# Dynamical Symmetry of the Hydrogen Atom

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**Abstract**

We examine a hidden symmetry of the hydrogen atom in quantum mechanics using the quantum mechanical analog of the classical Lenz vector and its relation to the generators of the familiar SO(3) rotational group. We use as our tool some basic concepts in algebraic group theory. The result is a mathematical description of a higher-dimension symmetry under the SO(4) group, or the rotational group in four dimensions. We use this insight to derive the energy eigenvalues of the hydrogen atom, and many of its constraints and features, free of the Schrödinger wave equation.

# 1 Introduction

The problem of the hydrogen atom central potential in quantum mechanics is usually solved by explicit separation of variables in the governing Schrödinger Equation. Formidable calculations then yield the resulting eigenstates of the Hamiltonian uniquely characterized by the quantum numbers $n$, $l$, and $m$, where $n$ is the energy level, $l$ is a measure of the magnitude of the orbital angular momentum, and $m$ contains information about its direction. We find that for a given $n$, $l$ can range from $n-1$ to 0 in integer steps, and $m$ can range from $-l$ to $+l$, again in integer steps. The corresponding energy eigenvalues of the system is also quantized with,

$$E_n = -\frac{\mu Z^2 e^4}{2\hbar^2 n^2}, \qquad \text{for } n = 1, 2, 3... \tag{1}$$

It is at once evident that this spectrum is many-fold degenerate in $l$ and $m$, dependent only on the quantum number, $n$. Seeking further physical insight into this important system, we examine the conditions and conserved properties that lead to this remarkable degeneracy.

# 2 Symmetry and Degeneracy

## 2.1 Conservation Laws

To begin, consider the relation between symmetry and conserved quantities in classical mechanics. We see that geometric symmetries in the environment, such as translationally and rotationally invariant potentials, allow for conservation of linear and angular momentum, respectively. The analog is true in quantum mechanics. We motivate this by a simple example. Consider a state $|\alpha\rangle$ with wavefunction $\psi_\alpha(\vec{r})$. We can create a displaced wavefunction $\psi_{\alpha'}(\vec{r})$ that is related to $\psi_\alpha(\vec{r})$ by a linear transformation such that,

$$\psi_{\alpha'}(\vec{r}) = \psi_\alpha(\vec{r} - \vec{r_0}) \tag{2}$$

We look for unitary transformation $U(\vec{r_0})$ that takes the state $\psi_\alpha(\vec{r})$ to the state $\psi_{\alpha'}(\vec{r})$. This matter can be simplified by a rotation of the coordinate system, such that the direction of translation is along the $x-$axis, see Figure 1. $U(\vec{r_0})$ then satisfies the following relation:

$$U(x_0)\psi_\alpha(x, y, z) = \psi_\alpha(x - x_0, y, z) = \psi_{\alpha'}(x, y, z) \tag{3}$$

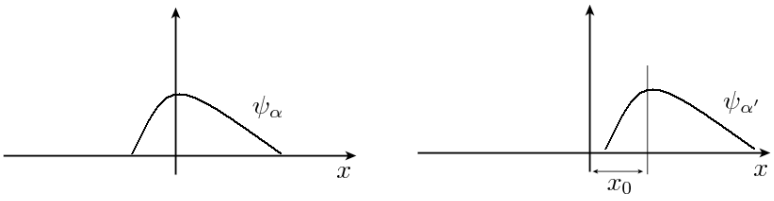


Figure 1: Displacement of wavefunction $\psi_\alpha$ through distance $x_0$.

A Taylor Expansion of the right-hand side of the equation yields,

$$\begin{aligned}\psi_\alpha(x-x_0,y,z) = \quad & \psi_\alpha(x,y,z) - x_0\frac{\partial}{\partial x}\psi_\alpha(x,y,z) \\ & + \frac{x_0^2}{2!}\frac{\partial^2}{\partial x^2}\psi_\alpha(x,y,z) - ... \end{aligned} \tag{4}$$

which we recognize as the power series expansion of

$$e^{-x_0(\frac{\partial}{\partial x})}\psi_\alpha(x,y,z) \tag{5}$$

Inserting the momentum operator $p_x = -i\hbar\frac{\partial}{\partial x}$, we compare equations 3 and 5, and find the explicit expression for a $U(x_0)$ to be $e^{-\frac{ip_x x_0}{\hbar}}$. The operator generalizes easily to three dimensions. By allowing the translation to be along any vector in space and invoking $\vec{p} = -i\hbar\nabla$ in our derivation, we arrive at the space-translation operator,

$$U(\vec{r_0}) = e^{-\frac{i\vec{p}\cdot\vec{r_0}}{\hbar}} \tag{6}$$

Since $\vec{p}$ is the invoked operator here, we say $\vec{p}$ *generates* translation. Though this operator can be applied to any state, $\psi_\alpha(\vec{r})$, the resulting state, $\psi_{\alpha'}(\vec{r})$, need not satisfy the same Hamiltonian; in other words, the translated state does not necessarily represent a possible motion of the system. But, it's enlightening to consider under what circumstances the new state $|\alpha'\rangle$ also solves the time-dependent Schrödinger Equation, $i\hbar\frac{\partial\psi}{\partial t} = H\psi$. Here, we use the reverse translation, $U^\dagger(\vec{r_0})|\alpha'\rangle = |\alpha\rangle$, where $U^\dagger = U^{-1}$ is the adjoint of $U$ and equal to $e^{\frac{i\vec{p}\cdot\vec{r_0}}{\hbar}}$.

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}|\alpha'\rangle &= i\hbar\frac{\partial}{\partial t}U|\alpha\rangle \\ &= Ui\hbar\frac{\partial}{\partial t}|\alpha\rangle \\ &= UH|\alpha\rangle \\ &= UHU^\dagger|\alpha'\rangle \end{aligned} \tag{7}$$

We learn that in order for translated state $|\alpha'\rangle$ to be a possible solution to the Schrödinger Equation with Hamiltonian, $H$, the following relation must be satisfied,

$$\begin{aligned} UHU^\dagger &= H \\ \Rightarrow UH &= HU \end{aligned} \tag{8}$$

which is only true if $\vec{p}$ commutes with $H$, i.e. $[\vec{p}, H] = 0$. We see that solution eigenstates to a space-displacement symmetric (translationally invariant) system can have a well-defined, constant momentum. Likewise, for rotationally invariant systems, possible states can be found (through the use of a rotation operator) to have a well-defined, constant angular momentum.

## 2.2 Relation to Degeneracy

Now, consider a system governed by Hamiltonian, $H$. Suppose this system possesses a symmetry defined by operator, $\Omega$, such that $[\Omega, H] = 0$. We see that this necessarily leads to energy degeneracy of linearly independent states.

$$\begin{aligned} H|\alpha\rangle &= E_\alpha|\alpha\rangle \\ \Omega H|\alpha\rangle &= \Omega E_\alpha|\alpha\rangle \\ H(\Omega|\alpha\rangle) &= E_\alpha(\Omega|\alpha\rangle) \end{aligned} \tag{9}$$

If $|\alpha\rangle$ is an eigenstate of the Hamiltonian with eigenvalue, $E_\alpha$, $\Omega|\alpha\rangle$ will be another eigenstate with the same energy eigenvalue. When it is linearly independent of $|\alpha\rangle$, as often is the case with operators of the kind discussed in the previous section, the result is energy degeneracy in states related by operator $\Omega$, or, degeneracy as a consequence of symmetry.

Considering a particle in a translationally symmetric potential, we have arrived at an intuitively obvious result. Energy of such a particle is independent of its location. Its momentum is a conserved quantity. If we think of a free particle, this makes sense. The translation of a free particle in space can correspond to a possible time evolution of the system, and time evolution is generated by the Hamiltonian, in this case with no effect on its momentum. The usefulness of this connection between symmetry and degeneracy, however, goes beyond the derivation of conservation laws; it can be applied to dynamically as well as geometrically symmetric systems. We will use it in later sections to derive the energy eigenvalues of the hydrogenic atom.

# 3 Motivating Dynamical Symmetry

## 3.1 Extra Degeneracy in the Hydrogen Atom Problem

Equation 1 gives the energy levels of a hydrogen atom under a central coulombic potential. As noted earlier, the energy is degenerate with respect to both $m$ and $l$. In light of the previous discussion on the correlation of symmetry with degeneracy, one would expect to find each of the observed degeneracies to be related to a symmetry of the system. This is true for $m$ and spherical symmetry of the coulomb potential, since a rotational operator that commutes with the Hamiltonian can be constructed to map states onto other states with different $m$ values. This result is also intuitive. A rotationally symmetric potential should have no preferred direction in space. The orientation of the angular momentum vector has no impact on the energy of the system. Furthermore, we see that this degeneracy in $m$ is lifted by application of a directional field, such as an external magnetic field. In this case, we have now set a preferred axis and the potential is no longer spherically symmetric, resulting in energy dependence on $m$.

However, an extra degeneracy remains unexplained in the hydrogen atom. This is a degeneracy in *magnitude* of the total angular momentum, $l$. We suspect the culprit to be an overlooked conserved quantity in the system, yet we've exhausted all geometrical symmetries. We motivate the concept of dynamical symmetries from the Kepler problem in classical mechanics.

## 3.2 The Lenz Vector and The Kepler Problem

The gravitational potential in classical mechanics holding celestial bodies in orbit is a direct analog of the coulombic potential of such great interest in quantum mechanics. The Hamiltonian, rewritten in relative coordinates, is given below.

$$H = \frac{\vec{p}^{\,2}}{2\mu} - \frac{\kappa}{r} \tag{10}$$

In the case of the gravitational potential, $\kappa = GMm$, while the coulomb potential requires that $\kappa = Ze^2$. Here, $Z$ is the atomic number of the atom.

A classical kepler orbit with a distinct energy has many closed-orbit stable solutions, circular and elliptical solutions among them. Eccentricity, $e$, is a measure of an orbit's deviation from a perfect circle. $e = \frac{\sqrt{a^2 - b^2}}{a}$, with $a$ as the semimajor axis,

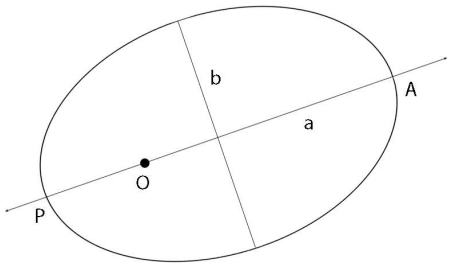


Figure 2: A closed non-circular orbit in a central potential. $O$ is the location of the center of gravity.

half the distance from perihelion to aphelion (labeled by P and A in Figure 2), and $b$ as the semiminor axis, perpendicular to $a$. A circular orbit has eccentricity zero.

The time-displacement symmetry of the Hamiltonian and the rotational symmetry of the potential motivates the conclusion that both total energy and angular momentum are conserved quantities. Using classical arguments and $\vec{L} = \vec{r} \times \vec{p}$, we find that [4],

$$E = -\frac{\kappa}{2a} \tag{11}$$

$$\vec{L}^{\,2} = \mu\kappa a(1 - e^2) \tag{12}$$

In a perfect inverse-square central force potential, there is one further conserved quantity. This vector, known as the *Lenz Vector*, first derived by Jakob Hermann, lies in the plane of the orbit, pointing from $O$ to $P$ in Diagram 2, and defines direction and orientation of the major axis. More explicitly, its construction can be seen in Figure 3. Mathematically, it is,

$$\vec{M} = \frac{\vec{p} \times \vec{L}}{\mu} - \frac{\kappa}{r}\hat{r} \tag{13}$$

Defined in this way, its magnitude is,

$$|\vec{M}| = \kappa e \tag{14}$$

We assume the vector to be fixed in direction and magnitude for our purposes, however, small perturbations to the inverse-square central force potential have been known to cause the vector to rotate [4]. The Lenz Vector characterizes the dynamical symmetry, a hidden symmetry, of the Kepler problem, acts as an additional constraint

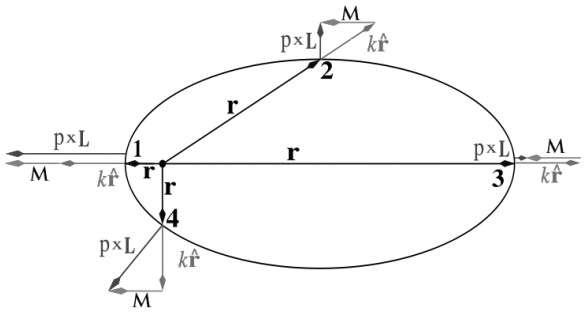


Figure 3: The Lenz Vector (given here by $\vec{M}$) at several points during an orbit [5]. Center of gravity indicated by a point.

on the system. Classically it obeys the following relations [1]:

$$\vec{L} \cdot \vec{M} = 0 \qquad \vec{M}^2 = \frac{2H}{\mu}\vec{L}^2 + \kappa^2 \tag{15}$$

where $H$ is the Hamiltonian given in 10.

# 4 Construction of the SO(4) Rotation Group

## 4.1 Quantum Mechanical Analog to the Lenz Vector

With $\vec{r}$, $\vec{p}$, and $\vec{L}$ as already established quantum mechanical operators, the classical Lenz vector, $\vec{M}$ can be translated directly into quantum mechanics. There is only a bit of subtlety involved in its construction. Since $[\vec{p}, \vec{L}] \neq 0$, we must take care in defining the cross product. We notice that $\vec{p} \times \vec{L} \neq -\vec{L} \times \vec{p}$; Equation 13 is not Hermitian. Following the derivation in Schiff [1], we redefine $\vec{M}$ as a symmetric average,

$$\vec{M} = \frac{1}{2\mu}(\vec{p} \times \vec{L} - \vec{L} \times \vec{p}) - \frac{\kappa}{r}\vec{r} \tag{16}$$

As expected, this vector commutes with the Hamiltonian,

$$[\vec{M}, H] = 0 \tag{17}$$

and defines a conserved quantity in the quantum mechanical analog of the classical Kepler problem. We find that with a small modification, the relations in 15 still apply:

$$\vec{L} \cdot \vec{M} = \vec{M} \cdot \vec{L} = 0 \qquad \vec{M}^2 = \frac{2H}{\mu}(\vec{L}^2 + \hbar^2) + \kappa^2 \tag{18}$$

From here, we look to understand how this new commuting operator relates to the physical system and all the existing conserved quantities. We examine the system in light of well-established geometrical symmetries.

## 4.2 Angular Momentum as Generators of the SU(2) or SO(3) Rotation Group

It is known that spherical symmetry of the coulomb potential leads to degeneracy in orientation of the angular momentum vector, $\vec{L}$. More rigorously, this consequence can be viewed in light of some basic concepts in algebraic group theory.

We perform a rotation of ket $|\alpha\rangle$ into $|\alpha'\rangle$, such that $|\alpha'(\vec{r})\rangle = |\alpha(R^{-1}(\vec{r}))\rangle$. Here, $R$ is simply a 3-dimensional linear operator which rotates vectors $\vec{r}$ in space; $R^{-1}(\vec{r})$ is analogous to writing $x - x_0$ in the linear case. Incidentally, the set of all rotation $R$ form the SO(3) group, or, the special orthogonal group in three dimensions: the set of all rotations in 3-dimensions which preserves orientation and vector length. They are most commonly represented by 3×3 real, orthonormal matrices with a determinant of +1. The SO(3) group shares three independent parameters, or *generators*, one for each orthogonal axis of rotation.

With a little bit of work, which can be examined in detail in Schiff [1], and using the results from Section 2.1, we can find an explicit expression for the rotation operator $U_R(\vec{\phi})$, which takes $\psi_\alpha(\vec{r})$ to $\psi_{\alpha'}(\vec{r})$.

$$U_R(\vec{\phi}) = e^{-\frac{i\vec{\phi}\cdot\vec{L}}{\hbar}} \tag{19}$$

We find unsurprisingly that the set of all rotation operators $U_R$ also form a closed group, we call it SU(2), or the special unitary group most simply represented by 2x2 matrices. This SU(2) group is related to the SO(3) rotation group by a *homomorphism* [6], indicative of identical group structure. SU(2) has the same number of generators as, and *doubly covers* SO(3). Each two element of the cover maps onto to exactly one element of the rotation group.

From $\vec{L} = \vec{r} \times \vec{p}$, we can decompose the angular momentum vector along its components and find more explicit expressions for each operator,

$$L_x = yp_z - zp_y \qquad L_y = zp_x - xp_z \qquad L_z = xp_y - yp_x \tag{20}$$

These are the three generators of the rotation group discussed above, in the sense that they "generate" infinitesimal rotations about their respective axes. Their commutators can be calculated easily in this form, and the commutation relations that

follow are consistent with the closed algebra of the generators of the SO(3) or SU(2) group. An algebra is said to be closed if the commutator of any pair of generators is a linear combination of the generators.

$$[L_x, L_y] = i\hbar L_z \quad [L_y, L_z] = i\hbar L_x \quad [L_z, L_x] = i\hbar L_y \tag{21}$$

All others are zero. This is equivalent to,

$$[L_i, L_j] = i\hbar\epsilon_{ijk}L_k, \text{ for } i, j, k = 1, 2, 3 \tag{22}$$

Considering that,

$$[L_i, H] = 0, \text{ for } i, j, k = 1, 2, 3 \tag{23}$$

the angular momentum operators can be diagonalized simultaneously with the Hamiltonian and their eigenvalues conserved. The SO(3) group, and similarly the SU(2) group, is of rank 1, due to the fact that no two of its generators commute. Therefore, it has one independent *Casimir operator*, or, one operator that commutes with all generators of this group. This operator, a bilinear combination of the three generators, is,

$$\vec{L}^2 = L_x^2 + L_y^2 + L_z^2 \tag{24}$$

with eigenvalues,

$$\vec{L}^2\psi_l = l(l+1)\hbar^2\psi_l \tag{25}$$

where in general $l$ can be any integer or half integer. In the known case of hydrogen, they take on only integral values.

Cast in the framework of group theory, we can think of angular momentum conservation and energy degeneracy in $m$ as due to the invariance or symmetry of the potential under rotations by the SO(3) group. We will soon see that dynamical symmetry is associated with rotations under the SO(4) group. These operations take place in a higher dimension and allow for quantum mechanical mixing of states of different angular momenta without change in energy.

## 4.3 The Algebra of the M, L Generators

The three components of $\vec{M}$, like the three components of $\vec{L}$, can be treated as generators of some infinitesimal transformations. With the goal of exploring the

algebraic structure of these new generators, we work out their commutation relations. There are 36 in all,

Some are known: $[L_i, L_j]$ are given above in Equation 22. The rest, for $i, j, k = 1, 2, 3$ are:

$$[M_i, L_i] = 0 \quad [M_i, L_j] = i\hbar\epsilon_{ijk}M_k \tag{26}$$

and,

$$[M_i, M_j] = -\frac{2i\hbar}{\mu}\epsilon_{ijk}HL_k \tag{27}$$

The first set of these commutators (26) shows $\vec{M}$ to be a vector. $M_i$ commutes with $L_i$ since $L_i$ generates rotation about the $i$ axis, which intuitively should have no effect on a vector pointed along that axis. On the other hand, if $j$ is perpendicular to $i$, $L_j$ induces a change in the direction of $M_i$. This is characterized by the second commutation relation in 26, which tells that the direction of this change in $M_j$ should be orthogonal to both $M_j$ and $L_i$, in this case, along $-M_k$.

The relation in Equation 27 breaks the closed algebra of the $\vec{L}$ and $\vec{M}$ operators together, since $H$ is introduced. To go on with our analysis, we make a useful substitution. First, we consider a degenerate subspace of $H$ with energy eigenvalue $E$. We can replace the Hamiltonian with the resulting energy eigenvalue. It is now illuminating to define $\vec{M}'$, such that,

$$\vec{M}' \equiv \left(-\frac{\mu}{2E}\right)^{1/2}\vec{M} \tag{28}$$

The commutation relations remain unchanged with $M$ everywhere replaced by $M'$, with the exception of 27, which now becomes, quite simply,

$$[M'_i, M'_j] = i\hbar\epsilon_{ijk}L_k \tag{29}$$

We realize we're dealing with a closed algebraic system. Though the fog is certainly clearing up, we still remain on the lookout for some physical insight to help us solve this system.

## 4.4 Identifying the Closed Algebra of L and M' with the SO(4) Group

Returning to our definition, in 20, for the angular momentum operator, $L$, we can relabel $L_x$, $L_y$, and $L_z$ in terms of orbital plane as opposed to vector orientation. Substituting 1,2,3 for $x$, $y$, and $z$ allows us to naturally extend beyond three dimensions.

Armed with the commutators of $[r_i, p_j]$, which is only zero when $i \neq j$, we claim the $\vec{M}'$ matrices are merely generalized angular momentum operators in the fourth dimension. In four dimensions we predict $\begin{pmatrix} 4 \\ 2 \end{pmatrix} = 6$ orthogonal angular momentum generators. We make the following associations,

$$\begin{aligned} M'_x = L_{14} \quad M'_y = L_{24} \quad M'_z = L_{34} \\ L_x = L_{23} \quad L_y = L_{31} \quad L_z = L_{12} \end{aligned} \tag{30}$$

It's then easy to verify that the commutation relations in Section 4.3 still hold. With four dimensions, a degree of decoupling emerges that we have yet to observe: the maximal commuting set of operators includes two angular momentum operators at the same time: $L_{12}$ with $L_{34}$, etc. We expect this decoupling since the two operators generate rotations in orthogonal, non-intersecting planes, defined by completely disjoint sets of vectors. This fact alludes to the structure of the group generated by the operators in 30: the rotational group in four-dimensions, SO(4), of rank 2, which contains as a subgroup the rank 1 SO(3) rotational group.

We've succeeded in generalizing the closed algebra of $\vec{L}$ and $\vec{M}'$ operators. The remarkable insight here is that dynamical symmetry in a system (as physically evidenced by eccentricities in closed orbits) is a mere artifact of rotational symmetry in a higher dimension!

# 5 Energy Levels of the Hydrogenic Atom

The rest is a matter of algebra. The partial decoupling of angular momentum operators in four dimensions hints at a possible decomposition of the SO(4) group generated by the operators $\vec{L}$ and $\vec{M}$. In fact, this is exactly the case. Rank 2 SO(4) groups by rule can be broken down into two completely decoupled rank 1 SU(2) groups with three noncommuting generators each.

$$\begin{aligned} SO(4) &\rightarrow SU(2) \times SU(2) \\ 6 \quad &\rightarrow \quad 3 \quad + \quad 3 \end{aligned} \tag{31}$$

We make a careful, judicious choice of basis,

$$\vec{I} = \frac{1}{2}(\vec{L} + \vec{M}') \qquad \vec{K} = \frac{1}{2}(\vec{L} - \vec{M}') \tag{32}$$

with,

$$\begin{aligned}
&[I_i, I_j] = i\hbar\epsilon_{ijk} I_k \qquad\qquad [K_i, K_j] = i\hbar\epsilon_{ijk} K_k \\
&[\vec{I}, \vec{K}] = 0 \\
&[\vec{I}, H] = [\vec{K}, H] = 0
\end{aligned} \tag{33}$$

Since it's clear that both $\vec{I}$ and $\vec{K}$ constitute an SU(2) known algebra, it is isomorphic to the group generated by angular momentum in three dimensions. We use our findings in Section 4.2, and immediately arrive at their possible eigenvalues.

$$\begin{aligned}
\vec{I}^{\,2}\psi = i(i+1)\hbar^2\psi \qquad \vec{K}^2\psi = k(k+1)\hbar^2\psi \\
\text{for } i, k = 0, 1/2, 1, ...
\end{aligned} \tag{34}$$

The corresponding Casimir operators, $\vec{I}^{\,2}$ and $\vec{K}^2$, are,

$$\vec{I}^{\,2} = \frac{1}{4}(\vec{L} + \vec{M}')^2 \qquad \vec{K}^2 = \frac{1}{4}(\vec{L} - \vec{M}')^2 \tag{35}$$

It's important to remember that Casimir operators are in a category distinct from generators. They are functions of generators and though they commute with all the generators of a group, are not considered to be in the set of commuting generators itself. A linear combination of these operators is automatically diagonal in the basis of their common eigenvectors. We construct operators $C_1 = \vec{I}^{\,2} - \vec{K}^2$ and $C_2 = \vec{I}^{\,2} + \vec{K}^2$ in this way and derive a constraint on our eigenvalues,

$$\begin{aligned}
C_1 &= \vec{I}^{\,2} - \vec{K}^2 \\
&= \frac{1}{4}(\vec{L}^2 + \vec{M}'^{\,2} + \vec{L}\cdot\vec{M}' + \vec{M}'\cdot\vec{L}) - \frac{1}{4}(\vec{L}^2 + \vec{M}'^{\,2} - \vec{L}\cdot\vec{M}' - \vec{L}\cdot\vec{M}') \\
&= \vec{L}\cdot\vec{M}' \\
&= 0
\end{aligned}$$

The last two lines used the relation from Equation 18. This implies $\vec{I}^{\,2} = \vec{K}^2$. We find the eigenvalues for operator $C_2$ with ease using equation 34.

$$\begin{aligned}
C_2 = \vec{I}^{\,2} + \vec{K}^2 = 2\vec{K}^2 \\
C_2\psi = 2k(k+1)\hbar^2\psi
\end{aligned} \tag{36}$$

Combining equations 35, 28, and 18, we derive a constant expression for $C_2$.

$$\begin{aligned} C_2 &= \vec{I}^{\,2} + \vec{K}^2 \\ &= \frac{1}{2}(\vec{L}^2 + \vec{M'}^2) \\ &= \frac{1}{2}(\vec{L}^2 - \frac{\mu}{2E}\vec{M}^2) \\ &= \frac{1}{2}\left(\vec{L}^2 - \frac{\mu}{2E}\left(\frac{2E}{\mu}(\vec{L}^2 + \hbar^2) + \kappa^2\right)\right) \\ &= -\frac{1}{2}\hbar^2 - \frac{\mu\kappa^2}{4E} \end{aligned} \tag{37}$$

Equating 36 and 37 we obtain our permissible energy levels for a hydrogenic atom:

$$\begin{aligned} 2k(k+1)\hbar^2 &= -\frac{1}{2}\hbar^2 - \frac{\mu\kappa^2}{4E} \\ \Rightarrow E_k &= -\frac{\mu\kappa^2}{2\hbar^2(2k+1)^2}, \qquad \text{for k = 0, 1/2, 1,...} \end{aligned} \tag{38}$$

Recalling that $\kappa = Ze^2$ and setting $2k+1 = n$, for $n = 1, 2, 3...$, we recover the wave equation result given at the very beginning of this paper,

$$E_n = -\frac{\mu Z^2 e^4}{2\hbar^2 n^2}, \qquad \text{for } n = 1, 2, 3... \tag{39}$$

### 5.1 Discussion

As further proof of the elegance of this method, not only have we reproduced the hydrogen energy spectrum, the relevant constraints on various quantum numbers have emerged quite naturally as well. For instance, consider $\vec{L} = \vec{I} + \vec{K}$ from the initial definition of $\vec{I}$ and $\vec{K}$. Since $\vec{I} = \vec{K}$, we see that by the triangular rule of vector addition, $L$ can be at most $k + k = n - 1$, and its values extend to $|k - k| = 0$ in integer steps, as expected. In addition, following the algebra of SU(2) groups, $I_z$ nad $K_z$, analogous to $L_z$ would each have $2k + 1 = n$ independent eigenvalues and eigenstates. The total degeneracy of an energy level, $n \times n = n^2$, is also recovered.

## 6 Conclusions

Historically, the method outlined in this paper actually preceded any others in the derivation of the energy levels of the hydrogen atom. It was originally established by

Pauli prior to the proposal of the Schrödinger wave equation. Since then, it has faded out of use in the general quantum mechanics curriculum. However, the incredible physical insights gained en route makes this approach anything but outdated. Not only have we arrived at the desired quantities and constraints, we've additionally discovered and characterized a hidden symmetry specific to the $1/r$ coulomb potential, and deepened our understanding of this physical system, and in turn, many others.


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