Part III

Deformation and Strain

Chapter 4

How Strain Gages Work

Matter *deforms* when subjected to stresses. While stresses relate to momentum conservation (see Chapter 2), deformation relates to change in geometry of a material or structural system. The deformation of an object can be measured by mechanical, optical, acoustical, pneumatic, and electrical means. The task of Engineering Mechanics, or more precisely of the Kinematics of Deformation, is to translate these measurements into physical quantities that can be used to link stresses, forces and moments to deformation, strains and displacements, and ultimately energy transformations. As far as Engineering Mechanics of Solids is concerned, the description of deformation is rather an application of some differential geometrical description tools than an appropriation of physics laws. But without these tools, it would be impossible to go beyond momentum conservation, and to describe how materials and structures deform when subjected to load.

4.1 Strain Measurements

The deformation of an object can be measured by mechanical, optical, acoustical, pneumatic, and electrical means. The earliest devices were mechanical devices that measured the change in length $\Delta\mathcal{L}$ of an object of original length $\mathcal{L}$. The amount of deformation per unit length of the object was coined *strain*:

$$\epsilon = \frac{\Delta\mathcal{L}}{\mathcal{L}} \tag{4.1}$$

For example, the extension meter (extensometer) uses a series of levers to amplify strain to a readable value. In general, however, mechanical devices tend to provide low resolutions, and are bulky and difficult to use. Optical sensors are sensitive and accurate, but are delicate and not very popular in industrial applications. They use interference fringes produced by optical flats to measure strain. Optical sensors operate best under laboratory conditions.

The most widely used characteristic that varies in proportion to strain is electrical resistance. It was Lord Kelvin who first reported in 1856 that metallic conductors subjected to mechanical strain exhibit a change in their electrical resistance R. This phenomenon was first put to practical use in the 1930s. The first bonded, metallic wire-type strain gage was developed in 1938. The metallic foil-type strain gage consists of a grid of wire filament (a resistor) of approximately 0.025 mm thickness, bonded directly to the strained surface by a thin layer of epoxy resin (Fig. 4.1). When the surface deforms, for instance in response to an applied load, the foil diaphragm and the adhesive bonding agent deform as well, and with them the embedded wire filaments. The change in length $\Delta\mathcal{L}$ of the wire filaments also induces a change in the wire's cross-sectional area. This change in cross-sectional area affects the amount of electrical charge that can flow through the wire. But if connected to an electrical circuit to which, from an outside source, a constant electrical current intensity I (which is the amount of electrical charge per unit time) is applied, so that the amount of electrical charge that flows per unit time through the wire remains the same, then the drift speed of the electrons must increase or decrease. As a consequence, the measurable input - output voltage changes ΔU , which is translated by means of Ohm's Law, $U = RI$, into a measurable change of the electrical resistance ΔR of the wire. Finally, the change in resistance is translated into strain, by means of a calibrated strain sensitivity factor of the wire's resistance, also called Gage Factor:

$$GF = \frac{\Delta R}{R} \times \left(\frac{\Delta\mathcal{L}}{\mathcal{L}}\right)^{-1} \tag{4.2}$$

Each strain gage wire material has its characteristic Gage Factor, resistance, and sensitivity to change in environmental conditions[1]. Typical materials include Constantan (copper-nickel alloy), Nichrome V (nickel-chrome alloy),

[1] The ideal strain gage would change resistance only due to the deformation of the surface to which the sensor is attached. However, in real applications, temperature, material properties, the adhesive that bonds the gage to the surface, and the stability of the metal all affect the detected resistance. Therefore, when selecting a strain gage, one must consider not only the strain characteristics of the sensor, but also its stability and temper-

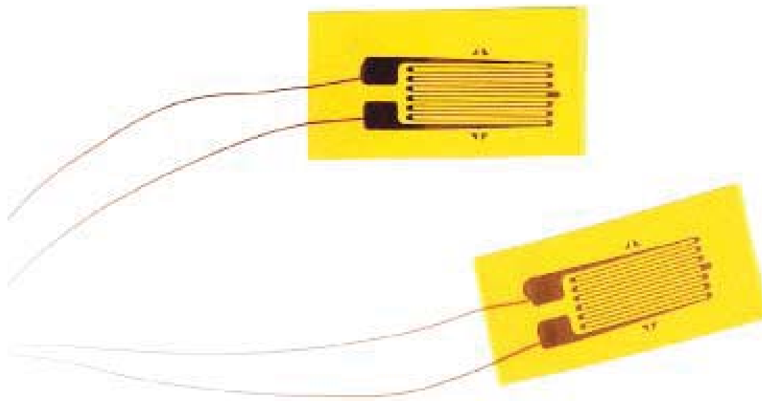

Figure 4.1: Typical metal-foil strain gages.

platinum alloys (usually tungsten), Isoelastic (nickel-iron alloy), or Karma-type alloy wires (nickel-chrome alloy), foils, or semiconductor materials. The most popular alloys used for strain gages are copper-nickel alloys and nickel-chromium alloys.

4.1.1 Electricity Background: 'Ohmic' Material

Let us remind us of some basics of electricity theory. The electric current I quantifies the flux of electrical charge through a surface A oriented by unit normal $\vec{n}$:

$$I = jA; \; j = Ne\, v_D \tag{4.3}$$

where j is the free electron density of N charge carriers (electrons) per volume of electron charge $e = 1.60217733 \times 10^{-19}\,\mathrm{C}$ (Coulomb) subjected to a drift velocity $v_D = \vec{v}_D \cdot \vec{n}$ in the length direction of the conductor. When electric current I is proportional to voltage U, the material is said to be 'ohmic', or to obey Ohm's Law, $U = R\,I$, where R is the resistance, which can be

ature sensitivity. Unfortunately, the most desirable strain gage materials are also sensitive to temperature variations and tend to change resistance as they age. For tests of short duration, this may not be a serious concern, but for continuous industrial measurement, one must include temperature and drift compensation.

expressed in terms of the conductors *resistivity* ϱ as:

$$R = \varrho \frac{\mathcal{L}}{A} \tag{4.4}$$

At the material scale of the conductor, Ohm's law relates the free electron density linearly to the gradient of the electric potential difference $U/\mathcal{L}$:

$$j = \frac{1}{\varrho} \frac{U}{\mathcal{L}} = \varkappa \frac{U}{\mathcal{L}} \tag{4.5}$$

where $\varkappa = 1/\varrho$ is the specific electric conductance of the material. It is a material property. Equation (4.5) constitutes a constitutive law for charge flow.

4.1.2 Dimensional Analysis: An Additional Base Dimension

To get a good feel for strain gage measurements based on the change of electrical resistance, it is instructive to perform a dimensional analysis of the electrical charge flow through a conductor. From a dimensional point of view, electromagnetic problems require the consideration of an additional base dimension that captures the physics problem at stake: a number of free electrons that have dissociated from their atomic structure, transport electrical charge. Electrical charge which is able to do work and change energy, is not of mechanical origin, which is why one needs to consider an additional base dimension[2]. In the SI system, this additional base dimension

[2] The introduction of an additional base dimension for electromagnetic problems is credited to the French Engineer Aimée Vaschy (1857–1899). In his '*Traité d'électricité et de magnetisme: théorie et applications : instruments et méthodes de mesure électrique : cours professé a l'École Supérieure de Télégraphie*', dated 1890, Vaschy applied for the first time the homogeneity principle of dimensional analyis to the problem of the telegraph line, following the formulation adopted by Lord Kelvin in his mathematical solution presented during the Atlantic Line project. Claiming for the necessity of an additional base dimension in electromagnetic problems, Vaschy states that 'the homogeneity principle will become more valuable than if there were only three fundamental units L, M and T'; thus recognizing the inadmissibility of a deficient base dimension system for dimensional analysis. On this basis, Vaschy made a dimensional analysis of the telegraph line problem. This was in 1890, 24 years before Buckingham in 1914 published his famous paper on the Pi-Theorem (see Section 1.1.3). The Pi-Theorem is, therefore, often referred to as Vaschy-Buckingham Theorem.

is the current (in unit of Ampere (A); see also Box 1.1):

$$\boxed{[q] = L^{\alpha} M^{\beta} T^{\gamma} I^{\delta}} \tag{4.6}$$

In this base dimension system, the electrical current intensity has dimension function $[I] = I$; the free electron density has dimension $[j] = L^{-2}I$. The voltage U is the electrical potential difference, and has dimension of energy per unit time (that is, *power*, in unit of Watt, $\mathrm{W} = \mathrm{kg\,m^2\,s^{-3}} = \mathrm{N\,m\,s^{-1}}$) and per unit of current intensity:

$$[U] = \frac{[E]}{T} I^{-1} = L^2 M T^{-3} I^{-1} \tag{4.7}$$

From Ohm's Law, $U = RI$, it is readily recognized that the dimension function of the electrical resistance (in unit of Ohm $\Omega = \mathrm{W\,A^{-1}}$) is:

$$[R] = \frac{[U]}{[I]} = L^2 M T^{-3} I^{-2} \tag{4.8}$$

Finally, the dimension function of the material's resistivity ϱ respectively of its inverse, the conductance is:

$$[\varrho] = L^3 M T^{-3} I^{-2};\ [\varkappa] = L^{-3} M^{-1} T^3 I^2 \tag{4.9}$$

4.1.3 D-Analysis of a Strain Gage Wire in a Circuit

Let us now consider a strain gage wire in a circuit, to which a constant current I is applied. Given that the wire length is typically much longer than the wire's section dimension, we will adopt a distorted length base dimension system (see Section 1.3), in which $L_{(x)}$ stands for the length dimension in the wire direction, and $L_{(S)}$ for the section's length dimension; hence, instead of (4.6):

$$\boxed{[q] = L_{(x)}^{\alpha_x} L_{(S)}^{\alpha_S} M^{\beta} T^{\gamma} I^{\delta}} \tag{4.10}$$

In this extended base dimension system, $[\mathcal{L}] = L_{(x)}$, while the dimension function of the free electron density is $[j] = L_{(S)}^{-2} I$. Similarly, the dimension function of the material's specific conductance is:

$$[\varkappa] = L_{(x)}^{-1} L_{(S)}^{-2} M^{-1} T^3 I^2 \tag{4.11}$$

It follows then from the constitutive law (4.5), that the voltage has the following dimension function:

$$[U] = \frac{[j]\,[\mathcal{L}]}{[\varkappa]} = L^2_{(x)} M T^{-3} I^{-1} \tag{4.12}$$

Finally, from Ohm's Law $U = R\,I$, the dimension function of the electrical resistance is:

$$[R] = \frac{[U]}{[I]} = L^2_{(x)} M T^{-3} I^{-2} \tag{4.13}$$

Therefore, dimensional analysis reveals that the resistance, in response to an increase of the wire's length, should scale with the length square, and not linearly as suggested by the strain factor relation (4.2). This motivates a refined analysis of the deformation. As we shall see, the linear relation between change in resistance and strain is valid only for small strains within the context of a linear deformation theory.

4.2 Finite Deformation Theory

We consider the strain gage wire as an infinitessimal *material vector* that is attached to the surface of an r.e.v. with which it deforms. We remind ourselves (see Section **??**) that the r.e.v. is the elementary material point of a continuum material system. In the undeformed configuration, the wire is located by the position vector $\vec{X} = X_\alpha \vec{e}_\alpha$ in a global coordinate system in an orthogonal frame defined by the base vectors $\vec{e}_\alpha$ ($\alpha = 1, 2, 3$). After deformation, the same material point is located by the position vector $\vec{x} = x_i \vec{e}_i$, and the deformation is described by (Fig. 4.2):

$$\vec{x} = \vec{x}\left(\vec{X}, t\right) \tag{4.14}$$

The time t in (4.14) indicates that $\vec{x}$ is defined in the *current*, *ie.* deformed configuration.

4.2.1 Deformation Gradient

During the deformation, the material vector attached to the surface of the r.e.v. has deformed. If we denote by $d\vec{X}$ and $d\vec{x}$ the same material vector

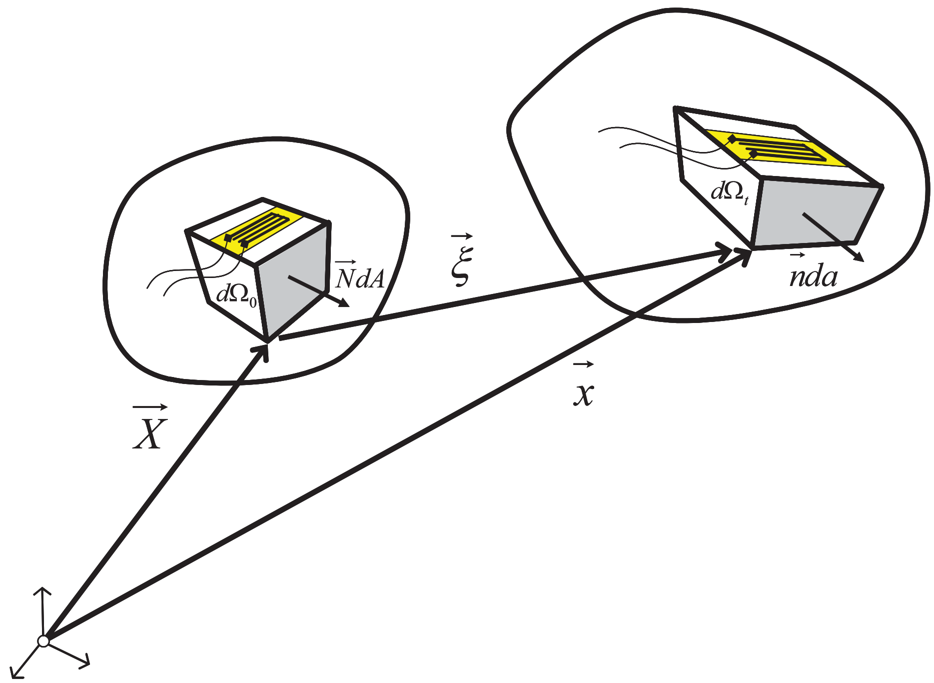

Figure 4.2: Description of deformation: a material vector (strain gage) attached to an r.e.v. is transported and transformed into the deformed configuration.

(strain gage) in the underformed and the deformed configuration, respectively, Eq. (4.14) immediately suggests that the deformation of the material vector $d\vec{X} \to d\vec{x}$ is described by:

$$\boxed{d\vec{x} = \frac{\partial \vec{x}}{\partial \vec{X}} \cdot d\vec{X} = \mathbf{F} \cdot d\vec{X}} \tag{4.15}$$

The second order tensor $\mathbf{F}$ that transports and transforms the material vector from the undeformed configuration into the deformed configuration, is called *deformation gradient*:

$$\mathbf{F} = \operatorname{Grad} \vec{x} = F_{i\alpha} \vec{e}_i \otimes \vec{e}_\alpha \tag{4.16}$$

In a Cartesian coordinate system, the components of the deformation gradient $\mathbf{F}$ are $F_{i\alpha} = \partial x_i / \partial X_\alpha$; while one would need to consider, for other coordinate systems, the appropriate expressions of the gradient operator of a vector field (Box 4.1). Furthermore, instead of using the position vectors, we could have also identified the position of the r.e.v. in the deformed configuration by the displacement vector, defined by (Fig. 4.2):

$$\vec{\xi} = \vec{x} - \vec{X} \tag{4.17}$$

In this case, the expression of the deformation gradient is readily obtained from a substitution of (4.17) in (4.16):

$$\mathbf{F} = \operatorname{Grad}\left(\vec{X} + \vec{\xi}\right) = \mathbf{1} + \operatorname{Grad} \vec{\xi} \tag{4.18}$$

4.2.2 Volume and Surface Transport Formulas

As a first application of the deformation gradient, let us consider the volume variation of the r.e.v. from its initial volume $d\Omega_0$ into its deformed configuration $d\Omega_t$. The volume transport is described by the so-called Jacobian of deformation; that is:

$$\boxed{J = \frac{d\Omega_t}{d\Omega_0} = \det \mathbf{F}} \tag{4.19}$$

It is readily understood that $J > 0$, which ensures that the deformation gradient can be inverted; that is:

$$d\vec{X} = \mathbf{F}^{-1} \cdot d\vec{x} \tag{4.20}$$

Grad **operator of a vector field** $\vec{\xi}$ **in Cylinder and Spherical Co-ordinates**
Cylinder Coordinates (r, θ, z): The vector field $\vec{\xi}$ is defined by the components ξ_r, ξ_θ and ξ_z in a local orthonormal basis $(\vec{e}_r, \vec{e}_\theta, \vec{e}_z)$. In this base, $$\text{Grad}\,\vec{\xi} = \nabla\vec{\xi} = \begin{bmatrix} \frac{\partial \xi_r}{\partial r} & \frac{1}{r}\left(\frac{\partial \xi_r}{\partial \theta} - \xi_\theta\right) & \frac{\partial \xi_r}{\partial z} \\ \frac{\partial \xi_\theta}{\partial r} & \frac{1}{r}\left(\frac{\partial \xi_\theta}{\partial \theta} + \xi_r\right) & \frac{\partial \xi_\theta}{\partial z} \\ \frac{\partial \xi_z}{\partial r} & \frac{1}{r}\frac{\partial \xi_z}{\partial \theta} & \frac{\partial \xi_z}{\partial z} \end{bmatrix}$$
Spherical Coordinates (r, θ, φ): The vector field $\vec{\xi}$ is defined by the components ξ_r, ξ_θ and ξ_φ in a local orthonormal basis $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi)$. In this base, $$\text{Grad}\,\vec{\xi} = \nabla\vec{\xi} = \begin{bmatrix} \frac{\partial \xi_r}{\partial r} & \frac{1}{r}\left(\frac{\partial \xi_r}{\partial \theta} - \xi_\theta\right) & \frac{1}{r}\left(\sin\theta\frac{\partial \xi_r}{\partial \varphi} - \xi_\varphi\right) \\ \frac{\partial \xi_\theta}{\partial r} & \frac{1}{r}\left(\frac{\partial \xi_\theta}{\partial \theta} + \xi_r\right) & \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial \xi_\theta}{\partial \varphi} - \xi_\varphi \cot\theta\right) \\ \frac{\partial \xi_\varphi}{\partial r} & \frac{1}{r}\frac{\partial \xi_\varphi}{\partial \theta} & \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial \xi_\varphi}{\partial \varphi} - \xi_\theta \cot\theta + \xi_r\right) \end{bmatrix}$$

Table 4.1: Background: Gradient operator of a vector field in cylinder and spherical coordinates.

Furthermore, during deformation the infinitessimal surface changes both in nominal area, $dA \rightarrow da$, and in orientation, $\vec{N} \rightarrow \vec{n}$ (Fig. 4.2). To describe this surface deformation, we start out with a volume composed of the oriented surface, $\vec{N}dA$ and a 'thickness' material vector attached to the r.e.v., $d\Omega_0 = d\vec{U} \cdot \vec{N}dA$. After deformation, the oriented surface is $\vec{n}\,da$, and the material vector is $d\vec{u}$, so that the volume is $d\Omega_t = d\vec{u} \cdot \vec{n}\,da$. If we now apply the transport formula (4.15) to $d\vec{u} = \mathbf{F} \cdot d\vec{U} = d\vec{U} \cdot \mathbf{F}^T$, where superscript T stands for transpose, and substitute the result in (4.19), we obtain:

$$d\vec{U} \cdot \mathbf{F}^T \cdot \vec{n}\,da = J d\vec{U} \cdot \vec{N}dA \tag{4.21}$$

The demonstration turns out to hold for any material vector $d\vec{U}$; whence the transport formula of an oriented surface:

$$\boxed{\vec{n}\,da = J\left(\mathbf{F}^T\right)^{-1} \cdot \vec{N}dA} \tag{4.22}$$

4.2.3 Length Dilation, Angle Distortion and Strain Tensor

During deformation, the material vector (strain gage) attached to the surface undergoes a length change, $\mathcal{L}_0 \rightarrow \mathcal{L}_t = (1+\lambda)\,\mathcal{L}_0$, where $\lambda = (\mathcal{L}_t - \mathcal{L}_0)/\mathcal{L}_0$ is the change in length normalized by the initial length, which is also called *length dilation*. The length variation of a vector is best captured by the scalar product:

$$\mathcal{L}_t^2 - \mathcal{L}_0^2 = d\vec{x} \cdot d\vec{x} - d\vec{X} \cdot d\vec{X} \tag{4.23}$$

Use of (4.15) in (4.23) yields:

$$\begin{aligned} \mathcal{L}_t^2 - \mathcal{L}_0^2 &= \mathbf{F} \cdot d\vec{X} \cdot \mathbf{F} \cdot d\vec{X} - d\vec{X} \cdot d\vec{X} \\ &= d\vec{X} \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot d\vec{X} - d\vec{X} \cdot d\vec{X} \\ &= d\vec{X} \cdot \left(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{1}\right) \cdot d\vec{X} \\ &= d\vec{X} \cdot 2\mathbf{E} \cdot d\vec{X} \end{aligned} \tag{4.24}$$

The length variation turns out to be entirely defined by the symmetric second order tensor $\mathbf{E}$, which is called *strain tensor*:

$$\boxed{\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{1}\right)} \tag{4.25}$$

or equivalently, using (4.18):

$$\boxed{\mathbf{E} = \frac{1}{2}\left(\operatorname{Grad}\vec{\xi} + \left(\operatorname{Grad}\vec{\xi}\right)^T + \left(\operatorname{Grad}\vec{\xi}\right)^T \cdot \operatorname{Grad}\vec{\xi}\right)} \tag{4.26}$$

Consider that the material vector (strain gage) in its undeformed configuration is defined as $d\vec{X} = \mathcal{L}_0 \vec{e}_\alpha$, where $\mathcal{L}_0 = \left\| d\vec{X} \right\|$ is the length in the undeformed configuration. Relation (4.24) provides a means to determine the relative length variation in the α direction, $\lambda(\vec{e}_\alpha)$:

$$\begin{gathered} \mathcal{L}_t^2 - \mathcal{L}_0^2 = \mathcal{L}_0^2\, \vec{e}_\alpha \cdot 2\mathbf{E} \cdot \vec{e}_\alpha = 2\mathcal{L}_0^2 E_{\alpha\alpha} \\ \Downarrow \\ \boxed{\lambda(\vec{e}_\alpha) = \frac{\Delta\mathcal{L}}{\mathcal{L}_0} = \sqrt{2E_{\alpha\alpha} + 1} - 1} \end{gathered} \tag{4.27}$$

where $E_{\alpha\alpha} = \vec{e}_\alpha \cdot \mathbf{E} \cdot \vec{e}_\alpha$ is the diagonal term of the strain tensor. Eqn. (4.26) and (4.27) reveal that the link between strain and length dilation is *a priori* non linear.

A second mode of deformation of the surface to which the material vector is attached, is the distortion. Distortion here means a change in angle. To illustrate this deformation consider two material vectors (or strain gages), $d\vec{X}_\alpha = dX_\alpha \vec{e}_\alpha$ and $d\vec{X}_\beta = dX_\beta \vec{e}_\beta$, which are perpendicular to each other, that is, the scalar product is zero, $\vec{e}_\alpha \cdot \vec{e}_\beta = 0 \Rightarrow d\vec{X}_\alpha \cdot d\vec{X}_\beta = 0$. This is no more the case after deformation, and the change in angle from 90° is called *distortion*. To evaluate the distortion, we recall, from vector algebra, that:

$$d\vec{x}_\alpha \cdot d\vec{x}_\beta = \|d\vec{x}_\alpha\|\, \|d\vec{x}_\beta\| \cos\left(\frac{\pi}{2} - \theta(\vec{e}_\alpha, \vec{e}_\beta)\right) = \|d\vec{x}_\alpha\|\, \|d\vec{x}_\beta\| \sin\theta(\vec{e}_\alpha, \vec{e}_\beta) \tag{4.28}$$

Now, if we apply to both material vectors in the deformed configuration the transport formula (4.15), that is, $d\vec{x}_\alpha = \mathbf{F} \cdot d\vec{X}_\alpha$ and $d\vec{x}_\beta = \mathbf{F} \cdot d\vec{X}_\beta$, and substitute these relations into (4.28), we readily obtain:

$$\begin{aligned}\sin\theta\left(\vec{e}_\alpha,\vec{e}_\beta\right) &= \frac{d\vec{X}_\alpha\cdot\mathbf{F}^T\cdot\mathbf{F}\cdot d\vec{X}_\beta}{\sqrt{\left(d\vec{X}_\alpha\cdot\mathbf{F}^T\cdot\mathbf{F}\cdot d\vec{X}_\alpha\right)\left(d\vec{X}_\beta\cdot\mathbf{F}^T\cdot\mathbf{F}\cdot d\vec{X}_\beta\right)}}\\ &= \frac{d\vec{X}_\alpha\cdot(2\mathbf{E}+\mathbf{1})\cdot d\vec{X}_\beta}{\sqrt{\left(d\vec{X}_\alpha\cdot(2\mathbf{E}+\mathbf{1})\cdot d\vec{X}_\alpha\right)\left(d\vec{X}_\beta\cdot(2\mathbf{E}+\mathbf{1})\cdot d\vec{X}_\beta\right)}}\\ &= \frac{d\vec{X}_\alpha\cdot(2\mathbf{E})\cdot d\vec{X}_\beta+d\vec{X}_\alpha\cdot d\vec{X}_\beta}{\left(1+\lambda\left(\vec{e}_\alpha\right)\right)\left\|d\vec{X}_\alpha\right\|\left(1+\lambda\left(\vec{e}_\beta\right)\right)\left\|d\vec{X}_\beta\right\|} \qquad (4.29)\end{aligned}$$

where we employed (4.24) and (4.27). Finally, if we remind ourselves that $d\vec{X}_\alpha\cdot d\vec{X}_\beta=0$, we obtain:

$$\boxed{\sin\theta\left(\vec{e}_\alpha,\vec{e}_\beta\right)=\frac{2E_{\alpha\beta}}{(1+\lambda\left(\vec{e}_\alpha\right))(1+\lambda\left(\vec{e}_\beta\right))}=\frac{2E_{\alpha\beta}}{\sqrt{(1+2E_{\alpha\alpha})(1+2E_{\beta\beta})}}} \qquad (4.30)$$

where $E_{\alpha\beta}=\vec{e}_\alpha\cdot\mathbf{E}\cdot\vec{e}_\beta$ are the out-of-diagonal terms of the strain tensor.

4.3 Linear Deformation Theory

The deformation theory so far developed holds for any finite deformation of continuum material systems, and it is readily recognized to be non linear. It is important to note that all differential operations here before defined are carried out w.r.t. the (known) undeformed configuration; that is, $\text{Grad}\,\vec{\xi}=\partial\vec{\xi}/\partial\vec{X}$, which we symbolize by the capital 'G' in the employed Grad operators. In contrast, if the differential operation is carried out w.r.t. the deformed configuration, we use a small 'g' in the grad operator. For any quantity (.), the two are related by:

$$\text{Grad}\,(.)=\frac{\partial\,(.)}{\partial\vec{x}}\cdot\frac{\partial\vec{x}}{\partial\vec{X}}=\text{grad}\,(.)\cdot\mathbf{F} \qquad (4.31)$$

From (4.18) and (4.31) it is readily recognized that the difference between the two operators is on the order of the magnitude of the displacement gradient $\text{Grad}\,\vec{\xi}$. Hence, provided that the norm of the displacement gradient is much smaller than unity, one could merge –as far as the differential operation is concerned– the undeformed and the deformed configuration. This constitutes the basis of a *linear* deformation theory; that is:

$$\boxed{\text{Linear: } \left\|\text{Grad}\,\vec{\xi}\right\| \ll 1 \Rightarrow \text{Grad}\,(.) \simeq \text{grad}\,(.)} \tag{4.32}$$

A second implication of the infinitessimal order of the displacement gradient is that the strains are infinitessimal as well. Indeed, the quadratic terms in (4.26) can be neglected w.r.t. the linear terms. The result is the *linearized strain tensor*:

$$\boxed{\text{Linear: } \left\|\text{Grad}\,\vec{\xi}\right\| \ll 1 \Rightarrow \|\mathbf{E}\| \ll 1 \Rightarrow \mathbf{E} \simeq \boldsymbol{\varepsilon} = \frac{1}{2}\left(\text{grad}\,\vec{\xi} + \left(\text{grad}\,\vec{\xi}\right)^T\right)} \tag{4.33}$$

In a Cartesian coordinate system, the linearized strain tensor reads:

$$\boldsymbol{\varepsilon} = \epsilon_{ij}\vec{e}_i \otimes \vec{e}_j;\ \epsilon_{ij} = \frac{1}{2}\left(\frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i}\right) \tag{4.34}$$

The linearized form of the strain tensor immediately implies that the length dilations λ are infinitessimal as well. Indeed, the infinitessimal order of the strain allows us to develop the square root in (4.27) in a power series; *ie.* $\sqrt{2E_{ii}+1} = 1 + E_{ii} + O\left(E_{ii}^2\right)$; thus:

$$\boxed{\text{Linear: } \lambda\left(\vec{e}_i\right) = \frac{\Delta\mathcal{L}}{\mathcal{L}_0} = \vec{e}_i \cdot \boldsymbol{\varepsilon} \cdot \vec{e}_i} \tag{4.35}$$

Similarly, given the infinitessimal order of the strain, the distortion as defined by (4.30) is infinitessimal as well. Then, a power series development of $\sin\theta = \theta + O\left(\theta^2\right)$ readily shows that the half-angle variation $\theta\left(\vec{e}_i, \vec{e}_j\right)/2$ in the linearized deformation theory is equal to the out-of-diagonal terms of the linearized strain tensor:

$$\boxed{\text{Linear: } \theta\left(\vec{e}_i, \vec{e}_j\right) = \vec{e}_i \cdot (2\boldsymbol{\varepsilon}) \cdot \vec{e}_j;\ i \neq j} \tag{4.36}$$

Hence, in contrast to the finite deformation theory, the components of the linearized strain tensor are directly linked to measurable length dilations (the diagonal terms) and measurable half- distortions (out of diagonal terms):

$$[\epsilon_{ij}] = \begin{array}{c|ccc} & (\overrightarrow{e}_1) & (\overrightarrow{e}_2) & (\overrightarrow{e}_3) \\ \hline (\overrightarrow{e}_1) & \lambda\left(\vec{e}_1\right) & & \text{sym} \\ (\overrightarrow{e}_2) & \gamma\left(\vec{e}_1, \vec{e}_2\right) & \lambda\left(\vec{e}_2\right) & \\ (\overrightarrow{e}_3) & \gamma\left(\vec{e}_1, \vec{e}_3\right) & \gamma\left(\vec{e}_2, \vec{e}_3\right) & \lambda\left(\vec{e}_3\right) \end{array} \tag{4.37}$$

where, to simplify the notation, we denote by $\gamma\left(\vec{e}_i, \vec{e}_j\right) = \frac{1}{2}\theta\left(\vec{e}_i, \vec{e}_j\right)$ the half-distortions.

Finally, if we develop $J = \det \mathbf{F}$ of the volume transport formula (4.19), we obtain:

$$J = \frac{d\Omega_t}{d\Omega_0} = \det(\mathbf{1} + \operatorname{grad}\vec{\xi}) \simeq 1 + \frac{\partial \xi_1}{\partial x_1} + \frac{\partial \xi_2}{\partial x_2} + \frac{\partial \xi_3}{\partial x_3} + ... = 1 + \operatorname{div}\vec{\xi} \quad (4.38)$$

or equivalently, making use of the geometrical significance of the components of ε_{ij}:

$$\boxed{\text{Linear: } J - 1 = \frac{d\Omega_t - d\Omega_0}{d\Omega_0} \simeq \operatorname{tr}\boldsymbol{\varepsilon} = \lambda\left(\vec{e}_1\right) + \lambda\left(\vec{e}_3\right) + \lambda\left(\vec{e}_3\right)} \quad (4.39)$$

where div is the divergence operator, and 'tr' is the trace of the strain matrix, that is the sum of the diagonal terms.

Last, a similar straightforward linearization is not possible for the transport formula of an oriented surface (4.22). In fact, on the basis of (4.32), the transport formula of a material vector (4.15) can be written as:

$$d\vec{u} = \mathbf{F} \cdot d\vec{U} \simeq \left(\mathbf{1} + \operatorname{grad}\vec{\xi}\right) \cdot d\vec{U} \quad (4.40)$$

and its inverse as:

$$d\vec{U} = \mathbf{F}^{-1} \cdot d\vec{u} = d\vec{u} - \operatorname{grad}\vec{\xi} \cdot d\vec{U} \simeq \left(\mathbf{1} - \operatorname{grad}\vec{\xi}\right) \cdot d\vec{u} \quad (4.41)$$

Use of (4.40) in (4.21) yields with the help of (4.39) and (4.41) the linearized version of the surface transport formula (4.22):

$$\boxed{\text{Linear: } \vec{n}\, da \simeq (1 + \operatorname{tr}\boldsymbol{\varepsilon}) \left(\mathbf{1} - \left(\operatorname{grad}\vec{\xi}\right)^T\right) \cdot \vec{N} dA} \quad (4.42)$$

4.4 Return to Strain Gage Measurement

We have now the tools in hand to address the question raised in this Chapter of 'how strain gages work'. Subjected to the deformation of the surface to which the strain gage is attached, the wire in the strain gage deforms. For purpose of illustration, let us consider that the wire is of cylindrical shape oriented in the z direction. It is a good assumption to consider that the

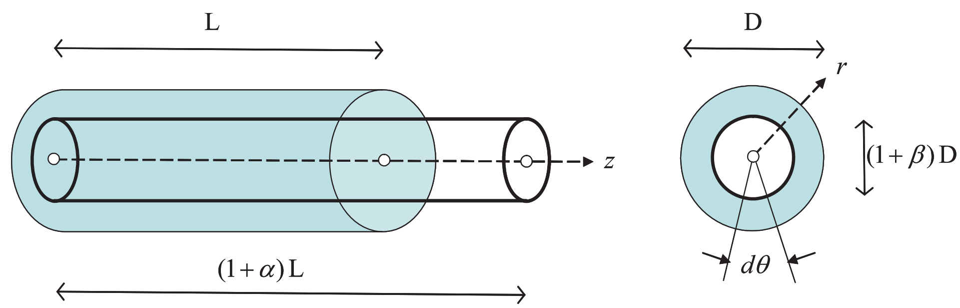

Figure 4.3: Assumed Deformation of a strain gage wire.

wire only deforms in length and radius. This is described by the following displacement field (Fig. 4.3):

$$\vec{\xi}\,(z,r) = \alpha z \vec{e}_z + \beta r \vec{e}_r \tag{4.43}$$

Let us note that the displacement is given in cylinder coordinates of orthonormal basis $(\vec{e}_r, \vec{e}_\theta, \vec{e}_z)$. The basis $\vec{e}_r, \vec{e}_\theta$ transform when spatial derivatives are applied, as required when determining *e.g.* the displacement gradient Grad $\vec{\xi}$ (see Box 4.1). However, we can circumvent these specific derivation rules for curvilinear coordinate systems, by exploring the geometry of the deformation. The deformation which is sketched in figure 4.3, shows that there are no angle distortions in the problem, *ie.* $\sin\theta\,(\vec{e}_i, \vec{e}_j) = 0$, but only length dilations $\lambda\,(\vec{e}_i)$. The two obvious ones are the length variations in the z and r direction which are scaled by the constants α and β, respectively; hence:

$$\lambda\,(\vec{e}_z) = \alpha;\ \lambda\,(\vec{e}_r) = \beta \tag{4.44}$$

Less obvious is the length change that occurs in the hoop-direction, which comes from a change of the circumferential arch-length. If we note that the arc length before deformation, $\mathcal{L}_\theta = r d\theta$, becomes $\ell_\theta = (r + \beta r)\, d\theta$ after deformation, we readily obtain the circumferential length dilation from:

$$\lambda\,(\vec{e}_\theta) = \frac{\ell_\theta - \mathcal{L}_\theta}{\mathcal{L}_\theta} = \beta \tag{4.45}$$

4.4.1 Finite Deformation Theory

To start with, we consider the problem in the framework of the finite deformation theory. In this case, the length dilations are related to the diagonal terms of the strain tensor by (4.27); that is:

$$E_{\alpha\alpha} = \frac{1}{2}\left(\left(\lambda\left(\vec{e}_\alpha\right)+1\right)^2 - 1\right) \tag{4.46}$$

Using (4.44) and (4.45) in (4.46), we obtain:

$$E_{zz} = \alpha + \frac{1}{2}\alpha^2;\ E_{rr} = E_{\theta\theta} = \beta + \frac{1}{2}\beta^2 \tag{4.47}$$

Furthermore, in the absence of distortions, Eq. (4.30) informs us that all out-of diagonal terms of the strain tensor are zero; hence for our problem:

$$[E_{\alpha\beta}] = \begin{bmatrix} \beta + \frac{1}{2}\beta^2 & & \text{sym} \\ 0 & \beta + \frac{1}{2}\beta^2 & \\ 0 & 0 & \alpha + \frac{1}{2}\alpha^2 \end{bmatrix} \tag{4.48}$$

If we remind us of the definition of the finite strain tensor (4.26), we readily recognize that the displacement gradient reads here:

$$\operatorname{Grad}\vec{\xi} = \beta\left(\vec{e}_r \otimes \vec{e}_r + \vec{e}_\theta \otimes \vec{e}_\theta\right) + \alpha \vec{e}_z \otimes \vec{e}_z \tag{4.49}$$

With $\operatorname{Grad}\vec{\xi}$ in hand, we can determine the deformation gradient from (4.18), and determine eventually the volume change and the change in surface from the transport formulas (4.19) and (4.22), respectively. In particular, the relative volume change is:

$$\begin{aligned} \frac{d\Omega_t - d\Omega_0}{d\Omega_0} &= J - 1 = \det\left(\mathbf{1} + \operatorname{Grad}\vec{\xi}\right) - 1 \\ &= (1+\alpha)(1+\beta)^2 - 1 \end{aligned} \tag{4.50}$$

Next, we evaluate the transport of the oriented surface (4.22) in vector form. We are interested in the surface oriented in the wire axis $\vec{N} = \vec{e}_z$; hence:

$$\begin{aligned} \begin{pmatrix} n_r \\ n_\theta \\ n_z \end{pmatrix} da &= (1+\alpha)(1+\beta)^2 \begin{bmatrix} \frac{1}{1+\beta} & 0 & 0 \\ 0 & \frac{1}{1+\beta} & 0 \\ 0 & 0 & \frac{1}{\alpha+1} \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dA \\ &= (1+\beta)^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dA \end{aligned} \tag{4.51}$$

From (4.51), we readily find that the wire after deformation is still oriented along $\vec{n} = \vec{e}_z$, and that all what changes, is the nominal section,

$$dA \rightarrow da = (1+\beta)^2 \, dA \tag{4.52}$$

4.4.2 Linear Deformation Theory

Let us now consider the linear deformation theory. The hypothesis of small deformation (4.32) can be expressed here as:

$$\left\| \mathrm{Grad}\, \vec{\xi} \right\| = \beta^2 + \alpha^2 \ll 1 \tag{4.53}$$

In this case, the diagonal terms of the linearized strain tensor are the length dilations, and the out-of-diagonal terms representing distortions, are zero:

$$\boldsymbol{\varepsilon} = \beta \left(\vec{e}_r \otimes \vec{e}_r + \vec{e}_\theta \otimes \vec{e}_\theta \right) + \alpha \vec{e}_z \otimes \vec{e}_z \tag{4.54}$$

The volume dilation is given by (4.39):

$$\frac{d\Omega_t - d\Omega_0}{d\Omega_0} \simeq \mathrm{tr}\, \boldsymbol{\varepsilon} = 2\beta + \alpha \tag{4.55}$$

Finally, the transport of the oriented surface is given in the linear theory by (4.42), and reads in vector components:

$$\begin{aligned} \begin{pmatrix} n_r \\ n_\theta \\ n_z \end{pmatrix} da &= (1 + 2\beta + \alpha) \begin{bmatrix} 1-\beta & 0 & 0 \\ 0 & 1-\beta & 0 \\ 0 & 0 & 1-\alpha \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dA \\ &= (1 + 2\beta + \alpha)(1-\alpha) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dA \end{aligned} \tag{4.56}$$

Retaining only the linear terms we find:

$$dA \rightarrow da = (1 + 2\beta) \, dA \tag{4.57}$$

4.4.3 The Gage Factor Revisited

Let us now recall that the resistance in a conductor is scaled by the length over the section of the wire as expressed by (4.4). If we assume the conductors *resistivity* ϱ as constant, the electrical resistance before and after deformation read:

$$R_0 = \varrho \frac{\mathcal{L}_0}{A} \rightarrow R_t = \varrho \frac{\mathcal{L}_t}{a} \tag{4.58}$$

We now employ the results of the deformation theory; that is $\mathcal{L}_t = (1+\alpha)\,\mathcal{L}_0$ and $a = (1+\beta)^2 A$ in the finite deformation theory, and $a = (1+2\beta)\,A$ in the linear deformation theory:

$$\text{Finite: } \frac{R_t}{R_0} = \frac{1+\alpha}{(1+\beta)^2} \text{ } vs. \text{ Linear: } \frac{R_t}{R_0} = \frac{1+\alpha}{1+2\beta} \tag{4.59}$$

Substituting these results in (4.2), we obtain a means to evaluate the Gage Factor as a function of α and β:

$$\text{Finite: } GF = \frac{\alpha - 2\beta - \beta^2}{\alpha\,(1+\beta)^2} \text{ } vs. \text{ Linear: } GF \simeq 1 - \frac{2\beta}{\alpha} \tag{4.60}$$

Both theories yield the same Gage Factor $GF = 1$ for $\beta = 0$, that is, for a wire that only deforms in length. On the other hand, an *incompressible* wire is one whose volume remains constant in the course of deformation; *ie.* $d\Omega_t = d\Omega_0$. The assumption of incompressibility provides a link between the two deformation parameters; *ie.* from (4.50) and (4.55):

$$\begin{gathered} d\Omega_t = d\Omega_0 \\ \Downarrow \\ \left\{ \begin{array}{c} \text{Finite: } \alpha = -\frac{2\beta+\beta^2}{(1+\beta)^2} \\ \Downarrow \\ GF = 2+\alpha \end{array} \right\} \; vs. \; \left\{ \begin{array}{c} \text{Linear: } \alpha = -2\beta \\ \Downarrow \\ GF \simeq 2 \end{array} \right\} \end{gathered} \tag{4.61}$$

Two observations deserve particular attention: (1) the gage factor depends on the link between the deformation parameters, α and β; that is on the wire's constitutive law of deformation; and (2) the gage factor is only constant in a completely linear theory which is linearized in deformation and linear in the constitutive law; *ie.* $\beta/\alpha = \text{const}$. In practice, the deformation and constitutive behavior of gage materials is not nearly so well behaved. This is amply evident in figure 4.4, which is a plot of fractional resistance $\Delta R/R_0$ against 'linear strain', $\alpha = \epsilon = \Delta\mathcal{L}/\mathcal{L}_0$ (that is the wire's length dilation) for common gage materials. In this figure the Gage Factor is simply the slope of

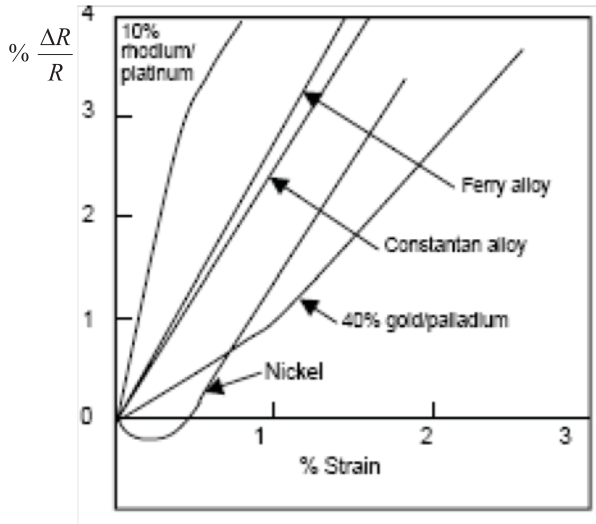

Figure 4.4: Change in resistance with 'linearized' strain for various strain gage element materials (from www). The slope of the curves is the Gage Factor.

the curve. The 10% Rhodium/Platinum alloy exhibits a desirably high $GF\tilde{}8$, but this changes abruptly at about 0.4% strain – an undesirable behavior in all but the most special linear cases. Pure nickel is also poorly behaved and exhibits a negative GF for small strain! This material is seldom used alone but is often employed as an element in other alloys. The most common material for static strain measure at room temperatures is the relatively well behaved constantan alloy, which has a gage factor on the order of 2 for small strains.

In summary, strain gage measurement based on resistance change appears as a reliable tool for small strain deformation. In engineering materials this strain level is typically from 2 to 10,000 microstrain or 0.000002 to 0.01. In this case, the Gage Factor is of order unity and therefore the resistance changes in the gage must be of the same order as the strain changes! Thus, changes in resistance in the gage of no more than 1% must be detected! Herein lies the challenge in the design of measuring circuitry and a problem in practical installations.

4.5 Strain Gage Rosette Measurements

We have seen that strain gage measurements go well along with the linear deformation theory, which we retain in all what follows. So far we were concerned with understanding how strain gages work, so that we can now apply them to the measurement of small strains. At this point, a basic difficulty has appeared: in the linear deformation theory distortions are small as well, so that it is difficult to measure the right-angle distortions with strain gages, as the example below illustrates.

4.5.1 Simple Shear Distortion

Consider a shear distortion of the material surface in the xy plane to which a strain gage is attached:

$$\vec{\xi} = \xi_x\,(y)\,\vec{e}_x = 2\gamma\,y\vec{e}_x \tag{4.62}$$

where $2\gamma = \theta\,(\vec{e}_x, \vec{e}_y)$ is the right-angle distortion of the surface, which is assumed to be small. Without difficulty we determine the strain tensor either analytically from (4.34) or from the geometrical significance of the distortion in the linear deformation theory as expressed by (4.37):

$$\boldsymbol{\varepsilon} = \gamma\,(\vec{e}_x \otimes \vec{e}_y + \vec{e}_y \otimes \vec{e}_x) \tag{4.63}$$

The linearized strain tensor informs us that the strain gage does not experience a linear dilation neither in the x direction, nor in the y direction, which is a first handicap to measure change in electrical resistance. Furthermore, we note that the (linear) volume dilation is zero, $\mathrm{tr}\,\boldsymbol{\varepsilon} = 0$, and that the displacement gradient of (4.62) has only one non-zero component that is:

$$\mathrm{grad}\,\vec{\xi} = \frac{\partial \xi_x}{\partial y}\vec{e}_x \otimes \vec{e}_y = 2\gamma\,\vec{e}_x \otimes \vec{e}_y \tag{4.64}$$

Use of these deformation features in (4.42) provides us with a means to determine the change in cross-section of the strain gage wire oriented e.g. in the $\vec{N} = \vec{e}_x$ direction:

$$\vec{n}\,da \simeq (\mathbf{1} - 2\gamma\,\vec{e}_y \otimes \vec{e}_x) \cdot \vec{e}_x dA = (\vec{e}_x - 2\gamma\,\vec{e}_y)\,dA \tag{4.65}$$

Taking the norm of both sides yields the change in surface area of the wire:

$$da = \sqrt{1 - 4\gamma^2} dA \tag{4.66}$$

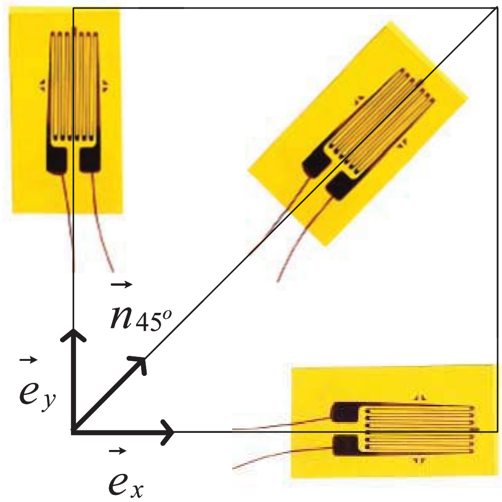

Figure 4.5: Strain Gage Rosette attached to a surface (xy plane).

Given the infinitessimal order of the distortion, $\gamma^2 \ll 1$, it is hardly possible to measure any change in the cross-section of the wire, and as a consequence a change in resistance. Therefore, it is almost impossible to measure infinitessimal distortions by strain gages.

4.5.2 Strain Gage Rosette

A clever idea that circumvents the direct measurement of distortions consists of multiplying length dilation measurements on a surface in different directions. The technology is known as Strain Gage Rosette. The simplest Strain Gage Rosette is one where three strain gages (or material vectors) are attached to the surface: two that are perpendicular to each other, and a third one at an angle of 45° (Fig. 4.5). From the geometrical significance of the strain tensor (see Eq. (4.37)), the two strain gages that are perpendicular to each other give access to the length dilations in the two perpendicular direction, that is, if we make measurements in the x and y directions (Fig. 4.5):

$$\lambda(\vec{e}_x) = \vec{e}_x \cdot \boldsymbol{\varepsilon} \cdot \vec{e}_x = \epsilon_{xx} \tag{4.67a}$$

$$\lambda(\vec{e}_y) = \vec{e}_y \cdot \boldsymbol{\varepsilon} \cdot \vec{e}_y = \epsilon_{yy} \tag{4.67b}$$

In return, the length dilation that is measured in the $\vec{n}_{45^\circ} = \sqrt{2}/2\,(\vec{e}_x + \vec{e}_y)$ direction is given by:

$$\begin{aligned}
\lambda\,(\vec{n}_{45^\circ}) &= \vec{n}_{45^\circ}\cdot\boldsymbol{\varepsilon}\cdot\vec{n}_{45^\circ} \\
&= \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{pmatrix} \begin{bmatrix} \epsilon_{xx} & & \text{sym} \\ \epsilon_{xy} & \epsilon_{yy} & \\ \epsilon_{xz} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix} \\
&= \frac{1}{2}\epsilon_{xx} + \epsilon_{xy} + \frac{1}{2}\epsilon_{yy}
\end{aligned} \tag{4.68}$$

Finally, if we substitute (4.67) in (4.68), we obtain the half-distortion in the plane from the three measured length dilations:

$$\epsilon_{xy} = \gamma\,(\vec{e}_x, \vec{e}_y) = \lambda\,(\vec{n}_{45^\circ}) - \frac{1}{2}\,(\lambda\,(\vec{e}_x) + \lambda\,(\vec{e}_y)) \tag{4.69}$$

For instance, for the single shear distortion example considered in Section 4.5.1, we would have measured with a strain gage rosette, $\lambda\,(\vec{e}_x) = \lambda\,(\vec{e}_y) = 0$ and $\lambda\,(\vec{n}_{45^\circ}) = \gamma$.

4.5.3 Generalization: The Mohr Strain Plane

We cannot conclude this Section on Deformation and Strain Measurements without taking the properties of the linearized strain tensor $\boldsymbol{\varepsilon}$ a step further, and display the 3-D strain state in a Mohr plane. Indeed, the symmetry of the strain tensor $\epsilon_{ij} = \epsilon_{ji}$ is a hallmark of the symmetry of the stress tensor. By analogy with the stress vector, we build a strain vector on a surface oriented by unit normal $\overrightarrow{n}$ (Fig. 4.6(a)):

$$\boxed{\overrightarrow{E}\,(\overrightarrow{n}) = \boldsymbol{\varepsilon}\cdot\overrightarrow{n} = \lambda\vec{n} + \gamma\vec{t} \begin{cases} \lambda = \epsilon_{nn} = \vec{n}\cdot\boldsymbol{\varepsilon}\cdot\vec{n} \\ \\ \gamma = \epsilon_{tn} = \vec{t}\cdot\boldsymbol{\varepsilon}\cdot\vec{n} \end{cases}} \tag{4.70}$$

Furthermore, by analogy with the principle stress definition (3.30), it is readily understood that the strain tensor has three real eigenvalues, the principle strains $\epsilon_I \geq \epsilon_{II} \geq \epsilon_{III}$, associated with the three principle strain directions $\overrightarrow{u}_J$ (Fig. 4.6(b)):

$$\boxed{\overrightarrow{E}\,(\overrightarrow{u}_J) = \boldsymbol{\varepsilon}\cdot\overrightarrow{u}_J = \epsilon_J\overrightarrow{u}_J;\ J = I, II, III} \tag{4.71}$$

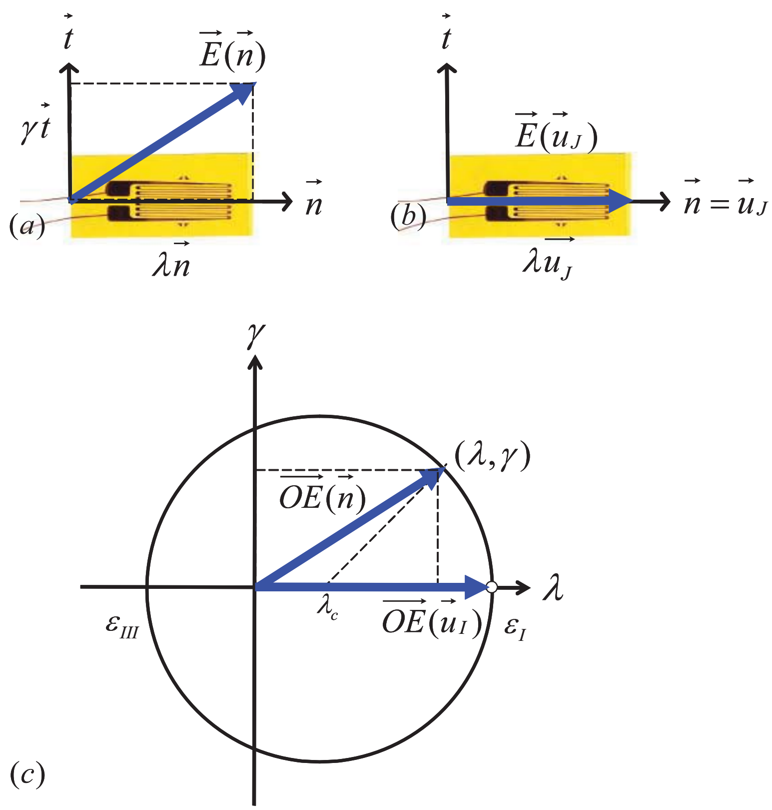

Figure 4.6: Strain vector and strain vector components: (a) length dilation and distortion; (b) only length dilation in the case of principal strain; (c) Mohr circle in the Mohr *strain* plane.

Moreover, the strain tensor permits a similar Mohr plane representation as the stress tensor, in a $\lambda - \gamma$ (ie. length dilation *vs.* half-distortion) parameter form:

$$\boxed{\overrightarrow{E}(\overrightarrow{n}) = \lambda\vec{n} + \gamma\vec{t}\begin{cases}\lambda = \overrightarrow{n}\cdot\overrightarrow{E}(\overrightarrow{n}) = \dfrac{\epsilon_I + \epsilon_{III}}{2} + \dfrac{\epsilon_I - \epsilon_{III}}{2}\cos 2\vartheta \\ \\ \gamma = \overrightarrow{t}\cdot\overrightarrow{E}(\overrightarrow{n}) = \dfrac{\epsilon_I - \epsilon_{III}}{2}\sin(-2\vartheta)\end{cases}} \tag{4.72}$$

where $\epsilon_I, \epsilon_{III}$ are the maximum and the minimum principle strain, and $\vartheta = \vartheta(\overrightarrow{u}_I, \overrightarrow{n})$ is the angle between the principle stress direction $\overrightarrow{u}_I$ and the orientation $\overrightarrow{n}$ of the surface on which the strain vector $\overrightarrow{E}(\overrightarrow{n})$ is built. In the *Mohr Strain plane*, (λ, γ) lie on a circle of center $\lambda_c = \frac{1}{2}(\epsilon_I + \epsilon_{III})$ and radius $\frac{1}{2}(\epsilon_I - \epsilon_{III})$ (Fig. 4.6(c)). While there are three Mohr circles spanning between the different principal strains, *the* Mohr circle is the one which has the greatest radius $\frac{1}{2}(\epsilon_I - \epsilon_{III})$, which represents the largest distortion the solid undergoes.

Example: Single Shear Distortion

To get a good feel for the strain Mohr plane, we reconsider the single shear distortion defined by (4.62), (see Fig. 4.7(a)), for which the strain tensor in an xyz coordinate system reads in matrix form:

$$[\epsilon_{ij}] = \begin{bmatrix} 0 & \epsilon_{xy} & 0 \\ \epsilon_{xy} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.73}$$

We now construct *the* Mohr circle. What we need are two points on the circle. We build a first strain vector on a surface oriented in the x direction (Fig. 4.7(b)):

$$\begin{aligned}\overrightarrow{n}^{(1)} &= \overrightarrow{e}_x; \overrightarrow{t}^{(1)} = \overrightarrow{e}_y : \\ \overrightarrow{E}^{(1)} &= \epsilon_{xy}\left(\overrightarrow{e}_x \otimes \overrightarrow{e}_y + \overrightarrow{e}_y \otimes \overrightarrow{e}_x\right)\cdot\overrightarrow{e}_x = \epsilon_{xy}\overrightarrow{e}_y = \epsilon_{xy}\overrightarrow{t}\end{aligned} \tag{4.74}$$

where we made use of (2.40), that is, $(\overrightarrow{e}_x \otimes \overrightarrow{e}_y)\cdot\overrightarrow{e}_x = 0$ and $(\overrightarrow{e}_y \otimes \overrightarrow{e}_x)\cdot\overrightarrow{e}_y = \overrightarrow{e}_y$. We then build a second strain tensor on a surface oriented in the y direction (Fig. 4.7(c)):

$$\begin{aligned}\overrightarrow{n}^{(2)} &= \overrightarrow{e}_y; \overrightarrow{t}^{(2)} = -\overrightarrow{e}_x : \\ \overrightarrow{E}^{(2)} &= \epsilon_{xy}\left(\overrightarrow{e}_x \otimes \overrightarrow{e}_y + \overrightarrow{e}_y \otimes \overrightarrow{e}_x\right)\cdot\overrightarrow{e}_y = \epsilon_{xy}\overrightarrow{e}_x = -\epsilon_{xy}\overrightarrow{t}\end{aligned} \tag{4.75}$$

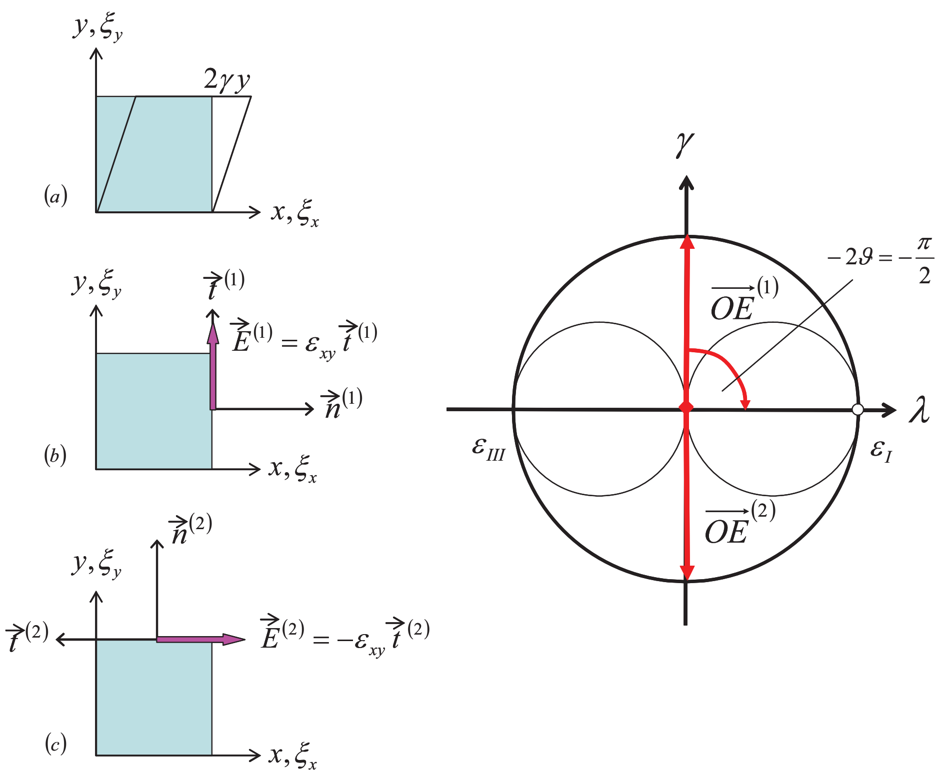

Figure 4.7: Single shear distortion: (a) Displacement field, $(b - c)$ construction of two strain vectors; and corresponding Mohr circle in the Mohr strain plane (λ, γ).

The negative sign comes from the fact that the tangent unit vector in a r.h.s. orthogonal system is oriented here in the $-x$ orientation (see Fig. 4.7(c)). Hence, the half-distortions $\gamma^{(1)} = \epsilon_{xy}$ and $\gamma^{(2)} = -\epsilon_{xy}$ of the two strain vectors $\overrightarrow{E} = \lambda\vec{n} + \gamma\vec{t}$ have opposite signs. In the Mohr strain plane, the two vectors are represented by their Mohr coordinates, $\overrightarrow{OE}^{(1)} = \left(\lambda^{(1)} = 0, \gamma^{(1)} = \epsilon_{xy}\right)$ and $\overrightarrow{OE}^{(2)} = \left(\lambda^{(2)} = 0, \gamma^{(2)} = -\epsilon_{xy}\right)$. These two points define a circle of center $\lambda_c = \frac{1}{2}\left(\lambda^{(1)} + \lambda^{(2)}\right) = 0$ and radius $R = \sqrt{\left(\lambda^{(1)} - \lambda_c\right)^2 + \left(\gamma^{(1)}\right)^2} = |\epsilon_{xy}|$. This Mohr circle construction is sketched in figure 4.7. To conclude, we readily determine the principle strains to be $\epsilon_{I,III} = \lambda_c \pm R = \pm|\epsilon_{xy}|$, and the intermediate principle strain is $\epsilon_I > \epsilon_{II} = 0 > \epsilon_{III}$. Finally, the angle between $\overrightarrow{n}^{(1)} = \overrightarrow{e}_x$ and the principle strain direction in the Mohr plane is $-2\vartheta\left(\overrightarrow{n}^{(1)}, \overrightarrow{u}_I\right) = -\pi/2$; whence in the real space $\vartheta\left(\overrightarrow{n}^{(1)}, \overrightarrow{u}_I\right) = +\pi/4$. This is also displayed in figure 4.7. Consistent with what we have found by strain gage rosette measurements, the principle strain $\epsilon_I = |\epsilon_{xy}|$ would have been measured here as the length dilation $\lambda\left(\vec{n}_{45^\circ}\right)$.

4.6 Beam Deformation Model

By way of application of the linear deformation theory, we investigate the linear deformation of beam elements viewed as dimensionally distorted systems in the sense of (1.37) [have a quick look back to Chapter 1, Section 1.3]: the span length ℓ is much greater than the cross-section dimensions (h, b). Based on this observation, the idea of the description of beam deformation is to define the position of any point situated within a beam section S w.r.t. the position of the beam's axis in both the undeformed and the deformed configuration. The resulting displacement field reads (Fig. 4.8):

$$\vec{\xi} = \vec{x} - \vec{X} = \vec{\xi}^{\,0} + \vec{\xi}^{\,S} \tag{4.76}$$

where $\vec{\xi}^{\,0} = \vec{\xi}\,(x)$ is the displacement vector along the beam's reference axis, while $\vec{\xi}^{\,S}$ is the additional displacement vector of point $\vec{X}_S$ situated within the cross-section S. Within the framework of the linear deformation theory, the linearized strains are recognized to be the sum of the strain state in the beam's axis and the additional strain due to the deformation of the cross-section:

$$\boldsymbol{\varepsilon} = \boldsymbol{\epsilon}^0 + \boldsymbol{\epsilon}^S \tag{4.77}$$

4.6.1 Dimensional Analysis

The displacement field of the beam's reference axis is recognized to depend only on the position x along the beam's axis, defining the section's centroid; that is:

$$\vec{\xi}^0 = \vec{\xi}^0 (x) \tag{4.78}$$

In a Cartesian coordinate system, the associated strains follow from an application of (4.34) to (4.78) entailing the following three non-zero strain components:

$$\epsilon^0_{xx} = \frac{\partial \xi^0_x}{\partial x};\ \epsilon^0_{xy} = \frac{1}{2}\frac{\partial \xi^0_y}{\partial x};\ \epsilon^0_{xz} = \frac{1}{2}\frac{\partial \xi^0_z}{\partial x} \tag{4.79}$$

where ξ^0_x, ξ^0_y and ξ^0_z are the Cartesian components of the displacement vector $\vec{\xi}^0$.

The corresponding strain vector built on the surface oriented along the beam's axis reads:

$$\overrightarrow{E}^0 (\vec{n} = \vec{e}_x) = \boldsymbol{\epsilon}^0 \cdot \overrightarrow{n} = \lambda^0 \vec{n} + \gamma^0 \vec{t} \tag{4.80}$$

where $\lambda^0 (\vec{e}_x) = \epsilon^0_{xx}$ is the length dilation, and $|\gamma^0| = \sqrt{\left(\epsilon^0_{xy}\right)^2 + \left(\epsilon^0_{xz}\right)^2}$ is the half-distortion of the cross-section S. It is instructive to perform a dimensional analysis of the strain components (4.79) in the extended base dimension system (1.37):

$$\left[\lambda^0\right] = \left[\epsilon^0_{xx}\right] = L_x L_x^{-1} = 1 \tag{4.81a}$$

$$\left[\gamma^0_{xy}\right] = \left[\epsilon^0_{xy}\right] = L_y L_x^{-1} = \left[\Lambda_y\right]^{-1} \tag{4.81b}$$

$$\left[\gamma^0_{xz}\right] = \left[\epsilon^0_{xz}\right] = L_z L_x^{-1} = \left[\Lambda_z\right]^{-1} \tag{4.81c}$$

where $\Lambda_y = \ell/b$ and $\Lambda_z = \ell/h$ stand for the beam's slenderness ratio; ℓ being the length of the beam and b, h standing for characteristic cross-section dimensions in the y and z directions. The dimensional analysis suggests that the section's distortions are inversely proportional to the slenderness ratio. Therefore, for large slenderness ratios $\Lambda_{(y,z)} \gg 1$, which characterize beam members, can be neglected w.r.t. to the dominating length dilation. The result is known as the Navier-Bernoulli assumption of beam deformation:

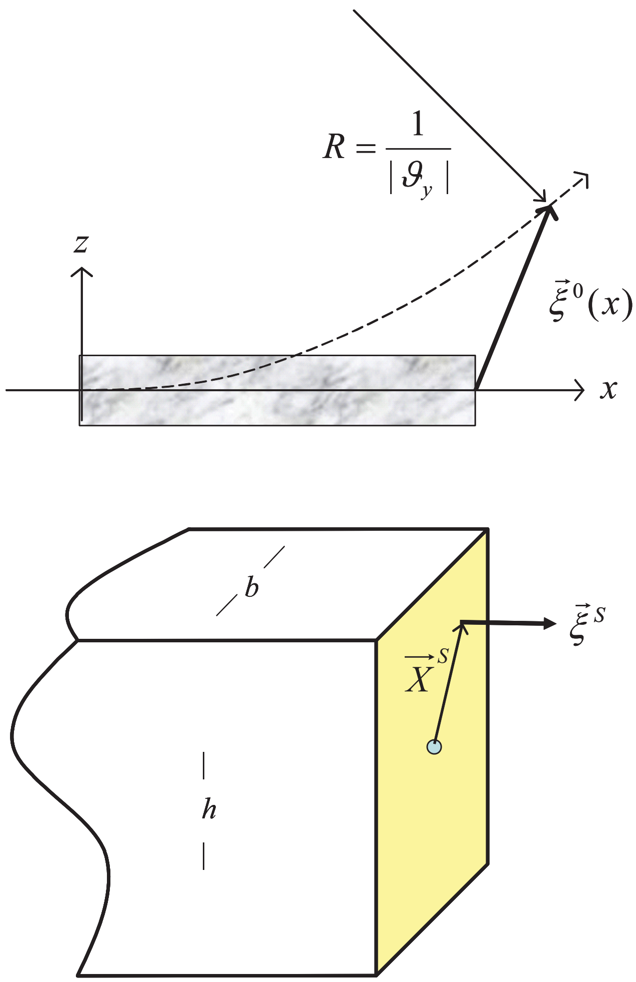

Figure 4.8: Beam deformation model: $\vec{\xi}^0$ is the displacement vector of the beam's axis of reference (centroid), while $\vec{\xi}^S$ is the additional displacement of the cross-section S. The figure also shows the curvature of the beam deformation.

An initially plane beam section which is perpendicular to the beam reference axis remains plane throughout the beam deformation and perpendicular to the beam's axis in the deformed configuration.

4.6.2 Navier-Bernoulli Assumption

Motivated by the dimensional analysis, we want to express the Navier-Bernoulli assumption mathematically. The first assumption that an initially plane cross-section remains plane throughout the deformation is expressed by the following displacement field:

$$\boxed{\text{Bernoulli: } \vec{\xi} = \vec{\xi}^{0}(x) + \vec{\omega}(x) \times \vec{X}_S} \tag{4.82}$$

where $\vec{\omega} = \vec{\omega}(x)$ is the vector of infinitessimal rotations, which describes the rotation of any cross-section point $\vec{X}_S = y\vec{e}_y + z\vec{e}_z$ around the beam's centroid axis. In components:

$$\xi_x = \xi_x^0(x) + z\ \omega_y(x) - y\ \omega_z(x) \tag{4.83}$$

$$\xi_y = \xi_y^0(x) - z\ \omega_x(x) \tag{4.84}$$

$$\xi_z = \xi_z^0(x) + y\ \omega_x(x) \tag{4.85}$$

The non-zero strain components defined by (4.82) are:

$$\epsilon_{xx} = \epsilon_{xx}^0 + \vartheta_y^0 z - \vartheta_z^0 y \tag{4.86a}$$

$$\epsilon_{xy} = \gamma_{xy}^0 - \frac{1}{2}\vartheta_x^0 z \tag{4.86b}$$

$$\epsilon_{xz} = \gamma_{xy}^0 + \frac{1}{2}\vartheta_x^0 y \tag{4.86c}$$

where γ_{xy}^0 and γ_{xy}^0 are the distortions of the cross-section along the beam's reference axis:

$$\gamma_{xy}^0 = \epsilon_{xy}^0 - \frac{1}{2}\omega_z;\ \gamma_{xz}^0 = \epsilon_{xz}^0 + \frac{1}{2}\omega_y \tag{4.87}$$

while $\vartheta_x^0, \vartheta_y^0$ and ϑ_z^0 stand for the infinitessimal changes of the beam's rotations:

$$\vartheta_x^0 = \frac{\partial \omega_x}{\partial x};\ \vartheta_y^0 = \frac{\partial \omega_y}{\partial x};\ \vartheta_z^0 = \frac{\partial \omega_z}{\partial x} \tag{4.88}$$

In addition, in order for the section not only to remain plane, but as well perpendicular to the beam's axis, the distortions γ_{xy}^0 and γ_{xy}^0 of the cross-section along the beam's reference axis ($y = z = 0$) must be zero:

$$\boxed{\text{Navier-Bernoulli: } \gamma_{xy}^0 = 0;\ \gamma_{xz}^0 = 0} \tag{4.89}$$

For this case, (4.79), (4.86) and (4.87) inform us that the in-plane rotations ω_z and ω_y must be related to the displacement components by:

$$\begin{array}{c} \boxed{\text{Navier-Bernoulli: } \omega_z = 2\epsilon_{xy}^0 = \frac{\partial \xi_y^0}{\partial x};\ -\omega_y = 2\epsilon_{xz}^0 = \frac{\partial \xi_z^0}{\partial x}} \\ \Downarrow \\ \boxed{\text{Navier-Bernoulli: } \vartheta_z^0 = \frac{\partial^2 \xi_y^0}{\partial x^2};\ -\vartheta_y^0 = \frac{\partial^2 \xi_z^0}{\partial x^2}} \end{array} \tag{4.90}$$

Eq. (4.90) which complements the Bernoulli plane-section assumption (4.82), is known as the *Navier-Bernoulli* assumption of beam deformation, and identifies the derivatives (4.88) as the cross-section's (infinitessimal) *curvatures*[3] ϑ_y^0 and ϑ_z^0 (Fig. 4.8).

Finally, let us note that the displacement field (4.82) only specifies the three strain components $\epsilon_{xx}, \epsilon_{xy}$ and ϵ_{xz}. This does not allow one to conclude, from e.g. dimensional analysis, that the strains ϵ_{yy}, ϵ_{yz} and ϵ_{zz} are zero. In fact, as we will see in the next Chapter, it requires a material law that links stresses to strains, to completely characterize all strain components due to beam deformation.

4.7 Summary: Deformation and Strain

In summary, the basis of the deformation theory is that deformation is *measurable*. On this basis, the tools of differential geometry provide a means to translate measurements into strains. The fundamental tool of this description is the material vector or strain gage, which provides access to length dilation and distortion, both experimentally and theoretically. Depending

[3] The curvature at a given point is a measure of how quickly the curve changes direction at that point. The direction of the beam's direction after deformation is decribed in the Cartesian coordinate system by ω_y and ω_z, which within the linear deformation theory are of infinitessimal order.

Name	**Finite Deformation**	**Linear Deformation**
Domain	no restriction	$\left\Vert \text{Grad}\,\vec{\xi} \right\Vert \ll 1$
Displacement	$\vec{\xi} = \vec{x} - \vec{X}$	
Material Vector	$d\vec{x} = \mathbf{F} \cdot d\vec{X}$	
Def. Gradient	$\mathbf{F} = \mathbf{1} + \text{Grad}\,\vec{\xi}$	$\mathbf{F} \simeq \mathbf{1} + \text{grad}\,\vec{\xi}$
Volume Dilation	$J = \det \mathbf{F}$	$J \simeq 1 + \text{tr}\,\boldsymbol{\varepsilon} = 1 + \text{div}\,\vec{\xi}$
Surface Transport	$\vec{n}\,da \simeq J\,\left(\mathbf{F}^T\right)^{-1} \cdot \vec{N} dA$	$\vec{n}\,da \simeq J \left(\mathbf{1} - \left(\text{grad}\,\vec{\xi}\right)^T\right) \cdot \vec{N}$
Strain Tensor	$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{1}\right)$	$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\text{grad}\,\vec{\xi} + \left(\text{grad}\,\vec{\xi}\right)^T\right)$
Length Dilations	$\lambda\left(\vec{e}_\alpha\right) = \sqrt{2E_{\alpha\alpha} + 1} - 1$	$\lambda\left(\vec{e}_i\right) = \epsilon_{ii} = \vec{e}_i \cdot \boldsymbol{\varepsilon} \cdot \vec{e}_i$
Angle Distortions	$\sin\theta\left(\vec{e}_\alpha, \vec{e}_\beta\right) = \frac{2E_{\alpha\beta}}{\sqrt{(1+2E_{\alpha\alpha})(1+2E_{\beta\beta})}}$	$\theta\left(\vec{e}_i, \vec{e}_j\right) = 2\epsilon_{ij} = \vec{e}_i \cdot (2\boldsymbol{\varepsilon}) \cdot \vec{e}_j$

Table 4.2: Summary of Finite and Linear Deformation Theory.

on the order of magnitude of the strains, we distinguish the finite deformation theory from the linearized deformation theory. The Box 4.2 summarizes the different quantities for both theories. In all what follows, we will primarily deal within the context of the linear theory, which is a good first-order approximations for engineering design.

But let there be no doubt, the tools of the deformation theory presented here just specify *how* matter deforms. They give no clue *when* and *to which extent* matter deforms. This is the focus of constitutive laws that link forces to displacements, stresses to strains, and so on. This is the focus of upcoming chapters, that tackle the ultimate goal of the Engineering Mechanics of solids: how does matter behave when subjected to forces and stresses?