^{-\frac{1}{\gamma}} \tag{37}$$

To characterize the value function of the central planner's problem completely, one must specify its asymptotic behavior as $\omega \nearrow \infty$. My analysis here is heuristic. Formal justification is given in Appendix, proof of Lemma 4. As ω increases, the possibility of using good one for investment becomes less and less important and the value function is asymptotically the same as it would be under the additional constraint $I_t \equiv 0$ (i.e., if the two sectors were completely isolated from each other). The value function under this additional constraint is denoted by $J^{LB}(K_1, K_2)$ (see Appendix):

$$J^{LB}(K_1, K_2) = \frac{\lambda_1}{1-\gamma} K_1^{1-\gamma} + b\frac{\lambda_2}{1-\gamma} K_2^{1-\gamma}, \tag{38}$$

$$\lambda_1 = \left(\alpha\frac{\gamma-1}{\gamma} - \frac{\sigma^2}{2}(\gamma-1) + \frac{\rho}{\gamma}\right)^{-\gamma}, \tag{39}$$

$$\lambda_2 = \left(-\delta(\gamma-1)+\rho\right)^{-1}. \tag{40}$$

Thus, as $\omega \nearrow \infty$,

$$f(\omega) \approx \lambda_1 + b\lambda_2 \exp((1-\gamma)\omega), \quad \gamma < 1, \tag{41}$$

$$f(\omega) \approx \lambda_1, \quad \gamma > 1. \tag{42}$$

A prerequisite for an econometric analysis of "average" (unconditional) behavior of economic variables is that their distribution is stationary over time. Thus, it is desirable to state a simple condition for stationarity of the ratio of capital stocks and other key economic variables in the model. Below, I state a condition that is sufficient and "almost" necessary: except for a knife-edge case, any violation of this condition ensures that the ratio of capital stocks converges to infinity.

First, I characterize the optimal investment process I_t^* and the resulting dynamics of the ratio of capital stocks. Under the optimal choice of the consumption/investment policy, ω_t is a reflected diffusion process, restricted to the half-line $[\omega^*, \infty)$. According to the Itô's formula for semimartingales (e.g., Chung and Williams 1990, (Th. 5.1) or Karatzas and

Shreve 1991, (Th. 3.3)),

$$d\omega_t = \mu_\omega(\omega_t)\, dt - \sigma\, dW_t + dL_t, \quad \omega_t \geq \omega^*, \tag{43}$$

$$\mu_\omega(\omega) = -\alpha - \delta + \widetilde{c}^*_{1t}(\omega) + \frac{\sigma^2}{2}, \tag{44}$$

where $\widetilde{c}^*_{1t} \equiv c^*_{1t}/K_{1t}$ is a function of ω only (see Eq. (31)) and L_t is the "reflection" process, preventing ω_t from falling below ω^*. The process L_t can be characterized as the local time of ω_t at ω^*. Gihman and Skorohod (1972, Ch. 5) and Chung and Williams (1990, Ch.8) provide background on reflected diffusion processes. Moreover, by the same formula, $dL_t = (1+\Omega^*)\, K^{*-1}_{2t} dI^*_t$, where K^*_{2t} is the stock of capital good two under the optimal consumption/investment policy. Thus, one can reconstruct the optimal investment process I^*_t as

$$I^*_t = I^*_0 + (1+\Omega^*)^{-1} \int_0^t K^*_{2u} dL_u, \tag{45}$$

where I^*_0 is the initial investment necessary to bring ω into the region $[\omega^*, \infty)$: if $\omega_0 < \omega^*$, $I^*_0 = K_{10}(\Omega_0 - \Omega^*)(1+\Omega^*)^{-1}$, otherwise $I^*_0 = 0$. The investment process I_t is singular: investment takes place only when $\Omega_t = \Omega^*$ ($\omega_t = \omega^*$). Formally, $I^*_t = I^*_0 + \int_0^t 1_{\{\omega_s = \omega^*\}} dI^*_s$, where $1_{\{\cdot\}}$ denotes the indicator function.

Given the nature of the optimal investment process, stationarity of the model is achieved under the following condition.

Proposition 2. *If parameters of the model satisfy*

$$\alpha + \delta - \left(\frac{\gamma - 1}{\gamma}\alpha - \frac{\sigma^2}{2}(\gamma - 1) + \frac{\rho}{\gamma} \right) - \frac{\sigma^2}{2} > 0, \tag{46}$$

the ratio of the capital stocks Ω_t possesses the long-run stationary distribution. If a strict inequality opposite to Eq. (46) holds, Ω_t almost surely tends to infinity and there does not exist a long-run stationary distribution.

2.3.2. Approximate solution

The control problem (Eqs. (31), (35), (36), (41), (42)) cannot be solved in closed form. Instead, I derive an explicit approximate solution based on an asymptotic expansion. This approximate solution takes form of a partial sum of a particular power series in b (see below). It is designed to approximate the solution of the central planner's problem for sufficiently small values of b. In equilibrium, small values of b imply that the second sector is not too large compared to the rest of the economy. I illustrate the range of values of b over which the asymptotic expansion remains accurate by comparing the approximation with the "exact" numerical solution. Unlike traditional power series, e.g., Taylor series, asymptotic expansions often fail to converge with the addition of extra terms. Instead, such series are constructed so that the relative difference between their partial sums and the exact solution vanishes as b approaches zero. For many problems, just a small number of terms in an asymptotic expansion can provide an excellent approximation to the true solution. Hinch (1991) provides further details. The accuracy of the approximation can be judged by how closely the approximate expression satisfies the original equation and the boundary conditions. Judd (1996) discusses methods that can be used to assess the accuracy of approximate solutions. He argues that for finite values of all model parameters, practical diagnostics are required even when theoretical convergence can be established. In this particular application, I compare the asymptotic approximation with the "exact" numerical solution of the problem.

While popular in physical sciences, such mathematical techniques have seen only a handful of applications in finance, e.g., Hull and White (1987), Atkinson and Wilmott (1995), Bertsimas, Kogan and Lo (1997), Whalley and Wilmott (1997). Judd (1996) reviews a similar set of techniques and their applications in economics. Thus, another objective of this section is to illustrate the methodology that might be applicable to other models with similar mathematical structure.

Assume that $\gamma \neq 1$. Analysis for the case of $\gamma = 1$ is similar and can be found in Kogan (1999). The unknown function $f(\omega)$ satisfies Eq. (31), subject to the boundary conditions (Eqs. (35), (36), (41), (42)). First, I introduce a rescaled independent variable Ξ equal to $b^{-1/\gamma}\Omega$. In terms of this new state variable, the optimal investment threshold is given by Ξ^*. Let ξ be the natural logarithm of Ξ: $\xi = \omega - \ln(b)/\gamma$. To economize on notation, denote the value of $f(\omega)$ corresponding to a particular value of ξ by $f(\xi)$. The new unknown function $f(\xi)$ satisfies

$$p_2 f''(\xi) + p_1 f'(\xi) + p_0 f(\xi) + \gamma \left(f(\xi) - \frac{1}{1-\gamma} f'(\xi) \right)^{1-1/\gamma} = -b^{1/\gamma} e^{(1-\gamma)\xi}, \tag{47}$$

subject to the boundary conditions at Ξ^*:

$$f'(\xi^*)(1 + b^{1/\gamma}\Xi^*) = f(\xi^*) b^{1/\gamma} \Xi^* (1-\gamma), \tag{48}$$

$$f''(\xi^*)(1 + b^{1/\gamma}\Xi^*) = f'(\xi^*)(1 + (1-\gamma) b^{1/\gamma}\Xi^*). \tag{49}$$

I look for $f(\xi)$ and Ξ^* in the form of power series

$$f(\xi) = \lambda_1 + b^{1/\gamma} \left(\lambda_2 e^{(1-\gamma)\xi} + f_{(0)}(\xi) + \cdots b^{n/\gamma} f_{(n)}(\xi) + \cdots \right), \tag{50}$$

$$\Xi^* = \Xi_{(0)} + b^{1/\gamma}\Xi_{(1)} + \cdots b^{n/\gamma}\Xi_{(n)} + \cdots . \tag{51}$$

I then substitute these expansions into Eq. (47) and the boundary conditions (Eqs. (48), (49)), replacing the only nonlinear component $\left(f(\xi) - (1-\gamma)^{-1} f'(\xi)\right)^{1-1/\gamma}$ by its Taylor series, and group the terms in the resulting expressions according to the corresponding powers of $b^{1/\gamma}$. As a result, the elements of the expansion $\left(f_{(0)}(\xi), \Xi_{(0)}\right)$, $\left(f_{(1)}(\xi), \Xi_{(1)}\right)$, etc., can be computed sequentially, as solutions of simple linear problems. Without going into further details (see appendix), I present the resulting expressions only for the first two terms in the asymptotic expansion. Adding higher-order terms is straightforward and provides no

further intuition. Moreover, a comparison with the numerical solution below shows that even one- and two-term expansions are accurate for economically significant values of b.

The first-order terms:

$$f_{(0)}(\xi) = A_{(0)} \exp\left(\kappa(\xi - \xi^*)\right), \tag{52}$$

$$A_{(0)} = \frac{\lambda_2 \gamma(1-\gamma)}{\kappa(\kappa-1)} \left(\frac{\lambda_1}{\lambda_2} \frac{\kappa - 1}{\kappa - (1-\gamma)}\right)^{1-1/\gamma}, \tag{53}$$

$$\Xi_{(0)} = \left(\frac{\lambda_1}{\lambda_2} \frac{\kappa - 1}{\kappa - (1-\gamma)}\right)^{-1/\gamma}, \tag{54}$$

$$\kappa = \frac{-q_1 - \sqrt{{q_1}^2 - 4q_2 q_0}}{2q_2}, \tag{55}$$

where q_0, q_1 and q_2 are given by

$$q_2 = p_2, \tag{56}$$

$$q_1 = p_1 + \lambda_1^{-1/\gamma}, \tag{57}$$

$$q_0 = -\lambda_1^{-1/\gamma} < 0. \tag{58}$$

and λ_1 and λ_2 are given by Eqs. (39) and (40).

The second-order terms:

$$f_{(1)}(\xi) = A_{(1)} \exp\left(\kappa(\xi - \xi^*)\right) + C_{(1)} \exp\left(2\kappa(\xi - \xi^*)\right), \tag{59}$$

$$C_{(1)} = \frac{1}{2\gamma\left(\gamma - 1\right)} \lambda_1^{-1-1/\gamma} \frac{A_{(0)}^2 \left(1 - \gamma - \kappa\right)^2}{4\kappa^2 q_2 + 2\kappa q_1 + q_0}, \tag{60}$$

where $A_{(1)}$ and $\Xi_{(1)}$ are characterized by the following system of linear equations

$$\kappa A_{(1)} + \left(\lambda_2 \left(1-\gamma\right)^2 \Xi_{(0)}^{-\gamma} - \lambda_1 \left(1-\gamma\right)\right) \Xi_{(1)} = -\left(\left(\kappa + \gamma - 1\right) A_{(0)} \Xi_{(0)} + 2\kappa C_{(1)}\right), \tag{61}$$

$$\left(\kappa^2 - \kappa\right) A_{(1)} - \lambda_2 \gamma \left(1-\gamma\right)^2 \Xi_{(0)}^{-\gamma} \Xi_{(1)} = -\left(\kappa \left(\kappa + \gamma - 1\right) A_{(0)} \Xi_{(0)} + \left(4\kappa^2 - 2\kappa\right) C_{(1)}\right). \tag{62}$$

Higher-order terms in the expansion Eq. (50) can be computed explicitly as well.

One can use the asymptotic results to derive a sequence of approximations to the optimal consumption policy. For instance, the first-order approximation is

$$\tilde{c}_1^*(\xi) \approx \lambda_1^{-1/\gamma} - b^{1/\gamma} \frac{\lambda_1^{-1/\gamma-1}}{\gamma} A_{(0)} \left(1 - \frac{\kappa}{1-\gamma}\right) e^{\kappa(\xi-\xi^*)}. \tag{63}$$

It is an increasing function of ξ and it approaches $\lambda_1^{-1/\gamma}$ as ξ approaches infinity. When b is small, the optimal consumption policy is approximately constant and equal to the one in an identical economy without the second sector.

2.3.3. Numerical solution

I now calibrate the model and compute the value function and the optimal investment policy numerically by solving the system (Eqs. (31) through (36), (41), (42)). The numerical solution allows one to assess the accuracy of the analytical approximation and can also be used when model parameters are such that the asymptotic expansion is not sufficiently accurate.

I set the technological parameters of the first sector to $\alpha = 0.07$ and $\sigma = 0.17$. When the first sector represents the bulk of the economy, this choice of parameter values implies that the first two moments of stock market returns match their historical values (see Campbell, Lo, and MacKinlay 1997 (Table 8.1)). Following Cooley and Prescott (1995), I set the depreciation rate δ and the subjective time-discount factor ρ to 5%. Qualitatively, the behavior of the model economy is not sensitive to the exact choice of these parameters. I consider three different values of the relative risk aversion parameter: 0.5, 1 and 1.5 and two values of b: 0.1 and 0.2.

To evaluate the accuracy of the asymptotic expansion, I compare the first- and the second-order approximations to the value function and its derivative with the numerical results. I plot the relative approximation error, defined as the difference between the exact and the

asymptotic solution divided by the exact solution, in Figures 1, 2. I also tabulate the optimal investment threshold Ξ^* and the approximations to Ξ^* in Table 1.

Fig. 1 and Fig. 2 show that both the first- and the second-order approximations are highly accurate for $b = 0.1$. The approximation error for $f(\xi)$ is well under 0.5% for both the first- and the second-order expansions. For $f'(\xi)$, the first-order expansion leads to an approximation error of 3% or less, while the error of the second-order expansion is less than 1%. The quality of the approximation is lowest for $\gamma = 0.5$. In equilibrium, given the choice of parameter values, the industry accounts for at least 4% of physical capital in the economy for $\gamma = 1.5$ and $\gamma = 1$, and for at least 2.5% for $\gamma = 0.5$ (see Table 1). In fact, the corresponding long-run average ratio of the two capital stocks, Ω, is between 3% and 6%, depending on the risk aversion parameter. Thus, the asymptotic approximation is accurate even when the size of the industry is far from negligible. Figs. 3 and 4 present results of the "stress test" of the approximation. The approximation error for $f(\xi)$ is generally under 2% and 1% for the first- and the second-order expansions respectively. The quality of the first-order approximation to $f'(\xi)$ is quite poor for $\gamma = 0.5$, but the approximation error for $\gamma = 1.5$ and $\gamma = 1$ is under 5%. The second-order approximation is still close to the numerical solution for $\gamma = 1.5$ and $\gamma = 1$ and produces less than 5% error for $\gamma = 0.5$. Given the value of $b = 0.2$, the capital stock of the industry under consideration is at least 6.6% relative to that of the first sector. The average ratio of the two capital stocks is between 9% and 11%. Thus, the industry under consideration is relatively large and the "small-industry" approximation based on $b \approx 0$ starts to break down, at least when the asymptotic expansion includes only one term.

Table 1 compares the approximations to the investment threshold with the numerical solution. It shows that the second-order approximation is extremely accurate for $b = 0.1$ and even for $b = 0.2$, except for $\gamma = 0.5$. Thus, even a small number of terms in the asymptotic expansion can provide an accurate approximation to the solution of the problem at hand. The

first-order approximation is accurate for $b = 0.1$, but breaks down for $b = 0.2$, particularly for $\gamma = 0.5$. Judd (1996) advocates the use of approximate methods in economics and their implementation through computer algebra software. As the development of this section illustrates, certain singular control problems arising in financial economics can be efficiently solved using asymptotic approximation techniques. By automating symbolic computations, one can carry out asymptotic analysis of arbitrarily high order.

2.4. The Competitive equilibrium

The solution of the central planner's problem can be used to construct a competitive equilibrium in a decentralized production economy. Similar results on equilibrium implementation of Pareto optimal allocations have been developed in several papers such as Lucas and Prescott (1971), Prescott and Mehra (1980), and Brock (1982).

Proposition 3. *There exists a competitive equilibrium with dynamically complete markets, satisfying the Definition 1. Processes K^*_{1t}, K^*_{2t}, c^*_{1t}, c^*_{2t} and I^*_t are given by the solution of the central planner's problem in section 2.2. The optimal portfolio policy is given by $(v^*_{bt}, v^*_{1t}, \pi^*_{Pt}) = (0, K^*_{1t}, 1)$. The stochastic discount factor is defined by*

$$\eta_{t,s} = e^{-\rho(s-t)} \frac{U_{c_1}(c^*_{1s}, c^*_{2s})}{U_{c_1}(c^*_{1t}, c^*_{2t})}. \tag{64}$$

Prices of financial assets satisfy

$$S_t = \frac{U_{c_2}(c^*_{1t}, c^*_{2t})}{U_{c_1}(c^*_{1t}, c^*_{2t})}, \tag{65}$$

$$r_t = -\frac{E_t\left[d\left(e^{-\rho t} U_{c_1}(c^*_{1t}, c^*_{2t})\right)\right]}{e^{-\rho t} U_{c_1}(c^*_{1t}, c^*_{2t}) dt}, \tag{66}$$

$$P_t = \frac{J_{K_2}(K^*_{1t}, K^*_{2t})}{J_{K_1}(K^*_{1t}, K^*_{2t})} K^*_{2t}. \tag{67}$$

2.5. The dynamics of investment and the price of capital

In this section I study the properties of the equilibrium investment process and the dynamics of the price of capital in the industry.

The market value of capital in the second sector is the same as the aggregate value of firm equity in that sector. Thus, according to Eq. (67), the value of a unit of capital is given by the ratio J_{K_2}/J_{K_1}, which is equal to average (or marginal) q of firms in the industry (the replacement cost of capital in my model is identically equal to one). q reaches one when investment takes place, and is less than one otherwise.

Fig. 5 illustrates the dependence of q on the logarithm of the ratio of the capital stocks ω, for the set of parameter values considered in section 2.3. q is bounded between zero and one and is decreasing in the state variable. The dynamics of q can be characterized using Itô's rule for semimartingales:

$$dq_t = \mu_q(\omega_t)q_t dt + \sigma_q(\omega_t)q_t dW_t - q_t\left(\frac{q'(\omega_t)}{q_t(\omega_t)}\right)dL_t, \tag{68}$$

$$\mu_q(\omega) = \mu_\omega(\omega)\frac{q'(\omega)}{q(\omega)} + \frac{\sigma^2}{2}\frac{q''(\omega)}{q(\omega)}, \tag{69}$$

$$\sigma_q(\omega) = -\sigma\frac{q'(\omega)}{q(\omega)}, \tag{70}$$

where $\mu_\omega(\omega)$ is defined by Eq. (44). Due to the fact that the state variable ω_t is reflected at ω^*, $dL_t = 1_{\{\omega_t=\omega^*\}}dL_t$ and the singular component of the process q_t equals $q_t\left(q'(\omega^*)/q(\omega^*)\right)dL_t$. It is easy to see that this is in fact equal to zero, since $q'(\omega^*) = 0$. The optimality conditions in Eq. (28) through Eq. (30) imply that

$$\left(\frac{J_{K_2}}{J_{K_1}}\right)_{K_2} = 0$$

at $\omega = \omega^*$. The result follows from

$$q'(\omega) = K_2 \left(\frac{J_{K_2}}{J_{K_1}} \right)_{K_2}.$$

Thus, q_t has no singular component and follows a regular diffusion process.

Further characterization of the dynamics of the price of capital can be obtained using the asymptotic solution in Eqs. (52–55). According to the first-order asymptotic expansion,

$$q = \frac{f'\Omega^{-1}}{(1-\gamma)f - f'} = \frac{\lambda_2}{\lambda_1} e^{-\gamma\xi} + \frac{\kappa}{\lambda_1} \frac{A_{(0)}}{1-\gamma} e^{(\kappa-1)\xi - \kappa\xi^*} + O(b^{1/\gamma}). \tag{71}$$

In the limit of $b \to 0$, one can check that q is a monotonically decreasing function of its argument, which in turn implies that there is a one-to-one correspondence between q and the state variable (the ratio of capital stocks).

The state variable ξ_t follows a diffusion process

$$\begin{aligned} d\xi_t &= \left(-\alpha - \delta + \lambda_1^{-1/\gamma} + \frac{\sigma^2}{2} + O(b^{1/\gamma}) \right) dt - \sigma\, dW_t \\ &= \left(-\frac{\alpha}{\gamma} - \delta + \frac{\rho}{\gamma} + \sigma^2 \left(1 - \frac{\gamma}{2} \right) \right) dt - \sigma\, dW_t \end{aligned} \tag{72}$$

reflected at ξ^*. In the limit of $b \to 0$, Eq. (72) describes a reflected Brownian motion with drift. Thus, the difference $\xi_t - \xi^* = \omega_t - \omega^*$ has an exponential long-run stationary distribution with mean $0.5\sigma^2 \left(-\alpha/\gamma - \delta + \rho/\gamma + \sigma^2 (1 - \gamma/2)\right)^{-1}$ (see Harrison, 1990 (§3.6)). This implies that the logarithm of the ratio of the capital stocks across the sectors, $\log(K_{2t}/K_{1t}) = \omega_t$, is more variable over time when α and δ are relatively small or ρ is relatively large. This result is intuitive: investment prevents the ratio of the two capital stocks from falling below an endogenously determined level ω^*. The rest of the time, the evolution of the ratio is determined by technological parameters and the optimal consumption policy c_{1t}^*. When the second capital stock depreciates at a high rate δ, the ratio K_{2t}/K_{1t} will tend to remain close

to its lower bound (investment threshold). High growth rate of the logarithm of the first capital stock, $\alpha/\gamma - \rho/\gamma - \sigma^2(1-\gamma/2)$, has the same effect. The impact of uncertainty, captured by σ, is ambiguous. The precautionary saving motive, and therefore the effect of σ on the optimal consumption policy c_{1t}^*, and the drift of the state variable ξ_t, depends on γ. Matters are further complicated by the fact that σ affects both the volatility and the drift of the state variable.

For sufficiently small values of b, the optimal investment threshold can be approximated by

$$\Omega^* = b^{1/\gamma}\Xi_{(0)} + O\left(b^{2/\gamma}\right) = b^{1/\gamma}\left(\frac{\lambda_1}{\lambda_2}\frac{\kappa-1}{\kappa-(1-\gamma)}\right)^{-1/\gamma} + O\left(b^{2/\gamma}\right), \tag{73}$$

where λ_1, λ_2, and κ are given by Eq. (39),Eq. (40) and Eq. (55) respectively. Asymptotically, the investment threshold is a power function of the preference parameter b, with the exponent equal to the inverse of the risk aversion parameter. Moreover, one can relax the assumption of $X = 1$ (X is the productivity parameter of the second sector) by simply replacing b with $bX^{1-\gamma}$. Thus, asymptotically, Ω^* is also a power function of X with an exponent of $1 - 1/\gamma$.

Next, I perform a comparative statics analysis of the optimal investment policy by plotting the leading term in the asymptotic approximation to Ω^*, i.e., $\Xi_{(0)}$, as a function of each of the parameters while holding other parameters fixed at their calibrated values. Fig. 6 presents the results for $\gamma = 3/2$. $\Xi_{(0)}$ is increasing in α and ρ and decreasing in δ and σ. In contrast, Fig. 7 shows the opposite dependence on the technological parameters α and σ: $\Xi_{(0)}$ is decreasing in α and increasing in σ. The dependence of the investment strategy on δ and ρ appears to be robust with respect to the risk aversion parameter and is intuitive. Higher depreciation rate δ makes investment into the industry less attractive and reduces the minimum acceptable level of the ratio K_2/K_1. Higher values of ρ imply that households value current consumption higher relative to future consumption and are therefore reluctant to transfer resources into the industry, since such investment is irreversible and generates a consumption flow which is spread out over time. However, the role of technological pa-

rameters of the first sector, α and σ, turns out to be ambiguous and conditional on γ. In particular, an economy-wide increase in uncertainty, captured by an increase in σ, raises the minimum level of K_2/K_1 for $\gamma = 1/2$ and lowers it for $\gamma = 3/2$. Thus, the comparative statics analysis of the impact of uncertainty on the investment policy produces ambiguous results. This stands in contrast to findings of Pindyck (1993), who argues that an increase in uncertainty creates an incentive to delay investment in a competitive-industry model with irreversible investment. The difference is due to the fact that asset prices in my model are endogenous and are affected by changes in σ. In particular, the interest rate is negatively related to the aggregate uncertainty, $r_t = \alpha - \gamma\sigma^2 + O(b^{1/\gamma})$. Thus, while an increase in aggregate uncertainty has the direct effect of increasing the value of the "option to wait" and hence delaying investment, it also has an indirect effect of reducing interest rates, which makes returns on investment appear relatively attractive. Fig. 7 illustrates that for certain combinations of model parameters the indirect effect might dominate.

3. Conclusion

In this paper I have developed a general equilibrium model of a two-sector production economy with irreversible real investment. Irreversibility of investment is the most prominent feature of the productive sector. It restricts capital accumulation, affecting firms' investment decisions, which in turn determine properties of asset prices. The primary focus of this paper is on the analysis of the equilibrium and the effects of irreversibility on investment. I use asymptotic methods to obtain accurate closed-form approximations to the equilibrium and perform comparative statics analysis of the optimal investment policy. Unlike in standard partial-equilibrium models of irreversible investment, the link between aggregate uncertainty and investment in general equilibrium is ambiguous and depends on households' preferences.

Appendix. Proofs and technical results

A.1. Proof of Proposition 1

To demonstrate that the conditions of the proposition are sufficient, I construct a lower and an upper bound on the value function and show that the conditions of the proposition imply that both bounds are finite.

Consider the lower bound first. It is provided by the solution of the problem in which the two sectors are completely isolated from each other. Consider the problem for an isolated sector one:

$$\max_{\{c_t\}} \mathrm{E}_0 \left[\int_0^\infty e^{-\rho t} \frac{1}{1-\gamma} c_t^{1-\gamma} dt \right], \quad \gamma \neq 1, \tag{A.1}$$

subject to

$$dK_t = (\mu K_t - c_t)dt + \sigma K_t dW_t, \quad K_t \geq 0, \quad c_t \geq 0. \tag{A.2}$$

The value function, $J_1(K)$, satisfies the dynamic programming equation

$$\rho J_1 = \max_{c \geq 0} \left\{ \frac{1}{1-\gamma} c^{1-\gamma} + J_{1K}(\mu K - c) + \frac{1}{2} J_{1KK} \sigma^2 K^2 \right\}. \tag{A.3}$$

Here I solve the dynamic programming equations in closed form. To verify that the resulting solution is indeed the value function of the original optimization problem, one can use the standard verification theorem, e.g., Fleming and Soner (1993, Th. 9.1). The solution of Eq. (A.3) is given by

$$J_1(K) = \frac{\lambda_1(\mu, \sigma, \rho)}{1-\gamma} K^{1-\gamma}, \tag{A.4}$$

$$c_t^* = (\lambda_1)^{-1/\gamma} K_t, \tag{A.5}$$

$$\lambda_1(\mu, \sigma, \rho) = \left(\mu \frac{\gamma - 1}{\gamma} - \frac{\sigma^2}{2}(\gamma - 1) + \frac{\rho}{\gamma} \right)^{-\gamma}. \tag{A.6}$$

Constraints $K_t \geq 0$, $c_t \geq 0$ are clearly satisfied. For the case $\gamma = 1$, it is easy to check that the value function has the form $J_1(K) = \rho^{-1} \ln(K) + const$.

For an isolated sector two, one must compute

$$\mathrm{E}_0 \left[\int_0^\infty e^{-\rho t} \frac{b}{1-\gamma} K_t^{1-\gamma} dt \right], \quad \gamma \neq 1, \tag{A.7}$$

subject to $dK_t = \mu K_t dt$. Since $K_t = K_0 \exp\left(\left(\alpha - \sigma^2/2\right) t\right)$, the solution is given by

$$J_2(K) = b \frac{\lambda_2(\mu, \rho)}{1-\gamma} K^{1-\gamma}, \tag{A.8}$$

$$\lambda_2(\mu, \rho) = (\mu(\gamma - 1) + \rho)^{-1}. \tag{A.9}$$

Similarly, for $\gamma = 1$, $J_1(K) = b\rho^{-1} \ln(K) + const$.

Thus, the lower bound on the value function is given by

$$J^{LB}(K_1, K_2) = \frac{\lambda_1}{1-\gamma} K_1^{1-\gamma} + b \frac{\lambda_2}{1-\gamma} K_2^{1-\gamma}. \tag{A.10}$$

The lower bound is finite if and only if λ_1 and λ_2 are finite. Thus, the condition given by Eq. (22) is sufficient.

An upper bound on the value function can be obtained by adding the solutions of the following two problems:

$$\max_{\{c_{1t},I_t\}} \mathrm{E}_0 \left[\int_0^\infty e^{-\rho t} \frac{1}{1-\gamma} c_{1t}^{1-\gamma} dt \right], \tag{A.11}$$

and

$$\max_{\{c_{1t},I_t\}} \mathrm{E}_0 \left[\int_0^\infty e^{-\rho t} \frac{b}{1-\gamma} K_{2t}^{1-\gamma} dt \right], \tag{A.12}$$

both subject to the same constraints as the original problem. The idea behind these two problems is clear: one computes an upper bound by maximizing each of the components of the original objective function separately. Economically, each of these auxiliary problems corresponds to maximizing the utility of consumption of only one of the goods. The first problem, Eq. (A.11), is identical to Eq. (A.1). Instead of solving the second problem, we consider an identical problem without uncertainty:

$$\max_{\{c_{1t},I_t\}} \mathrm{E}_0 \left[\int_0^\infty e^{-\rho t} \frac{b}{1-\gamma} K_{2t}^{1-\gamma} dt \right], \tag{A.13}$$

subject to

$$dK_{1t} = \alpha K_{1t}\, dt - dI_t, \tag{A.14}$$

$$dK_{2t} = -\delta K_{2t}\, dt + dI_t, \tag{A.15}$$

$$dI_t \geq 0. \tag{A.16}$$

One can show that the value function of this problem dominates the value function of (A.12) by repeating the proof of Theorem 12.2 in Shreve and Soner (1994). Moreover, for the value function of Eq. (A.13) to be finite it is necessary that $\alpha(1-\gamma) - \rho < 0$. This condition is also sufficient, as we now demonstrate.

The value function of Eq. (A.13) can be characterized by the differential inequality

$$\min\left(\rho V - \frac{1}{1-\gamma} K_2^{1-\gamma} - \alpha K_1 V_{K_1} + \delta K_2 V_{K_2}, V_{K_1} - V_{K_2} \right) = 0. \tag{A.17}$$

For a similar deterministic problem, Shreve, Soner and Xu (1991) established that the value function is a continuously differentiable solution of the corresponding differential inequality.

First, assume that $\alpha + \delta\gamma - \rho \geq 0$ and consider $V(K_1, K_2)$ satisfying the differential equation

$$\rho V - \frac{1}{1-\gamma} K_2^{1-\gamma} - \alpha K_1 V_{K_1} + \delta K_2 V_{K_2} = 0. \tag{A.18}$$

Using homotheticity of the problem, we represent the solution as $V(K_1, K_2) = 1/(1-\gamma) K_1^{1-\gamma} v(\omega)$, where $\omega = \ln(K_2/K_1)$. The new unknown function $v(\omega)$ satisfies the differential equation

$$(\alpha + \delta) v' - (\alpha(1-\gamma) - \rho) v - e^{(1-\gamma)\omega} = 0. \tag{A.19}$$

The general solution of this equation is given by $v(\omega) = A\exp\left(\frac{\alpha(1-\gamma)-\rho}{\alpha+\delta}\omega\right) + \frac{1}{\rho+\delta(\gamma-1)}\exp\left((1-\gamma)\omega\right)$. The inequality $V_{K_1} - V_{K_2} \geq 0$ implies $v - \frac{1}{1-\gamma}\left(1-e^{-\omega}\right)v' \geq 0$, which is equivalent to

$$e^{-\gamma\omega}\left(A\left(1-\frac{\alpha(1-\gamma)-\rho}{\alpha+\delta}\frac{1+e^{-\omega}}{1-\gamma}\right)\exp\left(\frac{\alpha+\delta\gamma-\rho}{\alpha+\delta}\omega\right) - \frac{1}{\rho+\delta(\gamma-1)}\right) \geq 0. \tag{A.20}$$

Since $\alpha(1-\gamma)-\rho < 0$, there exists a value of A and $\underline{\omega}$ such that Eq. (A.20) holds for any $\omega \geq \underline{\omega}$. In this case the no-investment region in Eq. (A.13) is nonempty and the value function of the original problem, Eqs. (15–20), is bounded above by

$$J^{UB}(K_1, K_2) = \frac{\lambda_1}{1-\gamma}K_1^{1-\gamma} + b\frac{1}{1-\gamma}\left(\lambda_2 K_2^{1-\gamma} + K_1^{1-\gamma}A\exp\left(\frac{\alpha(1-\gamma)-\rho}{\alpha+\delta}\omega\right)\right) \tag{A.21}$$

for a particular choice of A.

Next, consider the case $\alpha + \delta\gamma - \rho - \rho < 0$. It is straightforward to verify that $v(\omega) = \frac{1}{\rho+\delta(\gamma-1)}\left(1+e^{\omega}\right)^{1-\gamma}$ satisfies Eq. (A.17), therefore the upper bound is given by

$$J^{UB}(K_1, K_2) = \frac{\lambda_1}{1-\gamma}K_1^{1-\gamma} + b\frac{\lambda_2}{1-\gamma}\left(K_1+K_2\right)^{1-\gamma}. \tag{A.22}$$

Regardless of the exact combination of parameters, as long as $\alpha(1-\gamma)-\rho < 0$, the upper bound on the value function satisfies

$$J^{UB}(K_1, K_2) - \frac{\lambda_1}{1-\gamma}K_1^{1-\gamma} \approx b\frac{\lambda_2}{1-\gamma}K_2^{1-\gamma}, \quad K_2 \nearrow \infty. \tag{A.23}$$

A.2. Proof of Proposition 2

The proof relies on the following two lemmas.

Lemma 1. *The optimal solution of the central planner's problem satisfies*

$$E_t\left[\int_t^{\infty} e^{-\rho(s-t)}\frac{U_{c_2}\left(c_{1s}^*, c_{2s}^*\right)}{U_{c_1}\left(c_{1t}^*, c_{2t}^*\right)}\frac{R_{2s}}{R_{2t}}ds\right] = \frac{J_{K_2}\left(K_{1t}^*, K_{2t}^*\right)}{J_{K_1}\left(K_{1t}^*, K_{2t}^*\right)}, \tag{A.24}$$

where

$$R_{2t} = \exp\left(-\left(\delta+\sigma^2/2\right)t\right). \tag{A.25}$$

Equivalently,

$$E_t\left[\int_t^{\infty} e^{-\rho(s-t)}U_{c_2}\left(c_{1s}^*, c_{2s}^*\right)\frac{R_{2s}}{R_{2t}}ds\right] = J_{K_2}\left(K_{1t}^*, K_{2t}^*\right). \tag{A.26}$$

Relation in Eq. (A.24) has an intuitive interpretation. It states that the relative value of a marginal unit of capital two, expressed in terms of the numeraire good (the shadow price of capital), equals the present value of the entire future output produced by the marginal unit, taking into account the equilibrium capital accumulation dynamics (the marginal q).

Proof. Since $U_{c_1}\left(c_{1s}^*, c_{2s}^*\right) = J_{K_1}(K_1^*, K_2^*)$, it is sufficient to establish Eq. (A.26).

Without loss of generality, assume that in Eq. (A.26) $t = 0$. Consider an economy with initial capital stocks (K_{10}, K_{20}), $K_{20}/K_{10} \geq \Omega^*$. $\tilde{c}_{1t}^*$ and I_t^* are stochastic processes arising as a result of applying the optimal consumption/investment policy. Let (K_{1t}^*, K_{2t}^*) be the resulting process for the capital stocks. Consider now another economy with the initial condition $(K_{10}, K_{20}+\Delta)$.

Implement the consumption/investment policy defined by the pair of stochastic processes $(\tilde{c}_{1t}^*, I_t^*)$ (this amounts to "setting aside" Δ units of capital good two and treating the remaining capital stocks as if Δ did not exist). Let the corresponding indirect utility function be $\tilde{J}$. By construction,

$$\tilde{J} = \mathrm{E}_0 \left[\int_0^\infty e^{-\rho t} \left(\frac{1}{1-\gamma} (\tilde{c}_{1t}^* K_{1t}^*)^{1-\gamma} + \frac{b}{1-\gamma} (K_{2t}^* + \Delta R_{2t})^{1-\gamma} \right) dt \right]. \tag{A.27}$$

Since such a consumption/investment policy is suboptimal, $\tilde{J} \leq J(K_{10}, K_{20} + \Delta)$. Thus,

$$\frac{J(K_{10}, K_{20} + \Delta) - J(K_{10}, K_{20})}{\Delta} \geq \frac{\tilde{J} - J(K_{10}, K_{20})}{\Delta}.$$

One concludes, by taking the limit of $\Delta \searrow 0$ and using the dominated convergence theorem to pass the limit under the integral sign, that

$$J_{K_2}(K_{10}, K_{20}) \geq \mathrm{E}_0 \left[\int_0^\infty e^{-\rho t} b (K_{2t}^*)^{-\gamma} R_{2t} dt \right]. \tag{A.28}$$

Next, repeat the above argument for the economy with initial conditions $(K_{10}, K_{20} - \Delta)$, $\Delta < K_{20}$. Implementing $(\tilde{c}_{1t}^*, A_t^*)$ for this economy leads to the indirect utility function

$$\tilde{J} = \mathrm{E}_0 \left[\int_0^\infty e^{-\rho t} \left(\frac{1}{1-\gamma} (\tilde{c}_{1t}^* K_{1t}^*)^{1-\gamma} + \frac{b}{1-\gamma} (K_{2t}^* - \Delta R_{2t})^{1-\gamma} \right) dt \right]. \tag{A.29}$$

Note that $K_{2t}^* > \Delta K_{2s}^* / K_{2t}^*$, due to $\Delta < K_{20}$. Suboptimality of the constructed strategy implies $\tilde{J} \leq J(K_{10}, K_{20} - \Delta)$. Therefore,

$$-\frac{J(K_{10}, K_{20} - \Delta) - J(K_{10}, K_{20})}{\Delta} \leq -\frac{\tilde{J} - J(K_{10}, K_{20})}{\Delta} \tag{A.30}$$

and the limit of $\Delta \searrow 0$ yields

$$J_{K_2}(K_{10}, K_{20}) \leq \mathrm{E}_0 \left[\int_0^\infty e^{-\rho t} b (K_{2t}^*)^{-\gamma} R_{2t} dt \right]. \tag{A.31}$$

Inequalities in Eq. (A.28) and Eq. (A.31) imply

$$J_{K_2}(K_{10}, K_{20}) = \mathrm{E}_0 \left[\int_0^\infty e^{-\rho t} b (K_{2t}^*)^{-\gamma} R_{2t} dt \right], \tag{A.32}$$

which completes the proof.

Lemma 2. *The value function $J(K_1, K_2)$ satisfies*

$$J_{K_1 K_2} \leq 0, \quad \forall K_1 > 0, K_2 > 0. \tag{A.33}$$

The inequality in Eq. (A.33) can be justified intuitively. Note that the marginal unit of the numeraire good can be used in two ways: either for consumption, or for investment. Accordingly, its contribution to the indirect utility function can be decomposed into a sum of the discounted present value of the marginal utility of consumption and the value of the imbedded "investment option." While the utility derived from consumption does not depend on the size of the capital stock K_2, the value of the investment option does. Specifically, it decreases with K_2, since so does

the marginal indirect utility of K_2. This explains why the marginal value of the numeraire good decreases with K_2.

Proof. According to Lemma 1,

$$J_{K_2}(K_{1t}, K_{2t}) = \mathrm{E}_t \left[\int_t^\infty e^{-\rho(s-t)} b(K_{2s}^*)^{-\gamma} \frac{R_{2s}}{R_{2t}} dt \right]. \tag{A.34}$$

I start by characterizing the dynamics of the capital stock K_2^*, resulting from the optimal investment policy I_t^*. According to Eq. (2) and Eq. (45),

$$K_{2s}^* = K_{2t}^* \frac{R_{2s}}{R_{2t}} \exp\left(\frac{1}{1+\Omega^*} (L_s - L_t) \right), \quad t < s, \tag{A.35}$$

where R_{2t} is defined by Eq. (A.25). Then

$$J_{K_2}(K_{1t}, K_{2t}) = \mathrm{E}_t \left[\int_t^\infty e^{-\rho(s-t)} b(K_{2t})^{-\gamma} \left(\frac{R_{2s}}{R_{2t}} \right)^{1-\gamma} \exp\left(\frac{-\gamma}{1+\Omega^*} (L_s - L_t) \right) dt \right]. \tag{A.36}$$

It is therefore sufficient to demonstrate that $L_s - L_t$ is a nondecreasing function of the initial condition K_{1t}. Consider a pair of initial conditions (K_{1t}', K_{2t}) and (K_{1t}'', K_{2t}), $K_{1t}' < K_{1t}''$. Denote the corresponding reflection processes L_s' and L_s''. I will show that $L_s' \leq L_s''$ almost surely.

To this end, define ω_t' and ω_t'' as two solutions of Eq. (43) with initial conditions $\ln(K_{20}/K_{10}')$ and $\ln(K_{20}/K_{10}'')$ respectively. Suppose there exists a finite moment of time s at which $\omega_s' > \omega_s''$. Since $\omega_0' < \omega_0''$ and both solutions have almost surely continuous paths, there must exist $\tau \in (t, s)$, such that $\omega_\tau' = \omega_\tau''$. However, due to uniqueness of the solution of (43) (see Gihman and Skorohod (1972, §23, Th. 1)), this implies that $\omega_u' = \omega_u''$, for any $u > \tau$. I conclude that $\omega_s' \leq \omega_s''$ almost surely. Given the characterization of the reflection processes L_t' and L_t'' as local times, this last inequality implies that $L_t' \leq L_t''$ almost surely.

Lemma 2 provides an important insight into the behavior of the optimal consumption policy. Since $\tilde{c}_1^* = (K_1)^{-1} (J_{K_1})^{-1/\gamma}$, it is clear that $(\tilde{c}_1^*)_{K_2} \geq 0$. This should not be surprising by now: when deciding how much to allocate for consumption, the central planner trades off the direct benefit of consumption (the marginal utility), which is independent of K_2, against the indirect benefit of adding a marginal unit of the numeraire good to K_1, which has been shown to be a decreasing function of K_2. This results in the following lemma.

Lemma 3. *The optimal consumption policy $\tilde{c}_1^*(\omega)$ is a nondecreasing function of its argument: $\tilde{c}_1^{*\prime}(\omega) \geq 0$.*

Being a nondecreasing function of its argument, $\widetilde{c}_1^*(\omega)$ possesses a limit as ω approaches infinity. As suggested by Eqs. (41, 42), this limit is finite and equals $\lambda_1^{-1/\gamma}$ – the optimal consumption policy in the economy with completely isolated sectors. This result is formalized in the following lemma.

Lemma 4. *As the ratio of capital stocks increases, the optimal consumption policy $\widetilde{c}_1^*(\omega)$ approaches a finite limit:*

$$\lim_{\omega \to -\infty} \widetilde{c}_1^*(\omega) = \lambda_1^{-1/\gamma} = \frac{\gamma - 1}{\gamma} \alpha - \sigma^2 \frac{\gamma - 1}{2} + \frac{\rho}{\gamma}. \tag{A.37}$$

Proof. To show that $\lim_{\omega \to \infty} \tilde{c}^*(\omega) = \lambda_1^{-1/\gamma}$, it is sufficient to prove that

$$\lim_{K_2 \to \infty} J_{K_1}(K_1, K_2) = \lambda_1 K_1^{-\gamma}. \tag{A.38}$$

Since $J^{LB}(K_1, K_2) \leq J(K_1, K_2) \leq J^{UB}(K_1, K_2)$, where $J^{LB}(K_1, K_2)$ and $J^{UB}(K_1, K_2)$ are given by Eq. (A.10) and Eq. (A.23) respectively, one concludes that

$$\lim_{K_2 \to \infty} J(K_1, K_2) - \frac{b\lambda_2}{1-\gamma} K_2^{1-\gamma} = \frac{\lambda_1}{1-\gamma} K_1^{1-\gamma}, \tag{A.39}$$

convergence being uniform of compact subsets of $(0, \infty)$. This verifies Eqs. (41,42). Combined with the fact that J_{K_1} is a nonincreasing function of K_2 (see Lemma 2), this allows one to prove the statement of the lemma.

Consider an arbitrary value of K_1 and $h \in (0, K_1)$. Monotonicity of J_{K_1} and the mean value theorem imply

$$\frac{J(K_1 + h, K_2) - J(K_1, K_2)}{h} \leq J_{K_1}(K_1, K_2) \leq \frac{J(K_1, K_2) - J(K_1 - h, K_2)}{h}, \quad \forall K_2. \tag{A.40}$$

Combined with Eq. (A.39), this means that

$$\underline{\lim}_{K_2 \to \infty} J_{K_1}(K_1, K_2) \geq \frac{\lambda_1}{1-\gamma} \frac{(K_1 + h)^{1-\gamma} - K_1^{1-\gamma}}{h}, \tag{A.41}$$

$$\overline{\lim}_{K_2 \to \infty} J_{K_1}(K_1, K_2) \leq \frac{\lambda_1}{1-\gamma} \frac{K_1^{1-\gamma} - (K_1 - h)^{1-\gamma}}{h}. \tag{A.42}$$

The fact that this holds regardless of h guarantees that, as K_2 approaches infinity, $J_{K_1}(K_1, K_2)$ converges and the limit equals $\lambda_1 K_1^{-\gamma}$.

A brief discussion will clarify the intuition behind the main result. If parameters of the model satisfy Eq. (46), the drift of ω_t is strictly negative everywhere. By Bellman-Gronwall inequality (see Gihman and Skorohod 1972 (§16, Lemma 4)), this implies that ω_t is bounded from above by the process $\omega^* + (\alpha + \delta - \lambda_1^{-1/\gamma} - \sigma^2/2)t + \sigma W_t$, reflected at ω^*, which is equivalent in law to a reflected Brownian motion with the drift coefficient $\alpha + \delta - \lambda_1^{-1/\gamma} - \sigma^2/2$, the diffusion coefficient σ, and the initial condition ω_0. This process possesses the long-run stationary distribution, characterized by the exponential density (see Karlin and Taylor 1981). Thus, ω_t has zero probability of reaching infinity. Similarly, ω_t is bounded from below by a reflected Brownian motion with the drift $\alpha + \delta - \widetilde{c}_1^*(\omega^*) - \sigma^2/2$, therefore ω^* is not an absorbing boundary. Both the upper and the lower bounds on ω_t have long-run stationary distributions and so does ω_t.

If the inequality opposite to Eq. (46) holds, the drift of the process ω_t is uniformly positive for all values of ω exceeding a certain fixed value, therefore the ratio of capital stocks has a positive probability of reaching infinity (see Karlin and Taylor 1981). In the case under consideration an even stronger result holds, as stated in the proposition.

To prove the first statement of the proposition, define a new process $\zeta = \omega - \omega^*$. This process is a diffusion on $[0, \infty)$ reflected at zero and is equivalent in law to the solution of

$$d\zeta_t = \mu_\zeta dt + \sigma_\zeta dW_t, \quad \zeta_t \geq 0, \tag{A.43}$$

$$\mu_\zeta = \left(-\alpha - \delta + \widetilde{c}_{1t}^*(\omega_t) + \frac{\sigma^2}{2}\right), \tag{A.44}$$

$$\sigma_\zeta = \sigma. \tag{A.45}$$

Following Gihman and Skorohod (1972, §23), I introduce a function $u(\cdot)$ defined on $[0, \infty)$, decreasing, possessing bounded continuous derivatives $u'(\cdot)$, $u''(\cdot)$ and satisfying

$$u(0) = 0, \quad \mu_\zeta u'(0_+) + \frac{1}{2}\sigma_\zeta^2 u''(0_+) = 0, \quad \lim_{x \to \infty} u'(x) = 1, \quad \lim_{x \to \infty} u''(x) = 0. \tag{A.46}$$

Such a function can be constructed by "pasting" together a linear and an exponential function and smoothing the resulting function around the pasting point. Then $\eta_t \equiv u(\zeta_t)$ is itself a diffusion process reflected at zero. η_t satisfies

$$d\eta_t = \mu_\eta(\eta_t)dt + \sigma_\eta(\eta_t)dW_t \tag{A.47}$$

on the interval $(0, \infty)$, where

$$\mu_\eta(\eta) = \left(-\alpha - \delta + \widetilde{c}^*_{1t}(\omega^* + u^{-1}(\eta)) + \frac{\sigma^2}{2}\right) u'(u^{-1}(\eta)) + \frac{1}{2}\sigma_\zeta^2 u''(u^{-1}(\eta)), \tag{A.48}$$

$$\sigma_\eta(\eta) = \sigma_\zeta u'(u^{-1}(\eta)). \tag{A.49}$$

Thus, $\mu_\eta(\cdot)$ is bounded and it approaches $-\alpha - \delta + \lambda_1^{-1/\gamma} + \sigma^2/2$ as η approaches infinity. $\sigma_\eta(\cdot)$ is also bounded and approaches σ_ζ as η approaches infinity. Next, extend the functions $\mu_\eta(\cdot)$ and $\sigma_\eta(\cdot)$ to the whole real line by means of

$$\mu^*_\eta(x) = \mu_\eta(x), \quad x > 0, \tag{A.50}$$

$$\mu^*_\eta(x) = -\mu_\eta(x), \quad x \leq 0, \tag{A.51}$$

$$\sigma^*_\eta(x) = \sigma_\eta(-x), \quad x > 0, \tag{A.52}$$

$$\sigma^*_\eta(x) = \sigma_\eta(x), \quad x \leq 0. \tag{A.53}$$

Define η^*_t as a solution of

$$d\eta^*_t = \mu^*_\eta(\eta^*_t)dt + \sigma^*_\eta(\eta^*_t)dW^*_{1t.} \tag{A.54}$$

According to Gihman and Skorohod (1972, §23), the process $|\eta^*_t|$ is equivalent in law to η_t. Thus, given the one-to-one correspondence between η_t and ω_t, it suffices to establish the existence of the long-run stationary distribution of η^*_t. To do this, define a new function

$$\bar{\sigma}(x) \equiv \sigma^*_\eta(x) \exp\left(-\int_0^x \frac{2\mu^*_\eta(y)}{\sigma^{*2}_\eta(y)} dy\right). \tag{A.55}$$

According to Gihman and Skorohod (1972, §18, Th. 3), the process η^*_t possesses the long-run distribution if

$$\int_{-\infty}^{\infty} \frac{1}{\bar{\sigma}^2(y)} dy < \infty. \tag{A.56}$$

This last condition is clearly satisfied in view of the properties of the functions $\mu_\eta(\cdot)$ and $\sigma_\eta(\cdot)$ established above.

To prove the second statement, assume that $\text{Prob}\{\lim_{t\to\infty} \zeta_t = \infty\} < 1$. Then, there exists $M > 0$, such that for any $M_1 > M$,

$$\text{Prob}\left\{\exists\ \{t_k, k = 1, 2, \ldots\} : \lim_{k\to\infty} t_k = \infty, \zeta_{t_k} \leq M_1\right\} > 0. \tag{A.57}$$

To uncover a contradiction, define $M_1 > M$, such that $\mu_\zeta > 0$ for $\zeta \geq M_1$. Fix $M_2 > M_1$. Since the process ζ_t is continuous and bounded below by a reflected Brownian motion,

$$\text{Prob}\left\{\exists\ t \in (t_k, \infty) : \zeta_t = M_2 | \zeta_{t_k} \leq M_1\right\} = 1. \tag{A.58}$$

According to a well-known property of the Brownian motion with drift (e.g., Harrison (1990, §8), Karatzas and Shreve (1991, pp.196–197)),

$$\text{Prob}\left\{\exists\ s \in (t,\infty) : \zeta_s \leq M_1 | \zeta_t = M_2\right\} = \varepsilon < 1, \tag{A.59}$$

where ε is a function of M_1, M_2 and model parameters and does not depend on t. Thus, one concludes that

$$\text{Prob}\left\{\exists\ \{(t,s)_l, l = 1,2,\ldots\} : t_l < s_l < t_{l+1}, \zeta_{t_l} = M_2, \zeta_{s_l} \leq M_1\right\} = \lim_{l\to\infty} \varepsilon^l > 0, \tag{A.60}$$

which is a clear contradiction.

A.3. Proof of Proposition 3

One has to verify that the processes defined in Proposition 3 satisfy the Definition 1. I check each of the statements of the definition separately.

(i) Suppose that the household can attain a higher level of expected utility with another feasible consumption plan $(\tilde{c}_{1t}, \tilde{c}_{2t})$:

$$\text{E}_0\left[\int_0^\infty e^{-\rho t}\left(U\left(c_{1t}^*, c_{2t}^*\right) - U\left(\tilde{c}_{1t}, \tilde{c}_{2t}\right)\right) dt\right] < 0. \tag{A.61}$$

By concavity of the utility function,

$$U\left(c_1^*, c_2^*\right) - U\left(\tilde{c}_1, \tilde{c}_2\right) \geq U_{c_1}\left(c_1^*, c_2^*\right)\left(c_1^* - \tilde{c}_1\right) + U_{c_2}\left(c_1^*, c_2^*\right)\left(c_2^* - \tilde{c}_2\right), \tag{A.62}$$

and therefore

$$\text{E}_0\left[\int_0^\infty \eta_{0,t}\left(\left(c_{1t}^* - \tilde{c}_{1t}\right) + S_t\left(c_{2t}^* - \tilde{c}_{2t}\right)\right) dt\right] < 0. \tag{A.63}$$

As I will show in (ii), $\text{E}_0\left[\int_0^\infty \eta_{0,t} S_t c_{2t}^* dt - \int_0^\infty \eta_{0,t} dI_t^*\right] = P_0 K_{20}$. An entirely similar argument can be used to demonstrate that $\text{E}_0\left[\int_0^\infty \eta_{0,t} c_{1t}^* dt + \int_0^\infty \eta_{0,t} dI_t^*\right] = K_{10}$. It follows that

$$\text{E}_0\left[\int_0^\infty \eta_{0,t}\left(c_{1t}^* + S_t c_{2t}^*\right) dt\right] = P_0 K_{20} + K_{10} = V_0 \tag{A.64}$$

and therefore

$$\text{E}_0\left[\int_0^\infty \eta_{0,t}\left(\tilde{c}_{1t} + S_t \tilde{c}_{2t}\right) dt\right] > V_0. \tag{A.65}$$

Given (v), the dynamic budget constraint implies that

$$\text{E}_0\left[\int_0^T \eta_{0,t}\left(c_{1t} + S_t c_{2t}\right) dt\right] \leq V_0, \quad \forall T > 0. \tag{A.66}$$

for every feasible consumption plan (c_{1t}, c_{2t}) (e.g., Karatzas and Shreve 1991 (p. 374)). Using the monotone convergence theorem,

$$\text{E}_0\left[\int_0^\infty \eta_{0,t}\left(c_{1t} + S_t c_{2t}\right) dt\right] \leq V_0, \tag{A.67}$$

which is a clear violation of Eq. (A.65). Thus, $(\tilde{c}_{1t}, \tilde{c}_{2t})$ cannot be budget-feasible, which proves optimality of (c_{1t}^*, c_{2t}^*).

(ii) Without loss of generality, assume that there is only one firm in the industry, facing the problem

$$\max_{\{I_t\}} \mathrm{E}_0 \left[\int_0^\infty e^{-\rho t} \frac{U_{c_1}\left(c_{1t}^*, c_{2t}^*\right)}{U_{c_1}\left(c_{10}^*, c_{20}^*\right)} \frac{U_{c_2}\left(c_{1t}^*, c_{2t}^*\right)}{U_{c_1}\left(c_{1t}^*, c_{2t}^*\right)} K_{2t} dt - \int_0^\infty e^{-\rho t} \frac{U_{c_1}\left(c_{1t}^*, c_{2t}^*\right)}{U_{c_1}\left(c_{10}^*, c_{20}^*\right)} dI_t \right], \tag{A.68}$$

subject to Eq. (2) and $I_t \geq 0$, $dI_t \geq 0$. First, according to Eq. (A.24),

$$\mathrm{E}_s \left[\int_s^\infty e^{-\rho t} \frac{U_{c_2}\left(c_{1t}^*, c_{2t}^*\right)}{U_{c_1}\left(c_{1s}^*, c_{2s}^*\right)} \frac{R_{2t}}{R_{2s}} dt \right] = e^{-\rho s} \frac{J_{K_2}\left(K_{1s}^*, K_{2s}^*\right)}{J_{K_1}\left(K_{1s}^*, K_{2s}^*\right)}. \tag{A.69}$$

Second, the solution of Eq. (2) is given by

$$K_{2t} = K_{20} R_{2t} + \int_0^t \frac{R_{2t}}{R_{2s}} dI_s, \tag{A.70}$$

where R_{2t} is defined by Eq. (A.25). Finally,

$$\mathrm{E}_0 \left[\int_0^\infty e^{-\rho t} \frac{U_{c_1}\left(c_{1t}^*, c_{2t}^*\right)}{U_{c_1}\left(c_{10}^*, c_{20}^*\right)} \frac{U_{c_2}\left(c_{1t}^*, c_{2t}^*\right)}{U_{c_1}\left(c_{1t}^*, c_{2t}^*\right)} K_{2t} dt \right] = \tag{A.71}$$

$$\mathrm{E}_0 \left[\int_0^\infty e^{-\rho t} \frac{U_{c_2}\left(c_{1t}^*, c_{2t}^*\right)}{U_{c_1}\left(c_{10}^*, c_{20}^*\right)} \left(K_{20} R_{2t} + \int_0^t \frac{R_{2t}}{R_{2s}} dI_s \right) dt \right] = \tag{A.72}$$

$$\frac{J_{K_2}\left(K_{10}, K_{20}\right)}{J_{K_1}\left(K_{10}, K_{20}\right)} K_{20} + \mathrm{E}_0 \left[\int_0^\infty \int_s^\infty e^{-\rho t} \frac{U_{c_2}\left(c_{1t}^*, c_{2t}^*\right)}{U_{c_1}\left(c_{10}^*, c_{20}^*\right)} \frac{R_{2t}}{R_{2s}} dt dI_s \right], \tag{A.73}$$

where the first equality follows from Eq. (A.70) and the second is obtained using integration by parts and Lemma 1. According to the law of iterated expectations, the second term in the latter expression equals

$$\mathrm{E}_0 \left[\int_0^\infty \frac{U_{c_1}\left(c_{1s}^*, c_{2s}^*\right)}{U_{c_1}\left(c_{10}^*, c_{20}^*\right)} \mathrm{E}_s \left[\int_s^\infty e^{-\rho t} \frac{U_{c_2}\left(c_{1t}^*, c_{2t}^*\right)}{U_{c_1}\left(c_{1s}^*, c_{2s}^*\right)} \frac{R_{2t}}{R_{2s}} dt \right] dI_s \right] = \tag{A.74}$$

$$\mathrm{E}_0 \left[\int_0^\infty e^{-\rho s} \frac{U_{c_1}\left(c_{1s}^*, c_{2s}^*\right)}{U_{c_1}\left(c_{10}^*, c_{20}^*\right)} \frac{J_{K_2}\left(K_{1s}^*, K_{2s}^*\right)}{J_{K_1}\left(K_{1s}^*, K_{2s}^*\right)} dI_s \right], \tag{A.75}$$

the equality being an application of Eq. (A.24). Thus, the firm's problem is equivalent to

$$\max_{\{I_t\}} \mathrm{E}_0 \left[\int_0^\infty e^{-\rho t} \frac{U_{c_1}\left(c_{1t}^*, c_{2t}^*\right)}{U_{c_1}\left(c_{10}^*, c_{20}^*\right)} \left(\frac{J_{K_2}\left(K_{1t}^*, K_{2t}^*\right)}{J_{K_1}\left(K_{1t}^*, K_{2t}^*\right)} - 1 \right) dI_t \right]. \tag{A.76}$$

Note that, because of Eq. (26) and the constraint $dI_t \geq 0$, the last expression is nonpositive for all feasible investment policies. It equals zero if $1_{\{J_{K_2} \neq J_{K_1}\}} dI_t = 0$. Since the candidate solution I_t^* satisfies this condition, its optimality has been verified . Finally, the value of the firm under the optimal investment policy equals

$$\frac{J_{K_2}\left(K_{10}, K_{20}\right)}{J_{K_1}\left(K_{10}, K_{20}\right)} K_{20}. \tag{A.77}$$

(iii, iv) These conditions are satisfied by construction.

(v) The short-rate process r_t is well defined by Eq. (66) (this can also be verified by direct calculation or indirectly, as discussed in section 2.1) and satisfies Eq. (12) by construction. As I have demonstrated in (ii), $P_t = \mathrm{E}_t \left[\int_t^\infty \eta_{t,s} S_s K_{2s}^* ds - \int_t^\infty \eta_{t,s} dI_s^* \right]$, thus Eq. (13) is satisfied. Eq.

(14) holds, because corresponding cumulative return processes satisfy the Consumption CAPM (see Duffie (1996, p. 229)):

$$\alpha = r_t - \frac{U_{c_1c_1}(c_{1t}^*, c_{2t}^*)}{U_{c_1}(c_{1t}^*, c_{2t}^*)} \cdot \sigma \cdot \frac{d\langle c_{1t}^*, W_t\rangle}{dt} \tag{A.78}$$

This proves that $\eta_{t,s}$ is a valid stochastic discount factor.

I will now show that $\eta_{t,s}$ is the unique stochastic discount factor consistent with Eqs. (13) and (14). First, note that the instantaneous variance-covariance matrix of the cumulative return processes is constant and nonsingular. Second, according to Harrison and Kreps (1979), it is sufficient to demonstrate that the process for the market price of risk,

$$\theta_t \equiv -\frac{U_{c_1c_1}(c_{1t}^*, c_{2t}^*)}{U_{c_1}(c_{1t}^*, c_{2t}^*)} \frac{d\langle c_{1t}^*, W_t\rangle}{dt}, \tag{A.79}$$

is uniformly bounded. Using the relations $U_{c_1} = J_{K_1}$ and $c_1^* = (J_{K_1})^{-1/\gamma}$, rewrite Eq. (A.79) as

$$\theta_t = \sigma \frac{K_1 J_{K_1K_1}}{J_{K_1}}. \tag{A.80}$$

Since $J(K_1, K_2) = \frac{1}{1-\gamma} K_1^{1-\gamma} f(\omega)$, $K_1 J_{K_1K_1}/J_{K_1}$ is a function of ω only:

$$\frac{K_1 J_{K_1K_1}}{J_{K_1}} = -\gamma - \frac{f' - \frac{1}{1-\gamma} f''}{f - \frac{1}{1-\gamma} f'}. \tag{A.81}$$

The denominator equals $\tilde{c}_1^*(\omega)^{-\gamma}$, which is a bounded function (see Lemmas 2, 4). The following lemma implies that the numerator is also bounded.

Lemma 5. *As the ratio of capital stocks approaches infinity ($\omega \to \infty$), $f'(\omega) - \frac{1}{1-\gamma} f''(\omega)$ approaches zero.*

Proof. Define $g(\omega)$ using

$$g(\xi) \equiv b^{-1/\gamma} \left(f(\xi) - \lambda_1 - b^{1/\gamma} \lambda_2 \exp((1-\gamma)\xi) \right). \tag{A.82}$$

Then $f'(\omega) - \frac{1}{1-\gamma} f''(\omega)$ equals $b^{-1/\gamma} \left(g'(\omega) - \frac{1}{1-\gamma} g''(\omega) \right)$. According to Lemma 4,

$$\lim_{\omega\to\infty} \left(f(\omega) - \frac{1}{1-\gamma} f'(\omega) \right) = \lambda_1. \tag{A.83}$$

This implies that $\lim_{\omega\to\infty} \left(g(\omega) - \frac{1}{1-\gamma} g'(\omega) \right) = 0$. Next, Eq. (A.39) implies that $\lim_{\omega\to\infty} g(\omega) = 0$. Thus,

$$\lim_{\omega\to\infty} g(\omega) = \lim_{\omega\to\infty} g'(\omega) = 0. \tag{A.84}$$

Since $g(\omega)$ satisfies

$$p_2 g'' + p_1 g' + p_0 g = -\gamma b^{-1/\gamma} \left(\lambda_1 + b^{1/\gamma} \left(g - \frac{1}{1-\gamma} g' \right) \right)^{1-1/\gamma} - \lambda_1 p_0 b^{-1/\gamma}, \tag{A.85}$$

$$g' - \frac{1}{1-\gamma} g'' = g' + \frac{p_1 g' + p_0 g + \gamma b^{-1/\gamma} \left(\lambda_1 + b^{1/\gamma} \left(g - \frac{1}{1-\gamma} g' \right) \right)^{1-1/\gamma} + \lambda_1 p_0 b^{-1/\gamma}}{(1-\gamma) p_2}, \tag{A.86}$$

which converges to zero as ω approaches infinity (I used the fact that $p_0 = -\gamma \lambda_1^{-1/\gamma}$).

A.4. Asymptotic Expansion

Consider the case $\gamma \neq 1$. As I have stated in section 2.3, the first step in the analysis is to rescale the independent variable, introducing $\xi = \omega - \ln(b)/\gamma$. As a function of the new independent variable, $f(\xi)$ satisfies

$$p_2 f''(\xi) + p_1 f'(\xi) + p_0 f(\xi) + \gamma \left(f(\xi) - \frac{1}{1-\gamma} f'(\xi) \right)^{1-1/\gamma} = -b^{1/\gamma} e^{(1-\gamma)\xi}, \tag{A.87}$$

subject to the boundary conditions (Eqs. (35), (36), (41), (42)). I look for $f(\xi)$ and the optimal investment threshold Ξ^* in the form

$$f(\xi) = \lambda_1 + b^{1/\gamma} \left(\lambda_2 e^{(1-\gamma)\xi} + f_{(0)}(\xi) + \cdots b^{n/\gamma} f_{(n)}(\xi) + \cdots \right), \tag{A.88}$$

$$\Xi^* = \Xi_{(0)} + b^{1/\gamma} \Xi_{(1)} + \cdots b^{n/\gamma} \Xi_{(n)} + \cdots \tag{A.89}$$

The next step is to substitute the expansion given by Eq. (A.88) into Eq. (A.87), to expand the equation and the boundary conditions in powers of $b^{1/\gamma}$, and to collect the terms of the same order in $b^{1/\gamma}$. This leads to a sequence of equations on $f_{(0)}(\xi)$, $f_{(1)}(\xi)$, etc.:

$$q_2 f''_{(0)} + q_1 f'_{(0)} + q_0 f_{(0)} = 0, \tag{A.90}$$

$$q_2 f''_{(1)} + q_1 f'_{(1)} + q_0 f_{(1)} = \frac{\gamma - 1}{2\gamma} \lambda_1^{-1-1/\gamma} \left(f_{(0)} - \frac{1}{1-\gamma} f'_{(0)} \right)^2, \tag{A.91}$$

$$\vdots$$

where

$$q_2 = p_2 > 0, \tag{A.92}$$

$$q_1 = p_1 + \lambda_1^{-1/\gamma}, \tag{A.93}$$

$$q_0 = -\lambda_1^{-1/\gamma} < 0. \tag{A.94}$$

Each of these equations is a linear differential equation with a known general solution. The inhomogeneous term in equation on $f_{(k+1)}$ depends only on $f'_{(k)}$, therefore one can solve these equations sequentially. To specify the solution completely, one has to impose boundary conditions, obtained by substituting Eq. (A.88) and Eq. (A.89) into Eqs. (35, 36) and matching the terms according to their order in b. I solve for the first two terms in the asymptotic expansion given by Eq. (A.88).

The general solution of Eq. (A.90) is given by

$$f_{(0)}(\xi) = A_{(0)} \exp\left(\kappa_-(\xi - \xi^*)\right) + B_{(0)} \exp\left(\kappa_+(\xi - \xi^*)\right), \tag{A.95}$$

where

$$\kappa_\pm = \frac{-q_1 \pm \sqrt{{q_1}^2 - 4 q_2 q_0}}{2 q_2}, \quad \kappa_+ > 0, \quad \kappa_- < 0. \tag{A.96}$$

Since $f_{(0)}(\xi)$ inherits the condition $\lim_{\xi\to\infty} f_{(0)}(\xi) = \lambda_1$ from Eqs. (41, 42), one immediately concludes that $B_{(0)} = 0$. Next, substitute Eq. (A.95) and Eq. (A.89) into Eqs. (35, 36). This results in a system of two equations on $A_{(0)}$ and $\Xi_{(0)}$:

$$\lambda_2 (1-\gamma) \Xi_{(0)}^{1-\gamma} - \lambda_1 (1-\gamma) \Xi_{(0)} + \kappa A_{(0)} = 0, \tag{A.97}$$

$$-\lambda_2 \gamma (1-\gamma) \Xi_{(0)}^{1-\gamma} + (\kappa^2 - \kappa) A_{(0)} = 0. \tag{A.98}$$

where $\kappa = \kappa_-$. These equations can be solved explicitly:

$$A_{(0)} = \frac{\lambda_2 \gamma(1-\gamma)}{\kappa(\kappa-1)} \left(\frac{\lambda_1}{\lambda_2} \frac{\kappa - 1}{\kappa - (1-\gamma)} \right)^{1-1/\gamma}, \tag{A.99}$$

$$\Xi_{(0)} = \left(\frac{\lambda_1}{\lambda_2} \frac{\kappa - 1}{\kappa - (1-\gamma)} \right)^{-1/\gamma}. \tag{A.100}$$

$\Xi_{(0)}$ is well defined, because

$$\kappa - (1-\gamma) < 0. \tag{A.101}$$

Since

$$\kappa - (1-\gamma) = \frac{-q_1 - 2(1-\gamma)q_2 - \sqrt{q_1^2 - 4q_2 q_0}}{2q_2}$$

and $q_2 > 0$, it suffices to check the sign of

$$\left(q_1^2 - 4q_2 q_0\right) - \left(q_1 + 2(1-\gamma)q_2\right)^2.$$

The last expression equals

$$-4q_2 \left(q_0 + (1-\gamma)q_1 + (1-\gamma)^2 q_2\right) = \frac{4q_2}{\lambda_2} > 0.$$

This implies that Eq. (A.101) holds. Note that $sign(A_{(0)}) = sign(1-\gamma)$, which one would expect to hold, because $\left(b^{1/\gamma} A_{(0)}/(1-\gamma)\right) \exp(\kappa(\xi - \xi^*))$ approximates the difference between the optimal and a feasible solution of the maximization problem. Also, note that Eq. (A.101) implies that the first-order approximation to the value function is of the same sign as $1-\gamma$, which must be the case for the optimal solution. In the limit of γ approaching one, Eqs. (A.99, A.100) converge to their counterparts for the log-utility case.

Next, I determine $f_{(1)}(\xi)$ and $\Xi_{(1)}$. $f_{(1)}(\xi)$ equals

$$A_{(1)} \exp\left(\kappa(\xi - \xi^*)\right) + C_{(1)} \exp\left(2\kappa(\xi - \xi^*)\right), \tag{A.102}$$

where $C_{(1)}$ is found from (A.91):

$$C_{(1)} = \frac{1}{2\gamma(\gamma-1)} \lambda_1^{-1-1/\gamma} \frac{A_{(0)}^2 (1-\gamma-\kappa)^2}{4\kappa^2 q_2 + 2\kappa q_1 + q_0}. \tag{A.103}$$

$A_{(1)}$ and $\Xi_{(1)}$ satisfy a system of two linear equations, resulting from Eqs. (35, 36):

$$\kappa A_{(1)} + \left(\lambda_2 (1-\gamma)^2 \Xi_{(0)}^{-\gamma} - \lambda_1 (1-\gamma)\right) \Xi_{(1)} = -\left((\kappa+\gamma-1) A_{(0)} \Xi_{(0)} + 2\kappa C_{(1)}\right), \tag{A.104}$$

$$\left(\kappa^2 - \kappa\right) A_{(1)} - \lambda_2 \gamma (1-\gamma)^2 \Xi_{(0)}^{-\gamma} \Xi_{(1)} = -\left(\kappa(\kappa+\gamma-1) A_{(0)} \Xi_{(0)} + \left(4\kappa^2 - 2\kappa\right) C_{(1)}\right). \tag{A.105}$$

While it is clearly possible, I will not solve for $A_{(1)}$ and $\Xi_{(1)}$ explicitly here.

Table 1

The optimal investment threshold Ω^*, defined as the ratio of capital stocks in the two sectors of the economy, its first-order approximation $b^{1/\gamma}\Xi_{(0)}$ and the second-order approximation $b^{1/\gamma}\left(\Xi_{(0)} + b^{1/\gamma}\Xi_{(1)}\right)$ are tabulated for different values of the risk-aversion parameter γ and the utility parameter b, which controls the relative preference for the two consumption goods. The constants $\Xi_{(0)}$ and $\Xi_{(1)}$ used to construct the above approximations are defined by Eq. (A.100) and Eqs. (A.104, A.105).

γ	$b^{1/\gamma}\Xi_{(0)}$	$b^{1/\gamma}\left(\Xi_{(0)} + b^{1/\gamma}\Xi_{(1)}\right)$	Numerical Solution
		$\underline{b = 0.1}$	
1.5	0.044	0.042	0.043
1	0.043	0.041	0.041
0.5	0.028	0.026	0.026
		$\underline{b = 0.2}$	
1.5	0.070	0.066	0.066
1	0.086	0.077	0.078
0.5	0.111	0.083	0.088

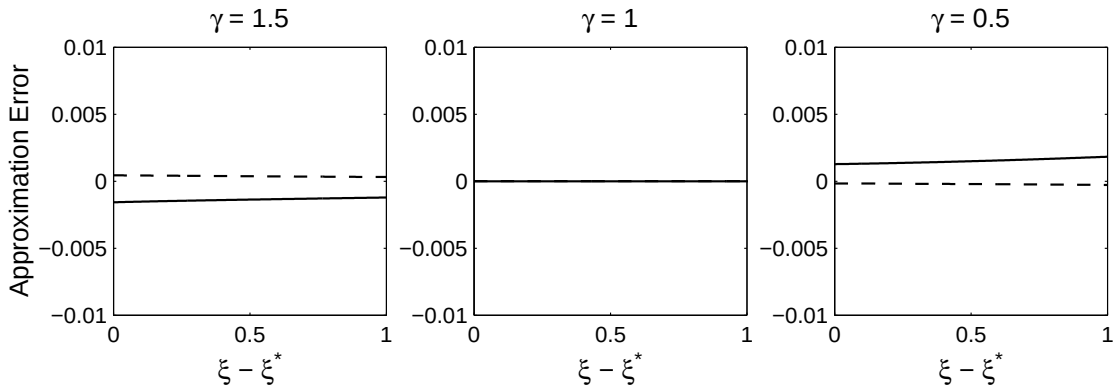

Fig. 1. Approximation error of the asymptotic expansion of the value function. The value function is characterized by $f(\xi)$ and the approximation error is defined as the difference between the exact value of $f(\xi)$ and its asymptotic expansion, divided by the exact value. The argument is the state variable $\xi - \xi^*$, defined as the difference between the logarithm of the ratio of the capital stocks in the two sectors of the economy and the logarithm of the critical value of such ratio, which is the equilibrium investment threshold. The first- and second-order asymptotic expansions are given by Eqs. (52–55) and Eqs. (59–62) respectively. The approximation error of the first-order expansion is plotted as a solid line and that of the second-order expansion as a dashed line. The relative preference for the two consumption goods is controlled by $b = 0.1$, the subjective time preference rate is $\rho = 0.05$, the expected return and the volatility of the production technology of the first sector are given by $\alpha = 0.07$ and $\sigma = 0.17$ respectively, and the depreciation rate of capital in the second sector is $\delta = 0.05$. The risk aversion parameter γ takes the values of 0.5, 1, and 1.5.

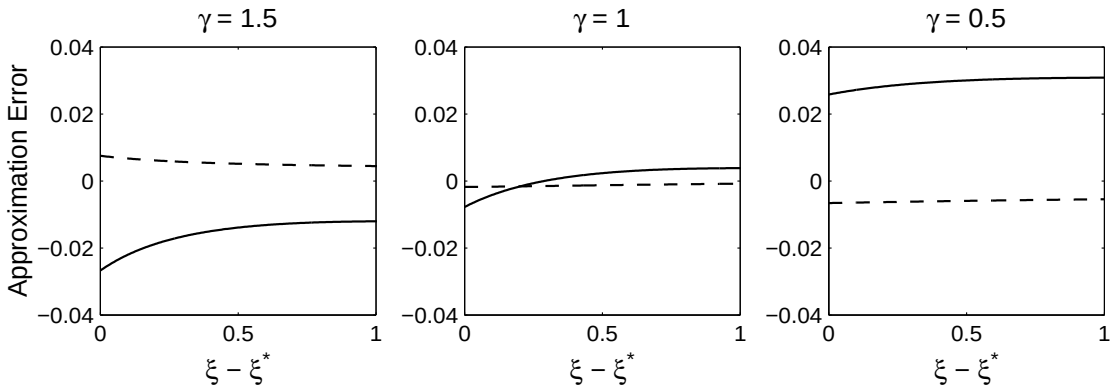

Fig. 2. Approximation error of the asymptotic expansion of the derivative of the value function. The value function is characterized by $f(\xi)$ and the approximation error of its derivative is defined as the difference between the exact value of $f'(\xi)$ and its asymptotic expansion, divided by the exact value. The argument is the state variable $\xi - \xi^*$, defined as the difference between the logarithm of the ratio of the capital stocks in the two sectors of the economy and the logarithm of the critical value of such ratio, which is the equilibrium investment threshold. The first- and second-order asymptotic expansions are given by Eqs. (52–55) and Eqs. (59–62) respectively. The approximation error of the first-order expansion is plotted as a solid line and that of the second-order expansion as a dashed line. The relative preference for the two consumption goods is controlled by $b = 0.1$, the subjective time preference rate is $\rho = 0.05$, the expected return and the volatility of the production technology of the first sector are given by $\alpha = 0.07$ and $\sigma = 0.17$ respectively, and the depreciation rate of capital in the second sector is $\delta = 0.05$. The risk aversion parameter γ takes the values of 0.5, 1, and 1.5.

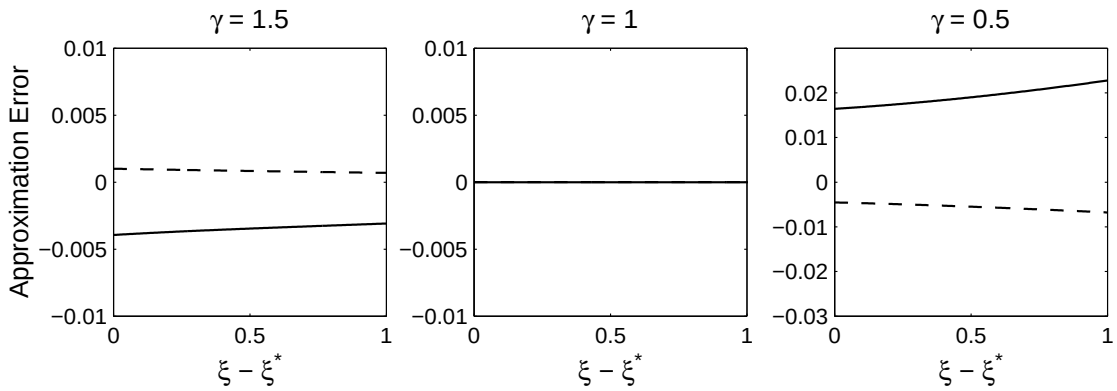

Fig. 3. Approximation error of the asymptotic expansion of the value function. The value function is characterized by $f(\xi)$ and the approximation error is defined as the difference between the exact value of $f(\xi)$ and its asymptotic expansion, divided by the exact value. The argument is the state variable $\xi - \xi^*$, defined as the difference between the logarithm of the ratio of the capital stocks in the two sectors of the economy and the logarithm of the critical value of such ratio, which is the equilibrium investment threshold. The first- and second-order asymptotic expansions are given by Eqs. (52–55) and Eqs. (59–62) respectively. The approximation error of the first-order expansion is plotted as a solid line and that of the second-order expansion as a dashed line. The relative preference for the two consumption goods is controlled by $b = 0.2$, the subjective time preference rate is $\rho = 0.05$, the expected return and the volatility of the production technology of the first sector are given by $\alpha = 0.07$ and $\sigma = 0.17$ respectively, and the depreciation rate of capital in the second sector is $\delta = 0.05$. The risk aversion parameter γ takes the values of 0.5, 1, and 1.5.

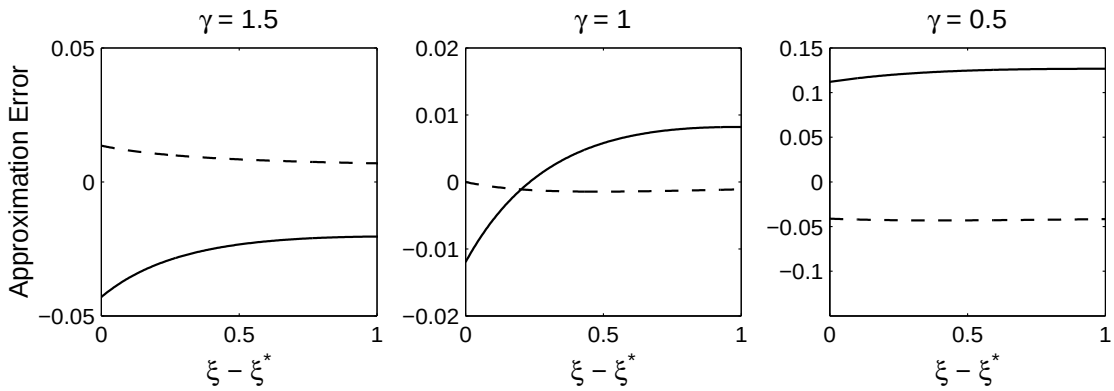

Fig. 4. Approximation error of the asymptotic expansion of the derivative of the value function. The value function is characterized by $f(\xi)$ and the approximation error of its derivative is defined as the difference between the exact value of $f'(\xi)$ and its asymptotic expansion, divided by the exact value. The argument is the state variable $\xi - \xi^*$, defined as the difference between the logarithm of the ratio of the capital stocks in the two sectors of the economy and the logarithm of the critical value of such ratio, which is the equilibrium investment threshold. The first- and second-order asymptotic expansions are given by Eqs. (52–55) and Eqs. (59–62) respectively. The approximation error of the first-order expansion is plotted as a solid line and that of the second-order expansion as a dashed line. The relative preference for the two consumption goods is controlled by $b = 0.2$, the subjective time preference rate is $\rho = 0.05$, the expected return and the volatility of the production technology of the first sector are given by $\alpha = 0.07$ and $\sigma = 0.17$ respectively, and the depreciation rate of capital in the second sector is $\delta = 0.05$. The risk aversion parameter γ takes the values of 0.5, 1, and 1.5.

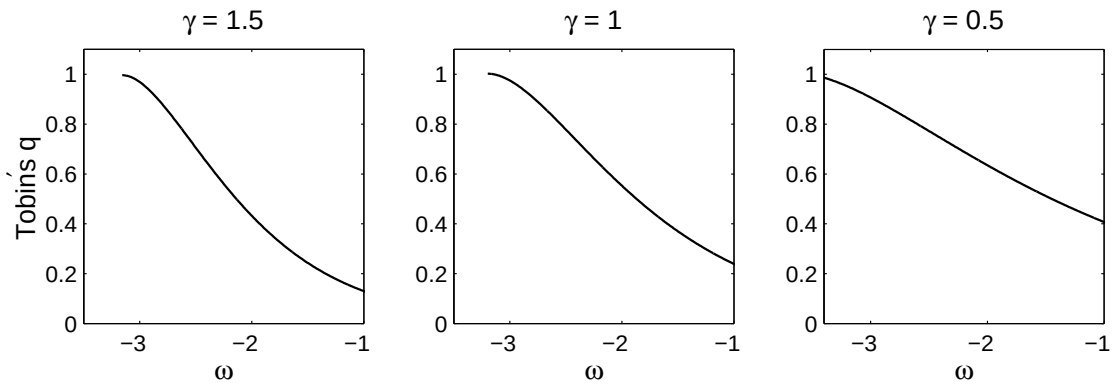

Fig. 5. The market value of capital as a function of the ratio of the capital stocks. Tobin's q, the market value of a unit of capital in the second sector, is given by $q = J_{K_2}/J_{K_1}$, where J is the value function of the central planner's problem and K_1 and K_2 denote the capital stocks in the two sectors of the economy. q is plotted as a function of the logarithm of the ratio of the two capital stocks, $\omega = \ln(K_2/K_1)$. The relative preference for the two consumption goods is controlled by $b = 0.1$, the subjective time preference rate is $\rho = 0.05$, the expected return and the volatility of the production technology of the first sector are given by $\alpha = 0.07$ and $\sigma = 0.17$ respectively, and the depreciation rate of capital in the second sector is $\delta = 0.05$. The risk aversion parameter γ takes the values of 0.5, 1, and 1.5.

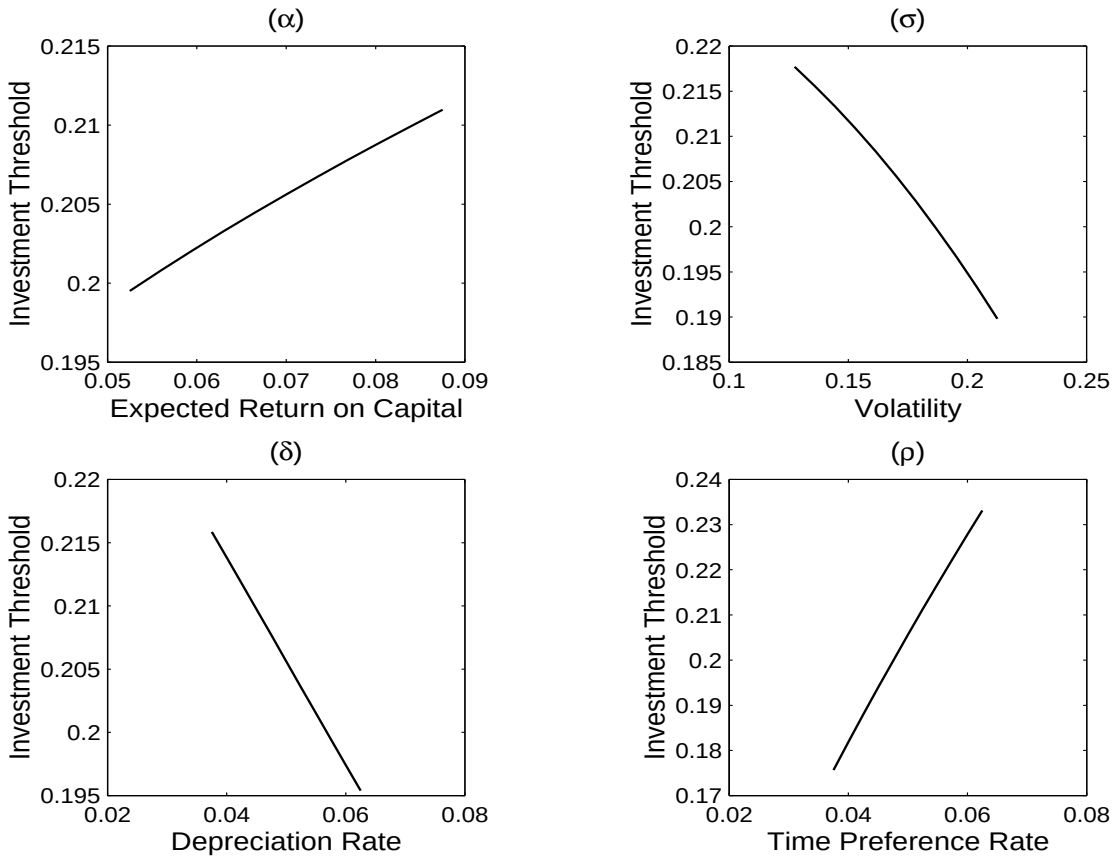

Fig. 6. The optimal investment threshold, comparative statics results. Each of the plots shows $\Xi_{(0)}$, the leading term in the asymptotic expansion of the optimal investment threshold, given by Eq. (73). The argument in each of the plots is one of the four model parameters: α and σ are the expected return and the volatility of the production technology of the first sector respectively, δ is the depreciation rate of capital in the second sector, and ρ is the subjective time preference rate. For each of the plots, one of these parameters is allowed to vary, while all other model parameters are fixed at their calibrated values: $\rho = 0.05$, $\alpha = 0.07$, $\delta = 0.05$, $\sigma = 0.17$. The risk aversion parameter is $\gamma = 1.5$.

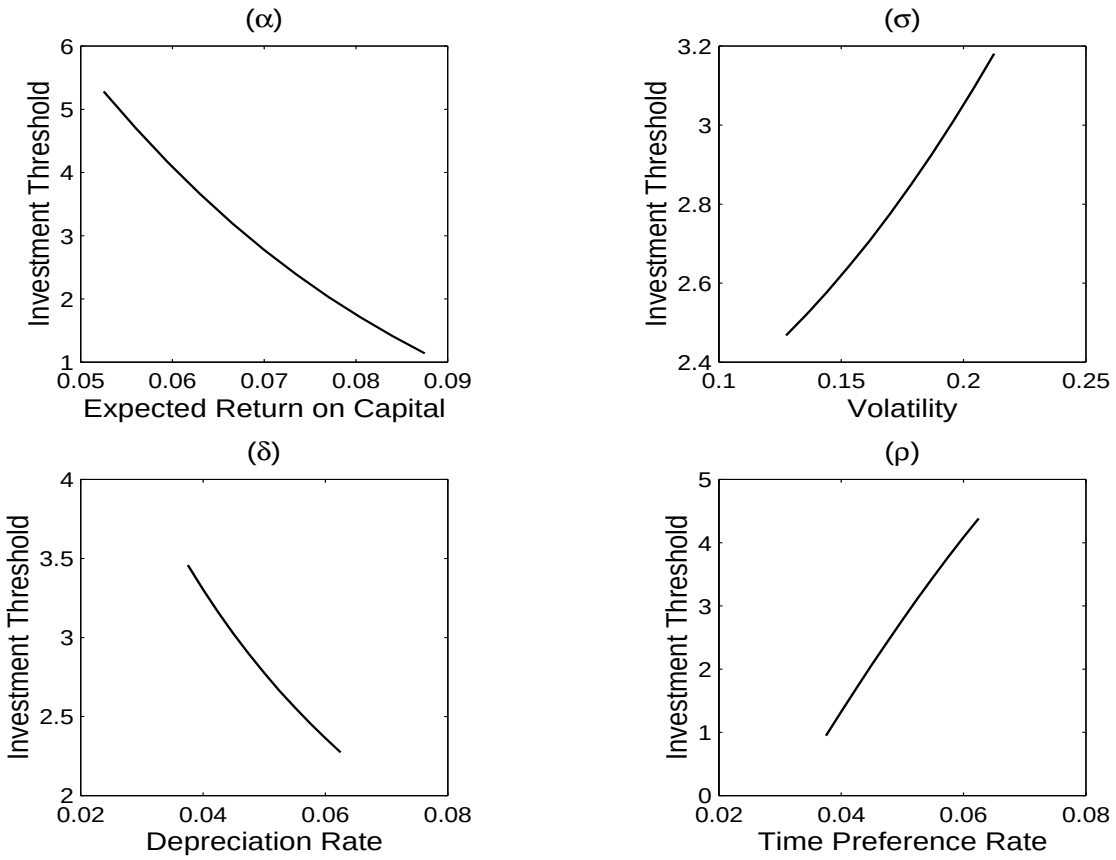

Fig. 7. The optimal investment threshold, comparative statics results. Each of the plots shows $\Xi_{(0)}$, the leading term in the asymptotic expansion of the optimal investment threshold, given by Eq. (73). The argument in each of the plots is one of the four model parameters: α and σ are the expected return and the volatility of the production technology of the first sector respectively, δ is the depreciation rate of capital in the second sector, and ρ is the subjective time preference rate. For each of the plots, one of these parameters is allowed to vary, while all other model parameters are fixed at their calibrated values: $\rho = 0.05$, $\alpha = 0.07$, $\delta = 0.05$, $\sigma = 0.17$. The risk aversion parameter is $\gamma = 0.5$.

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