Preprints of the 18th IFAC World Congress
Milano (Italy) August 28 - September 2, 2011

$\mathcal{H}_\infty$ State Feedback Control of Discrete-time Markov Jump Linear Systems through Linear Matrix Inequalities ⋆

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Abstract This paper addresses the $\mathcal{H}_\infty$ state-feedback control design problem of discrete-time Markov jump linear systems. First, under the assumption that the Markov parameter is measured, the main contribution is on the LMI characterization of all linear feedback controllers such that the closed loop output remains bounded by a given norm level. This results allows the robust controller design to deal with convex bounded parameter uncertainty, probability uncertainty and cluster availability of the Markov mode. For partly unknown transition probabilities, the proposed design problem is proved to be less conservative than one available in the current literature. An example is solved for illustration and comparisons.

Keywords: Markov models; Discrete-time systems; State-feedback control; Linear Matrix Inequalities.

1. INTRODUCTION

Dynamic systems that present sudden changes on their structures or parameters have been the subject of several studies in the last decades. Among the several ways to model such a dynamic system, one of increasing interest is the Markovian jump linear system (MJLS). There is a large amount of theory in the literature that extend the usual concepts of stability, observability, controllability, $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms to this special class, see Costa et al. [2005] and references therein.

An important assumption to consider for MJLS design problems is if the Markov chain state, often called mode, is available or not to the controller at every instant of time $k \geq 0$. Based on that information, the design is said to be either mode-dependent or mode-independent, respectively. A trade-off between the two approaches is to consider the Markov mode partially available, that is, it is possible to know if the current mode is within a *cluster* of the N modes. Such approach has been used by do Val et al. [2002] for $\mathcal{H}_2$ state-feedback control and it has been used in Gonçalves et al. [2010b] for the filtering problem with measurement transmitted over burst error channels.

Even though the $\mathcal{H}_\infty$ control problem has been considered long time ago for discrete-time MJLSs, it is interesting to note that there are no necessary and sufficient LMI conditions available in the literature for the elementary state feedback problem, to the best of our knowledge. The early papers approached the problem using coupled Riccati inequalities, where sufficient conditions were provided by Fragoso et al. [1995], Boukas and Shi [1997], Shi et al. [1999], and necessary and sufficient conditions by Costa and do Val [1996]. Discrete coupled Riccati equations are usually solved via iterative techniques [Abou-Kandil et al., 1995] which are difficult to be initialized. Also, transformation of the Riccati inequalities to LMIs via Schur complements [Ait-Rami and El Ghaoui, 1996] does not work directly in the discrete-time case.

Later papers have used LMIs for various $\mathcal{H}_\infty$ state feedback problems, for example with mixed $\mathcal{H}_2/\mathcal{H}_\infty$ criteria [Costa and Marques, 1998], norm-bounded uncertainty [Shi and Boukas, 1999], time-delays [Cao and Lam, 1999], polytopic uncertainties [Palhares et al., 2001], uncertain transition probabilities [Boukas, 2009], etc., however, none of them dealt with the basic problem, and only sufficient conditions were provided. This paper intends to fill this gap in the literature that went unnoticed for more than a decade, and to provide computationally more tractable approach using LMIs compared to classical Riccati inequalities.

In this paper, initially the set of all mode-dependent state feedback linear gains for bounded closed loop $\mathcal{H}_\infty$ norm is obtained. The use of LMIs also allows us to include

⋆ This research was supported by grants from “Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq” and “Fundação de Amparo à Pesquisa do Estado de São Paulo - FAPESP”, Brazil.

additional constraints to the basic problem, as for instance to design robust controllers able to face parameter or transition probability uncertainty, something that is not possible for the Riccati equation approach. In particular for the latter, we show our proposed design is less conservative than that from Zhang and Boukas [2009]. Another additional constraint to the main problem allows us to design controllers for partially available Markov modes.

The notation used throughout is standard. Capital letters denote matrices and small letters denote vectors. For scalars, small Greek letters are used. For real matrices or vectors $(')$ indicates transpose. For the sake of easing the notation of partitioned symmetric matrices, the symbol $(\bullet)$ denotes generically each of its symmetric blocks. The set of natural numbers is denoted by $\mathbb{N}$, while the finite set of the first N natural numbers $\{1, \cdots, N\}$ is denoted by $\mathbb{K}$. Given N^2 nonnegative real numbers p_{ij} satisfying $p_{i1} + \cdots + p_{iN} = 1$ for all $i \in \mathbb{K}$ and N real matrices X_j for all $j \in \mathbb{K}$, the convex combination of these matrices with weights p_{ij} is denoted by $X_{pi} = \sum_{j=1}^{N} p_{ij} X_j$. Similarly, for positive definite matrices, the inverse of the convex combination of inverses is denoted as

$$X_{qi} = \left(\sum_{j=1}^{N} p_{ij} X_j^{-1} \right)^{-1} \tag{1}$$

Clearly, X_{pi} depends linearly on matrices $X_1, \cdots, X_N$ while the dependence of X_{qi} with respect to the same matrices is highly nonlinear. The symbol $\mathcal{E}\{\cdot\}$ denotes mathematical expectation of $\{\cdot\}$. For any stochastic signal $z(k)$, defined in the discrete time domain $k \in \mathbb{N}$, the quantity $\|z\|_2^2 = \sum_{k=0}^{\infty} \mathcal{E}\{z(k)'z(k)\}$ is its squared norm.

2. PROBLEM FORMULATION

A discrete-time Markovian jump linear system (MJLS) is described by the following stochastic state-space model $\mathcal{G}$

$$\begin{aligned} x(k+1) &= A(\theta_k)x(k) + B(\theta_k)u(k) + J(\theta_k)w(k) \\ z(k) &= C(\theta_k)x(k) + D(\theta_k)u(k) + E(\theta_k)w(k) \end{aligned} \tag{2}$$

where $x(k) \in \mathbb{R}^n$ is the state, $w(k) \in \mathbb{R}^m$ is the external perturbation, $z(k) \in \mathbb{R}^r$ is the output to be controlled and $u(k) \in \mathbb{R}^q$ is the control input. It is assumed that the system evolves from $x(0) = 0$. The state space matrices (2) depend on the random variable $\theta_k \in \mathbb{K}$ governed by a Markov chain with the associated transition probability matrix $\mathbb{P} \in \mathbb{R}^{N \times N}$ whose elements are given by $p_{ij} = \mathrm{Prob}(\theta_{k+1} = j \mid \theta_k = i)$, which satisfies the normalized constraints $p_{ij} \geq 0$ and $\sum_{j=1}^{N} p_{ij} = 1$ for each $i \in \mathbb{K}$. To ease the presentation, the following notations $A(\theta_k) := A_i$, $B(\theta_k) = B_i$, $J(\theta_k) := J_i$, $C(\theta_k) := C_i$, $E(\theta_k) := E_i$ whenever $\theta_k = i \in \mathbb{K}$ are used. The next proposition from Seiler and Sengupta [2003] shows how the $\mathcal{H}_\infty$ norm can be calculated.

Lemma 1. The $\mathcal{H}_\infty$ norm of system $\mathcal{G}$, for $u(k) \equiv 0$, can be calculated by:

$$\|\mathcal{G}\|_\infty^2 = \inf_{(\gamma, P_i) \in \Phi} \gamma \tag{3}$$

where Φ is the set of all positive definite matrices $P_i \in \mathbb{R}^{n \times n}$ and $\gamma \in \mathbb{R}$ such that the following matrix inequality

$$\begin{bmatrix} P_i & \bullet & \bullet & \bullet \\ 0 & \gamma I & \bullet & \bullet \\ A_i & J_i & P_{pi}^{-1} & \bullet \\ C_i & E_i & 0 & I \end{bmatrix} > 0 \tag{4}$$

holds for each $i \in \mathbb{K}$.

We are now in position to state the control design problem to be dealt with in the rest of this paper. Associated to (2), consider the linear state feedback controller

$$u(k) = K(\theta_k)x(k) \tag{5}$$

The goal is to determine the controler gains K_i for all $i \in \mathbb{K}$ such that the $\mathcal{H}_\infty$ norm of the output is minimized. Hence, the problem to be solved is written in the form

$$\min_{K_i} \|\mathcal{G}_c\|_\infty^2 \tag{6}$$

where the closed loop system $\mathcal{G}_c$ satisfies:

$$\begin{aligned} x(k+1) &= (A(\theta_k) + B(\theta_k)K(\theta_k))x(k) + J(\theta_k)w(k) \\ z(k) &= (C(\theta_k) + D(\theta_k)K(\theta_k))x(k) + E(\theta_k)w(k) \end{aligned} \tag{7}$$

3. MODE-DEPENDENT DESIGN

Based on the previous results, our main purpose in this section is to calculate the global optimal solution of the mode-dependent $\mathcal{H}_\infty$ state-feedback control design problem, given by (6). Notice that matrices A_i, and C_i appearing in (4) are replaced by the closed loop matrices $A_i + B_iK_i$ and $C_i + D_iK_i$ respectively. The next theorem shows how to express problem (6) in LMI terms.

Theorem 2. There exists a state feedback control of the form (5) satisfying the constraint $\|\mathcal{G}_c\|_\infty^2 < \gamma$ if and only if there exist symmetric matrices X_i and matrices G_i, Y_i, H_i and Z_{ij} of compatible dimensions satisfying the LMIs

$$\begin{bmatrix} G_i + G_i' - X_i & \bullet & \bullet & \bullet \\ 0 & \gamma I & \bullet & \bullet \\ A_iG_i + B_iY_i & J_i & H_i + H_i' - Z_{pi} & \bullet \\ C_iG_i + D_iY_i & E_i & 0 & I \end{bmatrix} > 0 \tag{8}$$

$$\begin{bmatrix} Z_{ij} & \bullet \\ H_i & X_j \end{bmatrix} > 0 \tag{9}$$

for all $(i, j) \in \mathbb{K} \times \mathbb{K}$. In the affirmative case, suitable state-feedback gains are given by $K_i = Y_iG_i^{-1}$.

Proof. For the necessity, assume that (4) holds. Defining $X_i := P_i^{-1}$, $Y_i = K_iG_i$ and multiplying (4) to the right by $\mathrm{diag}[G_i, I, I, I]$ and to the left by its transpose we obtain

$$\begin{bmatrix} G_i'X_i^{-1}G_i & \bullet & \bullet & \bullet \\ 0 & \gamma I & \bullet & \bullet \\ A_iG_i + B_iY_i & J_i & X_{qi} & \bullet \\ C_iG_i + D_iY_i & E_i & 0 & I \end{bmatrix} > 0 \tag{10}$$

For $H_i = X_{qi}$ and $Z_{ij} = X_{qi}X_j^{-1}X_{qi} + \varepsilon I$ with $\varepsilon > 0$ we see that (9) is verified and we obtain

$$H_i + H_i' - Z_{pi} = X_{qi} - \varepsilon I \tag{11}$$

hence, taking $\varepsilon > 0$ sufficiently small, inequality (11) implies that (8) holds for $G_i = X_i$ and the claim follows.

For the sufficiency, assume that (8) and (9) hold. From (9) we have $Z_{ij} > H_i'X_j^{-1}H_i$ and consequently multiplying these inequalities by p_{ij} and summing up for all $j \in \mathbb{K}$ we obtain

$$
\begin{aligned}
H_i + H_i' - Z_{pi} &= H_i + H_i' - \sum_{j=1}^{N} p_{ij} Z_{ij} \\
&\le H_i + H_i' - H_i' X_{qi}^{-1} H_i \\
&\le X_{qi} - (H_i - X_{qi})' X_{qi}^{-1} (H_i - X_{qi}) \\
&\le X_{qi}
\end{aligned} \tag{12}
$$

which implies that (8) remains valid if the diagonal term on the second column and row is replaced by X_{qi}. On the other hand,

$$
G_i + G_i' - X_i \le G_i' X_i^{-1} G_i \tag{13}
$$

and the term $G_i + G_i' - X_i$ can be replaced by $G_i' X_i^{-1} G_i$ in (8) without changing its validity. Multiplying the inequality obtained after the replacements indicated by (12) and (13) to the right by $\text{diag}[G_i^{-1}, I, I, I]$ and to the left by its transpose we obtain

$$
\begin{bmatrix}
X_i^{-1} & \bullet & \bullet & \bullet \\
0 & \gamma I & \bullet & \bullet \\
A_i + B_i K_i & J_i & \left(\sum_{j=1}^{N} p_{ij} X_j^{-1} \right)^{-1} & \bullet \\
C_i + D_i K_i & E_i & 0 & I
\end{bmatrix} > 0 \tag{14}
$$

which is equivalent to (4) for the closed loop matrices $A_i + B_i K_i$, $C_i + D_i K_i$ and for $P_i = X_i^{-1}$. □

Based on Theorem 2, we can write problem (6) in LMI terms as

$$
\min_{\mathcal{X} \in \Omega} \gamma \tag{15}
$$

where $\mathcal{X} = (G_i, Y_i, X_i, H_i, Z_{ij})$ for all $(i, j) \in \mathbb{K} \times \mathbb{K}$ represents the matrix variables and Ω is the set of all feasible solutions from (8) and (9). Notice that the variables G_i for $i \in \mathbb{K}$ can be seen as slack variables for the optimization program. This fact is illustrated by the next corollary.

Corollary 1. There exists a state feedback control of the form (5) satisfying the constraint $\|\mathcal{G}_c\|_\infty^2 < \gamma$ if and only if there exist symmetric matrices X_i and matrices Y_i, H_i and Z_{ij} of compatible dimensions satisfying the LMIs

$$
\begin{bmatrix}
X_i & \bullet & \bullet & \bullet \\
0 & \gamma I & \bullet & \bullet \\
A_i X_i + B_i Y_i & J_i & H_i + H_i' - Z_{pi} & \bullet \\
C_i X_i + D_i Y_i & E_i & 0 & I
\end{bmatrix} > 0 \tag{16}
$$

$$
\begin{bmatrix}
Z_{ij} & \bullet \\
H_i & X_j
\end{bmatrix} > 0 \tag{17}
$$

for all $(i, j) \in \mathbb{K} \times \mathbb{K}$. In the affirmative case, suitable state-feedback gains are given by $K_i = Y_i X_i^{-1}$.

Proof. First assume that (16) and (17) are valid. For $G_i = X_i$ for all $i \in \mathbb{K}$, (8) and (9) are valid and sufficiency follows from Theorem 2. For the necessity, assume that (8) and (9) are valid. Based on (13) we can substitute the block diagonal term $G_i + G_i' - X_i$ by the upperbound $G_i' X_i^{-1} G_i$. After multiplying the obtained inequality by $\text{diag}[G_i^{-1} X_i, I, I, I]$ to the right and by its transpose to the left we get (16). □

For the mode-dependent controller with fully known parameters, including the transition probabilities, we may set $G_i = X_i$ without loss of generality. Nonetheless, those variables will play a key role in reducing conservatism for design under parameter uncertainty or cluster mode availability. Also notice that the inequality (9) is indexed in both $i \in \mathbb{K}$ and $j \in \mathbb{K}$ and it is needed to linearize the term P_{pi}^{-1} in (4). Those constraints are similar to the ones developed for output feedback at Geromel et al. [2009]. Another approach for such linearization could be the same adopted, for example by de Souza [2006] or Zhang and Boukas [2009], with the use of augumented blocks inside the LMIs. That strategy is used on the next theorem.

Theorem 3. There exists a state feedback control of the form (5) satisfying the constraint $\|\mathcal{G}_c\|_\infty^2 < \gamma$ if and only if there exist symmetric matrices X_i and matrices G_i, Y_i of compatible dimensions satisfying the LMIs

$$
\begin{bmatrix}
G_i + G_i' - X_i & \bullet & \bullet & \bullet \\
0 & \gamma I & \bullet & \bullet \\
\mathfrak{L}_i (A_i G_i + B_i Y_i) & \mathfrak{L}_i J_i & \mathfrak{X} & \bullet \\
C_i G_i + D_i Y_i & E_i & 0 & I
\end{bmatrix} > 0 \tag{18}
$$

where

$$
\mathfrak{X} := \text{diag}[X_1, \cdots, X_N] \tag{19}
$$

$$
\mathfrak{L}_i := [\sqrt{p_{i1}} I, \cdots, \sqrt{p_{iN}} I]' \tag{20}
$$

for all $i \in \mathbb{K}$. In the affirmative case, suitable state-feedback gains are given by $K_i = Y_i G_i^{-1}$.

Proof. For the necessity, assume that (4) holds. Defining $X_i := P_i^{-1}$, $Y_i = K_i G_i$ and multiplying (4) to the right by $\text{diag}[G_i, I, I, I]$ and to the left by its transpose we obtain (10). Based on the definitions (19) and (20) we can write

$$
X_{qi}^{-1} = \mathfrak{L}_i' \mathfrak{X}^{-1} \mathfrak{L}_i \tag{21}
$$

Substituting (21) in (10) and applying the Schur complement over the diagonal block X_{qi} we get (18).

For the sufficiency, assume that (18) holds. Based on (13) the term $G_i + G_i' - X_i$ can be replaced by $G_i' X_i^{-1} G_i$ in (18) without changing its validity. Applying the Schur complement over the block diagonal term $\mathfrak{X}$ and taking (21) into account we obtain an LMI equivalent to (4) for the closed loop matrices $A_i + B_i K_i$, $C_i + D_i K_i$ and for $P_i = X_i^{-1}$. □

In terms of the conditions from Theorem 3, the problem (6) can be written as

$$
\min_{\mathcal{Y} \in \Xi} \gamma \tag{22}
$$

where $\mathcal{Y} = (X_i, G_i, Y_i)$ for all $i \in \mathbb{K}$ represents the matrix variables and Ξ is the set of all feasible solutions for (18). Theorems 2 and 3 are equivalent and both can be used to determine the gains K_i that minimize $\|\mathcal{G}_c\|_\infty^2$. Using each of them is a matter of computational effort and programming ease. For a complete discussion on the computational effort, refer to Borchers and Young [2007]. Like we did for Theorem 2, we can state a corollary of Theorem 3 for the case with no parameter uncertainty and mode-dependent feedback control. The proof is very similar to that of Corollary 1 and is thus omitted.

Corollary 2. There exists a state feedback control of the form (5) satisfying the constraint $\|\mathcal{G}_c\|_\infty^2 < \gamma$ if and only if there exist symmetric matrices X_i and matrices Y_i of compatible dimensions satisfying the LMIs

$$\begin{bmatrix} X_i & \bullet & \bullet & \bullet \\ 0 & \gamma I & \bullet & \bullet \\ \mathfrak{L}_i(A_iX_i + B_iY_i) & \mathfrak{L}_iJ_i & \mathfrak{X} & \bullet \\ C_iX_i + D_iY_i & E_i & 0 & I \end{bmatrix} > 0 \tag{23}$$

where $\mathfrak{X}$ and $\mathfrak{L}_i$ are given by (19) and (20) for all $i \in \mathbb{K}$. In the affirmative case, suitable state-feedback gains are given by $K_i = Y_iX_i^{-1}$.

In the next section, we introduce some constraints on the parameters knowledge or on the mode $\theta_k \in \mathbb{K}$ availability to the state-feedback controller.

4. ROBUST CONTROLLER AND CLUSTER AVAILABILITY

4.1 Robust Controller

Suppose the system matrices appearing in (2) are not exactly known, but belong to the convex polytope

$$\mathcal{R} = \mathrm{co}\left\{ \begin{bmatrix} A_i^l & B_i^l & J_i^l \\ C_i^l & D_i^l & E_i^l \end{bmatrix} , \ l = 1, \cdots , N_r \right\} \tag{24}$$

defined by the convex hull of N_r vertices. A guaranteed cost is given from the optimal solution of problem (15), with $\Omega = \bigcap_{l=1}^{N_r} \Omega_l$ where Ω_l is the set defined by respective LMIs calculated at each vertex of the uncertain polytope (24). The same can be said for problem (22), with $\Xi = \bigcap_{l=1}^{N_r} \Xi_l$. As it can be easily verified, this claim follows from the linear dependence of those LMIs with respect to the plant state space matrices.

4.2 Uncertain probabilities

As already discussed at Gonçalves et al. [2010a], a crucial point to be noticed concerning (4) is that for each fixed $i \in \mathbb{K}$, it depends *exclusively* on the i-th row of the transition matrix $\mathbb{P}$. In other words, for each $i \in \mathbb{K}$ fixed, these constraints are not coupled by p_{ij} appearing in different rows of $\mathbb{P}$. Based on this observation, we split the matrix $\mathbb{P}$ by rows as follows: $\mathbb{P}' = [\mathbb{P}_1', \cdots , \mathbb{P}_N'] \in \mathbb{R}^{N\times N}$ where $\mathbb{P}_i \in \mathbb{R}^N$. We can also use the fact that the LMIs (8) and (9) are affine with respect to the transition probabilities to calculate a guaranteed cost from the optimal solution of problem (15) calculated at each vertex of the convex polytope

$$\mathcal{P}_i = \mathrm{co}\left\{ \mathbb{P}_i^{(l)} \ , \ l = 1, \cdots , N_p \right\} \tag{25}$$

for $i \in \mathbb{K}$.

Another model for probability uncertainty, introduced in Zhang and Boukas [2009], considers some of the elements from $\mathbb{P}$ unknown. In this case, the transition probability matrix for system (2) has $[p_{ij}] =?$ for some $(i, j) \in \mathbb{K} \times \mathbb{K}$, meaning that the corresponding element is completely unknown. We denote $\mathbb{J}_{\mathrm{K}}^i$ and $\mathbb{J}_{\mathrm{UK}}^i$ as the sets

$$\mathbb{J}_{\mathrm{K}}^i := \{j : p_{ij} \text{ is known}\} \tag{26}$$

$$\mathbb{J}_{\mathrm{UK}}^i := \{j : p_{ij} \text{ is not known}\} \tag{27}$$

and $\rho_i := \sum_{j\in\mathbb{J}_{\mathrm{K}}^i} p_{ij}$ the sum of all known transition probabilities from a given mode $i \in \mathbb{K}$. Indeed, such an assumption can be seen as a particular case of polytopic uncertainty like the one represented by (25). For example, considering the uncertain transition probability matrix

$$\mathbb{P} = \begin{bmatrix} ? & 0.7 & ? \\ 0.9 & ? & ? \\ 0.3 & 0.3 & 0.4 \end{bmatrix} \tag{28}$$

the following rows define each of the uncertain polytope vertices

$$\mathbb{P}_1^{(1)} = [0.3 \ 0.7 \ 0.0] \, , \ \mathbb{P}_1^{(2)} = [0.0 \ 0.7 \ 0.3]$$
$$\mathbb{P}_2^{(1)} = [0.9 \ 0.1 \ 0.0] \, , \ \mathbb{P}_2^{(2)} = [0.9 \ 0.0 \ 0.1]$$
$$\mathbb{P}_3^{(1)} = [0.3 \ 0.3 \ 0.4]$$

It is important to realize that whenever a row contains none or one "?" then, due to the normalization constraint, only one row is generated. Furthermore, if two or more "?" appear in a row, the same number of vertices is generated. Moreover, working directly with the elements of the probability transmission matrix, it is possible to take into account more general situations, for instance, the one defined by $p_{ij} \in [\underline{p}_{ij}, \ \overline{p}_{ij}]$, for some $(i, j) \in \mathbb{K} \times \mathbb{K}$. The normalization conditions impose that $\sum_{j=1}^N \underline{p}_{ij} \leq 1$ and $\sum_{j=1}^N \overline{p}_{ij} \geq 1$ for each $i \in \mathbb{K}$, otherwise the set of possible row elements from $\mathbb{P}$ such that the sum on the i-th row is equal to one will be an empty set. We are now in position to compare our results with that of Theorem 3.4 from Zhang and Boukas [2009].

Corollary 3. Let γ° and $\gamma^\bullet$ be the minimum value of the $\mathcal{H}_\infty$ level provided by Theorem 3.4 of Zhang and Boukas [2009] and by theorems 2 or 3, respectively. Then $\gamma^\bullet \leq \gamma^\circ$.

Proof. The proof follows by showing that for $\gamma > 0$ given, the feasibility of LMIs of Theorem 3.4, Zhang and Boukas [2009], implies the same for the LMIs of theorems 2 or 3. For the closed loop matrices, the $\mathcal{H}_\infty$ condition used on Zhang and Boukas [2009] can be given by the following LMIs, after applying Schur complement, rearranging rows and columns and adopting our notation

$$\begin{bmatrix} \rho_iX_i & \bullet & \bullet & \bullet \\ 0 & \rho_i\gamma I & \bullet & \bullet \\ \Pi_i(A_iX_i + B_iY_i) & \Pi_iJ_i & \Pi_i & \bullet \\ \rho_i(C_iX_i + D_iY_i) & \rho_iE_i & 0 & \rho_iI \end{bmatrix} > 0 \tag{29}$$

$$\begin{bmatrix} X_i & \bullet & \bullet & \bullet \\ 0 & \gamma I & \bullet & \bullet \\ X_j^{-1}(A_iX_i + B_iY_i) & X_j^{-1}J_i & X_j^{-1} & \bullet \\ C_iX_i + D_iY_i & E_i & 0 & I \end{bmatrix} > 0 \tag{30}$$

for all $i \in \mathbb{K}$ and $j \in \mathbb{J}_{\mathrm{UK}}^i$, where $\Pi_i := \sum_{j\in\mathbb{J}_{\mathrm{K}}^i} p_{ij}X_j^{-1}$. If we multiply the LMIs (30) by $(1 - \rho_i)$ and sum them with (29) we have

$$\begin{bmatrix} X_i & \bullet & \bullet & \bullet \\ 0 & \gamma I & \bullet & \bullet \\ \Pi_{ij}(A_iX_i + B_iY_i) & \Pi_{ij}J_i & \Pi_{ij} & \bullet \\ C_iX_i + D_iY_i & E_i & 0 & I \end{bmatrix} > 0 \tag{31}$$

for all $i \in \mathbb{K}$ and $j \in \mathbb{J}_{\mathrm{UK}}^i$, where $\Pi_{ij} := \sum_{r\in\mathbb{J}_{\mathrm{K}}^i} p_{ir}X_r^{-1} + (1 - \rho_i)X_j^{-1}$. For each row $i \in \mathbb{K}$ from $\mathbb{P}$ and $j \in \mathbb{J}_{\mathrm{UK}}^i$ we can represent one vertex l of polytope (25). Defining $X_{qi}^{(l)}$ according to (1) with weights $p_{ij}^{(l)}$ from vertex $l \in \{1, 2, \cdots , N_p\}$, we have the following

$$X_{qi}^{(l)} = \Pi_{ij}^{-1} \tag{32}$$

Thus, (31) together with (32) allow us to follow the same linearization steps from the sufficiency part of the proofs from theorems 2 and 3 to show that the LMIs (8) and (9) or the LMIs (18) calculated for each vertex of the polytope (25) are valid. □

4.3 Cluster availability of the Markov mode

The assumption of the Markov mode $\theta_k \in \mathbb{K}$ availability may not be practical. In some applications, it may be more adequate to consider the cluster availability. Consider the set $\mathbb{L} = \{1, 2, \cdots, N_c\}$ with $N_c \leq N$ and define the set of Markov chain states $\mathbb{K}$ as the union of N_c disjoint sets, or clusters, that is, $\mathbb{K} \equiv \cup_{\ell \in \mathbb{L}} \mathbb{U}_\ell$ such that $\cap_{\ell \in \mathbb{L}} \mathbb{U}_\ell \equiv \emptyset$. Associated to (2) consider the state-feedback controller with cluster observations

$$u(k) = K_\ell x(k) \tag{33}$$

whenever $\theta_k \in \mathbb{U}_\ell$. This implies that the N modes are split into N_c clusters and we assume it is possible to measure to which cluster $\mathbb{U}_\ell$ a mode i belongs, even if the mode i itself is unknown. Clearly, the mode-dependent ($N_c = N$) and mode-independent ($N_c = 1$) design problems are special cases from this definition. If we add the equality constraints

$$G_i = G_\ell \ , \ Y_i = Y_\ell \tag{34}$$

for all $\ell \in \mathbb{L}$ and $i \in \mathbb{U}_\ell \subset \mathbb{K}$ to the LMI conditions of theorems 2 or 3, we can see that the feedback gains K_i will be the same for all the modes i belonging to the same cluster $\mathbb{U}_\ell \subset \mathbb{K}$. Naturally, necessity is lost with the addition of constraints (34) and the minimum γ such that inequalities (8)-(9) are satisfied is an upper bound for $\|\mathcal{G}_c\|_\infty^2$.

5. EXAMPLES AND COMPARISONS

As the first example, let us consider the system proposed by de Souza [2006]. The objective is to project a mode-independent stabilizing controller for the system modelled by $x(k+1) = A(\theta_k)x(k) + B(\theta_k)u(k)$ with $\theta_k \in \{1, 2, 3\}$ and state matrices given by

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1.2 \end{bmatrix} A_2 = \begin{bmatrix} 1.13 & 0 \\ 0.16 & 0.478 \end{bmatrix} A_3 = \begin{bmatrix} 0.3 & 0.13 \\ 0.16 & 1.14 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0 \end{bmatrix} \ B_2 = \begin{bmatrix} 0.2 & 0.1 \\ 0 & -0.1 \end{bmatrix} \ B_3 = \begin{bmatrix} 0 & 0.1 \\ 0 & -0.1 \end{bmatrix}$$

where the transition probability matrix is uncertain, belonging to a 2-vertices polytope with vertices, and given by

$$\mathbb{P}_1 = \begin{bmatrix} 0.15 & 0.85 & 0 \\ 0 & 0.45 & 0.55 \\ 0 & 0.2 & 0.8 \end{bmatrix} \quad \mathbb{P}_2 = \begin{bmatrix} 0.25 & 0.75 & 0 \\ 0 & 0.6 & 0.4 \\ 0 & 0.4 & 0.6 \end{bmatrix}.$$

In order to deal with the mode-independent design, we applied Theorem 2 with the additional constraints $G_1 = G_2 = G_3$ and $Y_1 = Y_2 = Y_3$. For any choice of J_i, C_i, D_i and E_i, we can successfully find a stabilizing controller. But as a major advantage, for a particular choice of those matrices, we can not only stabilize the system, but further impose a $\mathcal{H}_\infty$ level for the closed-loop system.

The second example is borrowed from Zhang and Boukas [2009]. It is described by equations (2) with the following state space realization

$$A_1 = \begin{bmatrix} 1 & -1.25 \\ 2.5 & 2.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.5 & -0.83 \\ 2.5 & 3.5 \end{bmatrix}$$
$$A_3 = \begin{bmatrix} 0.25 & -0.25 \\ 2.5 & 3 \end{bmatrix}, A_4 = \begin{bmatrix} 0.75 & -0.57 \\ 2.5 & 2.75 \end{bmatrix}$$

and $B_i = [0.5 \ 0.1]'$, $J_i = [0.08 \ 0.1]'$, $C_i = [1 \ 0]$, $D_i = 0.8$ and $E_i = 0.6$ for all $i \in (1, \cdots, 4)$.

Two distinct cases for the transition probability matrix are considered:

$$\mathbb{P}_3 = \begin{bmatrix} 0.3 & 0.2 & 0.1 & 0.4 \\ ? & ? & 0.3 & 0.2 \\ ? & 0.1 & ? & 0.3 \\ 0.2 & ? & ? & ? \end{bmatrix}, \quad \mathbb{P}_4 = \begin{bmatrix} 0.3 & ? & 0.1 & ? \\ ? & ? & 0.3 & 0.2 \\ ? & 0.1 & ? & 0.3 \\ 0.2 & ? & ? & ? \end{bmatrix}.$$

The guaranteed $\mathcal{H}_\infty$ costs for these two cases are summarized in Table 1, from where we illustrate the fact that the results we achieve are less conservative than the ones presented in Zhang and Boukas [2009].

Table 1. Minimum $\gamma^\star$ for different transition probability cases

	$\mathbb{P}_3$	$\mathbb{P}_4$
Zhang and Boukas [2009]	1.8782	1.9388
Theorem 2	1.3166	1.3283

Finally, let us consider a new transition probability matrix

$$\mathbb{P}_5 = \begin{bmatrix} 0.3 & 0.2 & 0.1 & 0.4 \\ ? & ? & 0.3 & 0.2 \\ ? & 0.1 & ? & 0.3 \\ 0.2 & 0.3 & 0.4 & 0.1 \end{bmatrix},$$

For this case, the resulting values for the state-feedback controllers for each mode are given by:

$$K_1 = \begin{bmatrix} -5.5291 \\ -2.1518 \end{bmatrix}', \quad K_2 = \begin{bmatrix} -5.0980 \\ -4.7362 \end{bmatrix}',$$
$$K_3 = \begin{bmatrix} -4.3659 \\ -4.2682 \end{bmatrix}', \quad K_4 = \begin{bmatrix} -5.5493 \\ -3.9748 \end{bmatrix}'.$$

Also, the upper-limit for the $\mathcal{H}_\infty$ norm for all possible transition probability matrices is $\sqrt{\gamma} = 1.2819$.

To test the conservativeness of the result, we performed the following simulation. For a grid of values inside the limits of $\mathbb{P}_5$, we calculated not only the real value of the $\mathcal{H}_\infty$ - norm of the closed loop system by the robust controller just obtained but also the norm of all the optimal controllers considering fully known transition probabilities, which obviously are not able to cope with the uncertainty in $\mathbb{P}_5$. Our interests are two-fold. First, compare the real values of the $\mathcal{H}_\infty$ norm with γ provided in the optimization step, in order to have a measure of how conservative the optimization method was. And comparing with the optimal controllers with full knowledge of the transition probabilities, we can try to measure the performance distance of our proposed controller to the very optimistic best result.

Figure 1 brings the results of the proposed experiment. The upper curve, in yellow/orange, is the $\mathcal{H}_\infty$ norm of

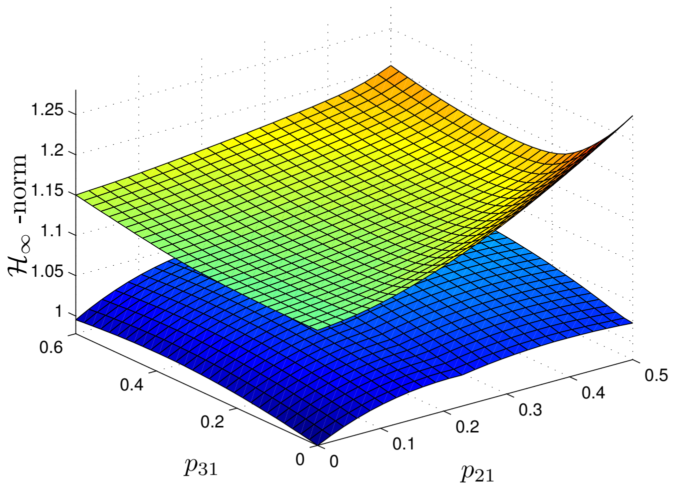

Figure 1. $\mathcal{H}_\infty$ -norms of Example 2 for $\mathbb{P}_5$.

the proposed controller whereas the lower one, in blue, is the result for all optimal controllers, each one calculated for a single point on the grid. First of all, we can see the proximity of both curves, indicating the good quality of the proposed controller. Indeed, the difference in performance between the robust controller and the optimal local ones never exceeds 25.2%.

More important, we can see that worse performance happens when $p_{21} = 0.5$ and $p_{31} = 0$. In this specific value, the $\mathcal{H}_\infty$ norm of the closed-loop system is 1.2807, which is less than 0.1% smaller than the provided upperbound $\sqrt{\gamma}$ given in the optimization process. This can illustrate that the conservatism of the proposed LMIs is very low.

6. CONCLUSION

In this paper, a LMI design method for mode-dependent $\mathcal{H}_\infty$ state-feedback controllers was presented. The main motivation for presenting this results was the observation that, to the best of our knowledge, there wasn't any work in the current literature considering the optimal case with the use of LMIs. The proposed design also allows the treatment of a wider class of problems of theoretical and practical importance, by simply including additional linear constraints to the basic problem. This is precisely the case for robust, cluster availability and transition probabilities uncertainty. For the partly unknown transition probabilities, it was shown theoretically that our method outperforms the one proposed in Zhang and Boukas [2009].

REFERENCES

H. Abou-Kandil, G. Freiling, and G. Jank. On the solution of discrete-time Markovian jump linear quadratic control problems. *Automatica*, 31(5):765–768, 1995.

M. Ait-Rami and L. El Ghaoui. LMI optimization for nonstandard Riccati equations arising in stochastic control. *IEEE transactions on automatic control*, 41(11):1666–1671, 1996.

B. Borchers and J. G. Young. Implementation of a primal-dual method for sdp on a shared memory parallel architecture. *Computational Optimization and Applications*, 37:355 – 369, 2007.

E. K. Boukas. $\mathcal{H}_\infty$ control of discrete-time Markov jump systems with bounded transition probabilities. *Optimal Control Applications and Methods*, 30(5):477–494, 2009.

E. K. Boukas and P. Shi. $\mathcal{H}_\infty$ control for discrete-time linear systems with Markovian jumping parameters. In *IEEE Conference on Decision and Control, 36 th, San Diego, CA*, pages 4134–4139, 1997.

Y. Y. Cao and J. Lam. Stochastic stabilizability and $\mathcal{H}_\infty$ control for discrete-time jump linear systems with time delay. *Journal of the Franklin Institute*, 336(8): 1263–1281, 1999.

O. L. V. Costa and J. B. R. do Val. Full Information $\mathcal{H}_\infty$ -Control for Discrete-Time Infinite Markov Jump Parameter Systems. *Journal of Mathematical Analysis and Applications*, 202(2):578–603, 1996.

O. L. V. Costa and R. P. Marques. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$-control of discrete-time Markovian jump linear systems. *IEEE Transactions on Automatic Control*, 43(1):95–100, 1998.

O. L. V. Costa, M. D. Fragoso, and R. P. Marques. *Discrete-Time Markov Jump Linear Systems*. Springer-Verlag, 2005.

C. E. de Souza. Robust stability and stabilization of uncertain discrete-time markovian jump linear systems. *IEEE Trans. on Automatic Control*, 51(5):836–841, 2006.

J. B. R. do Val, J. C. Geromel, and A. P. C. Gonçalves. The H_2-control for jump linear systems: cluster observations of the Markov state. *Automatica*, 38:343–349, 2002.

M. D. Fragoso, J. B. R. do Val, and D. L. Pinto Jr. Jump linear $\mathcal{H}_\infty$ control: the discrete-time case. *Control Theory and Advanced Technology*, 10:1459–1474, 1995.

J. C. Geromel, A. P. C. Gonçalves, and A. R. Fioravanti. Dynamic Output Feedback Control of Discrete-Time Markov Jump Linear Systems Through Linear Matrix Inequalities. *SIAM Journal on Control and Optimization*, 48:573–593, 2009.

A. P. C. Gonçalves, A. R. Fioravanti, and J. C. Geromel. Filtering of discrete-time markov jump linear systems with uncertain transition probabilities. *Int. Journal of Robust and Nonlinear Control*, 2010a. DOI:10.1002/rnc.1610.

A. P. C. Gonçalves, A. R. Fioravanti, and J. C. Geromel. Markov jump linear systems and filtering through network transmitted measurements. *Signal Processing*, 90: 2842–2850, 2010b.

R. M. Palhares, C. E. de Souza, and P. L. D. Peres. Robust $\mathcal{H}_\infty$ control for uncertain discrete-time state-delayed linear systems with markovian jumping parameters. In *Proc. 3rd IFAC Workshop on Time Delay Systems*, pages 107–112, 2001.

P. Seiler and R. Sengupta. A bounded real lemma for jump systems. *IEEE Trans. on Automatic Control*, 48: 1651–1654, 2003.

P. Shi and E. Boukas. Control of Markovian jump discrete-time systems with norm bounded uncertainty and unknown delay. *IEEE Transactions on Automatic Control*, 44(11):2139–2144, 1999.

P. Shi, E. K. Boukas, and R. K. Agarwal. Robust control for markovian jumping discrete-time systems. *Int. Journal of Systems Science*, 8:787–797, 1999.

L. X. Zhang and E. K. Boukas. H_∞ control for discrete-time markovian jump linear systems with partly unknown transition probabilities. *Int. Journal of Robust and Nonlinear Control*, 19(8):868–883, 2009.