A General Model and Analysis of a Discrete Two-Machine Production Line

Stanley B. Gershwin
Massachusetts Institute of Technology, gershwin@mit.edu

Saeideh Fallah-Fini
Northeastern University, fallahfini.s@neu.edu

We consider a two-machine, one-buffer, discrete time production line which is a generalization of earlier models. The machines have multiple up and down states. When a machine is not blocked or starved, the transitions among its up and down states are described by Markov chains. An analytical solution of the transition equations is summarized and numerical results are shown.

Key words: transfer lines; queues; finite buffers; Markov chains; eigenvalues

1. Introduction

We consider a two-machine, one-buffer, production line (Figure 1) which is a generalization of earlier models of Buzacott (1967), Gershwin and Schick (1983), Gershwin (1994), Tolio et al. (2002), and others. The system operates in discrete time and produces discrete material; it is modeled as a Markov chain. The buffer is finite. The machines have multiple up and down states. When a machine is not blocked or starved, the transitions among its up and down states are described by Markov chains.

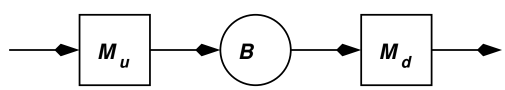

Figure 1 **Two-Machine, One-Buffer Line**

While there may be many applications of this class of models, there are two recent motivations for their study. One is the quality/quantity modeling of production lines in Kim and Gershwin (2005) and Colledani and Tolio (2005). The other is the multiple-part-type decompositions of Colledani et al. (2003), Jang (2007). In both cases, the decompositions require generalized two-machine models like those described here. In the first case, the multiple up states are needed to describe quality changes; in the second, the transitions among the down states are more general than those considered before.

This paper is a brief summary of an analytic solution of the transition equations. Numerical results are shown. The complete analysis and results, as well as a review of related literature, can be found in Gershwin and Fallah-Fini (2007). Tan and Gershwin (2007) have analyzed a two-machine, one-buffer model with continuous (fluid) material by a similar technique.

2. Model and Notation

In the following, u superscripts and subscripts refer to the upstream machine M_u and d refers to the downstream machine M_d. The state of the system is given by (n, α_u, α_d), where n is the number of parts in the buffer, α_u is the state of the upstream machine, and α_d is the state of the downstream machine. The set of possible values of these states are

- $n = 0, 1, ..., N$.
- $\alpha_u = \Upsilon_1^u, ..., \Upsilon_{I_u}^u$ if M_u is *up*; $\alpha_u = \Delta_1^u, ..., \Delta_{J_u}^u$ if M_u is *down*.
- $\alpha_d = \Upsilon_1^d, ..., \Upsilon_{I_d}^d$ if M_d is *up*; $\alpha_d = \Delta_1^d, ..., \Delta_{J_d}^d$ if M_d is *down*.

M_u is never starved and M_d is never blocked. When M_u is not blocked, α_u is the state of a Markov chain. For example, see Figure 2. Similarly for M_d when it is not starved.

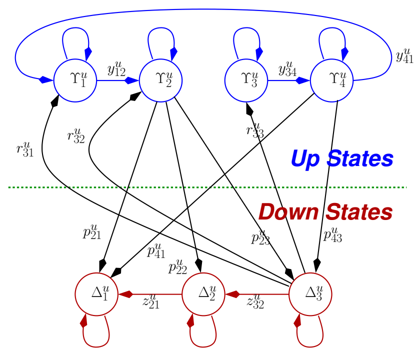

Figure 2 Markov Chain Graph of Upstream Machine

Operation-dependence: α_u cannot leave an up state if $n = N$. α_d cannot leave an up state if $n = 0$.

$$\begin{aligned}
\text{prob } \left[\alpha_u(t+1) \neq \alpha_u(t) \mid \alpha_u(t) \in \{\Upsilon_1^u, ..., \Upsilon_{I_u}^u\}, n(t) = N\right] &= 0 \\
\text{prob } \left[\alpha_d(t+1) \neq \alpha_d(t) \mid \alpha_d(t) \in \{\Upsilon_1^d, ..., \Upsilon_{I_d}^d\}, n(t) = 0\right] &= 0,
\end{aligned}$$

In the following, we use i and j with and without a prime for the upsream machine; and k and l for the downstream machine. The dynamics of α_u and α_d are given by

$$\begin{aligned}
r_{ji}^u &= \text{prob } (\alpha_u(t+1) = \Upsilon_i^u \mid \alpha_u(t) = \Delta_j^u) \\
r_{lk}^d &= \text{prob } (\alpha_d(t+1) = \Upsilon_k^d \mid \alpha_d(t) = \Delta_l^d) \\
p_{ij}^u &= \text{prob } (\alpha_u(t+1) = \Delta_j^u \mid \alpha_u(t) = \Upsilon_i^u \text{ and } n(t) < N) \\
p_{kl}^d &= \text{prob } (\alpha_d(t+1) = \Delta_l^d \mid \alpha_d(t) = \Upsilon_k^d \text{ and } n(t) > 0) \\
y_{ii'}^u &= \text{prob } (\alpha_u(t+1) = \Upsilon_{i'}^u \mid \alpha_u(t) = \Upsilon_i^u \text{ and } n(t) < N) \\
y_{kk'}^d &= \text{prob } (\alpha_d(t+1) = \Upsilon_{k'}^d \mid \alpha_d(t) = \Upsilon_k^d \text{ and } n(t) > 0) \\
z_{jj'}^u &= \text{prob } (\alpha_u(t+1) = \Delta_{j'}^u \mid \alpha_u(t) = \Delta_j^u) \\
z_{ll'}^d &= \text{prob } (\alpha_d(t+1) = \Delta_{l'}^d \mid \alpha_d(t) = \Delta_l^d)
\end{aligned}$$

where $i = 1, ..., I_u; j = 1, ..., J_u; k = 1, ..., I_d; l = 1, ..., J_d$ and similarly for the primed symbols.

Dynamics of n:

$$n(t+1) = n(t) + \mathcal{I}_u(t+1) - \mathcal{I}_d(t+1),$$

where

$$\mathcal{I}_u(t+1) = \begin{cases} 1 \text{ if } \alpha_u(t+1) \in \left\{\Upsilon_1^u, ..., \Upsilon_{I_u}^u\right\} \\ \quad \text{and } n(t) < N, \\ \\ 0 \text{ otherwise.} \end{cases} \qquad \text{and} \qquad \mathcal{I}_d(t+1) = \begin{cases} 1 \text{ if } \alpha_d(t+1) \in \left\{\Upsilon_1^d, ..., \Upsilon_{I_d}^d\right\} \\ \quad \text{and } n(t) > 0 \\ \\ 0 \text{ otherwise.} \end{cases}$$

2.1. Transition Equations

We seek $\mathbf{p}(n,\alpha_u,\alpha_d)$, the steady-state probabilities of the states (n,α_u,α_d). We define $\mathbf{p}^{\Upsilon\Upsilon}(n)$ as the matrix of $\mathbf{p}(n,\alpha_u,\alpha_d)$ in which $\alpha_u=\Upsilon_1^u,...,\Upsilon_{I_u}^u$ and $\alpha_d=\Upsilon_1^d,...,\Upsilon_{I_d}^d$. Similarly for $\mathbf{p}^{\Upsilon\Delta}(n)$, $\mathbf{p}^{\Delta\Upsilon}(n)$, and $\mathbf{p}^{\Delta\Delta}(n)$.

Internal Equations For $n=2,...,N-2$, there are a set of four equations for the steady-state probabilities. For brevity, we show only one:

$$\mathbf{p}^{\Delta\Upsilon}(n)=Z^{u^T}\mathbf{p}^{\Delta\Delta}(n+1)R^d+Z^{u^T}\mathbf{p}^{\Delta\Upsilon}(n+1)Y^d+P^{u^T}\mathbf{p}^{\Upsilon\Delta}(n+1)R^d+P^{u^T}\mathbf{p}^{\Upsilon\Upsilon}(n+1)Y^d \tag{1}$$

in which the coefficient matrices are made up of the transition probabilities.

Boundary Equations The equations for $\mathbf{p}(n,\alpha_u,\alpha_d)$ are considerably more complicated for $n=0,1,N-1,N$. For brevity again, we do not display them here.

3. Analysis

3.1. Internal Equations

A product or sum-of-products form solution has worked for the two-machine models cited above. Therefore, we assume the more general sum-of-products form

$$\mathbf{p}(n,\alpha_u,\alpha_d)=\sum_{m=1}^{M}c_m\pi_m(n,\alpha_u,\alpha_d), \qquad \text{for } n=2,...,N-2, \text{ where}$$

$$\begin{array}{rr} \pi_m(n,\Upsilon_i^u,\Upsilon_k^d)=X_m^nU_{m1i}U_{m2k} & \pi_m(n,\Upsilon_i^u,\Delta_l^d)=X_m^nU_{m1i}D_{m2l} \\ \pi_m(n,\Delta_j^u,\Upsilon_k^d)=X_m^nD_{m1j}U_{m2k} & \pi_m(n,\Delta_j^u,\Delta_l^d)=X_m^nD_{m1j}D_{m2l} \end{array}$$

and X_m, U_{m1i}, U_{m2k}, D_{m1j}, D_{m2l}, c_m $(m=1,...,M)$, and M are constants to be determined. The m subscript is suppressed wherever possible in the following. The internal form is then

$$\begin{array}{rr} \pi^{\Delta\Delta}(n)=\{\pi(n,\Delta_j^u,\Delta_l^d)\}=X^nD^uD^{d^T} & \pi^{\Delta\Upsilon}(n)=\{\pi(n,\Delta_j^u,\Upsilon_k^d)\}=X^nD^uU^{d^T} \\ \pi^{\Upsilon\Delta}(n)=\{\pi(n,\Upsilon_i^u,\Delta_l^d)\}=X^nU^uD^{d^T} & \pi^{\Upsilon\Upsilon}(n)=\{\pi(n,\Upsilon_i^u,\Upsilon_k^d)\}=X^nU^uU^{d^T} \end{array}$$

where U^u,D^u,U^d,D^d are vectors $U^u=\{U_{1i}\}$, $D^u=\{D_{1j}\}$, $U^d=\{U_{2k}\}$, $D^d=\{D_{2l}\}$. Then (1) is

$$\begin{aligned} \pi^{\Delta\Upsilon}(n) &= Z^{u^T}\pi^{\Delta\Delta}(n+1)R^d+Z^{u^T}\pi^{\Delta\Upsilon}(n+1)Y^d+P^{u^T}\pi^{\Upsilon\Delta}(n+1)R^d+P^{u^T}\pi^{\Upsilon\Upsilon}(n+1)Y^d \\ &= X^{n+1}\left[Z^{u^T}D^uD^{d^T}R^d+Z^{u^T}D^uU^{d^T}Y^d+P^{u^T}U^uD^{d^T}R^d+P^{u^T}U^uU^{d^T}Y^d\right] \end{aligned}$$

or

$$X^nD^uU^{d^T}=X^{n+1}\left[D^{u^T}Z^u+U^{u^T}P^u\right]^T\left[D^{d^T}R^d+U^{d^T}Y^d\right] \tag{2}$$

The same analysis can be performed for the other three internal equations. Let us define

$$\begin{array}{rr} L^u=D^{u^T}Z^u+U^{u^T}P^u & K^u=D^{u^T}R^u+U^{u^T}Y^u \\ L^d=D^{d^T}Z^d+U^{d^T}P^d & K^d=D^{d^T}R^d+U^{d^T}Y^d \end{array}$$

Then the internal transitions equations become

$$U^u U^{d^T} = K^u K^{d^T} \tag{3}$$
$$X U^u D^{d^T} = K^u L^{d^T} \tag{4}$$
$$X^{-1} D^u U^{d^T} = L^u K^{d^T} \tag{5}$$
$$D^u D^{d^T} = L^u L^{d^T} \tag{6}$$

Equations (3) and (6) can be transformed into the following, in which A and B are constants to be determined.

$$R^u D^u + Y^u U^u - AU^u = 0 \qquad\qquad Z^u D^u + P^u U^u - BD^u = 0$$

If A is such that $Y^u - AI_{I_u}$ is invertible (where I_{I_u} is the $I_u \times I_u$ identity matrix), then

$$U^u = -(Y^u - AI_{I_u})^{-1} R^u D^u \tag{7}$$

so

$$(Z^u - P^u(Y^u - AI_{I_u})^{-1} R^u - BI_{J_u}) D^u = 0$$

For this equation to have a solution, the coefficient matrix must have a 0 determinant, or

$$\det(Z^u - P^u(Y^u - AI_{I_u})^{-1} R^u - BI_{J_u}) = 0$$

This is one equation in the two unknowns A and B. B is an eigenvalue of the matrix

$$Z^u - P^u(Y^u - AI_{I_u})^{-1} R^u. \tag{8}$$

Then D^u is an eigenvector of (8), and U^u is obtained from (7). In the same manner, A and B must also satisfy

$$R^d D^d + Y^d U^d - A^{-1} U^d = 0 \qquad\qquad Z^d D^d + P^d U^d - B^{-1} D^d = 0$$

If A is such that $Y^d - A^{-1} I_{I_d}$ is invertible, then

$$U^d = -(Y^d - A^{-1} I_{I_d})^{-1} R^d D^d \tag{9}$$

and

$$(Z^d - P^d(Y^d - A^{-1} I_{I_d})^{-1} R^d - B^{-1} I_{I_d}) D^d = 0$$

Therefore as before, B^{-1} is an eigenvalue of

$$Z^d - P^d(Y^d - A^{-1} I_{I_d})^{-1} R^d, \tag{10}$$

D^d is an eigenvalue of (10), and U^d is obtained from (9).

These two systems of eigenvalue equations provide two sets of equations in two unknowns (A and B). Figure 3 shows examples. (Details of the cases are eliminated for brevity.) Each graph consists of multiple lines; the shades of the lines indicates whether they come from the first set of determinant equations or the second. In these graphs, all A and B are real; however, A and B are complex in general.

The values of A and B that are needed are those that simultaneously satisfy both sets of equations. They occur at the intersections of the lines in the examples of Figure 3 (because all eigenvalues are real). We have observed that there are always $M = I_u J_d + I_d J_u$ intersections (including possibly repeated roots). There is always an intersection at $A = B = 1$. From these M values of A_m, B_m, we can find X_m, $U^u_m, D^u_m, U^d_m, D^d_m$.

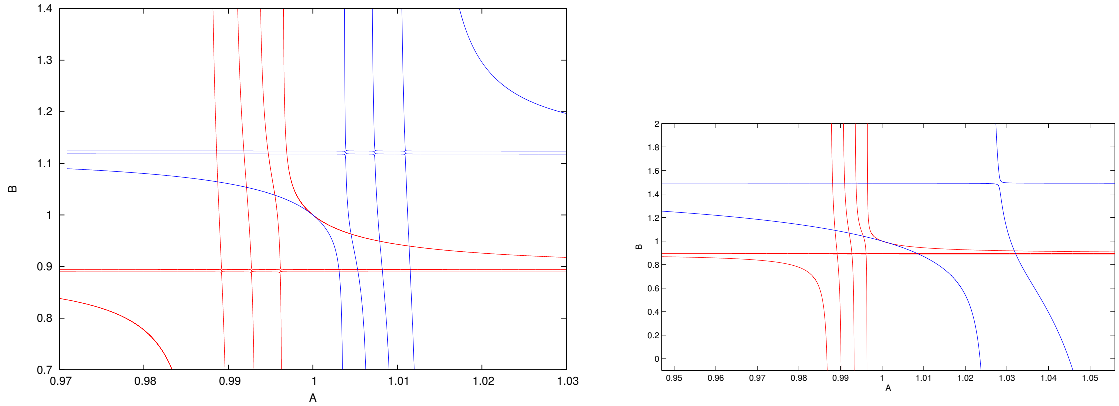

Figure 3 A–B **graphs**

3.2. Boundary Equations

The remaining transition equations are used to develop expressions for πs for some of the boundary states and for the c_m coefficients. From them, the probability distribution and performance measures can be derived.

4. Numerical Results

Consider a two-machine line whose machines are identical and have the Markov chain illustrated in Figure 4. The probability of transition from the up state to the first down state is .1; the transition probabilities from each down state to the next, and to the up state, is .98. In this case, the MTTF is 10 time steps and the MTTR is 14 time steps. Note that the up time is geometrically distributed while the down time is not. (It has a coefficient of variation of .3742.)

The graph on the right shows the probability distributions of buffer levels for buffer sizes 50, 100, and 150. The distributions have damped sinusoid-like appearance which is quite striking. We attribute this to the near-deterministic down times of the machines.

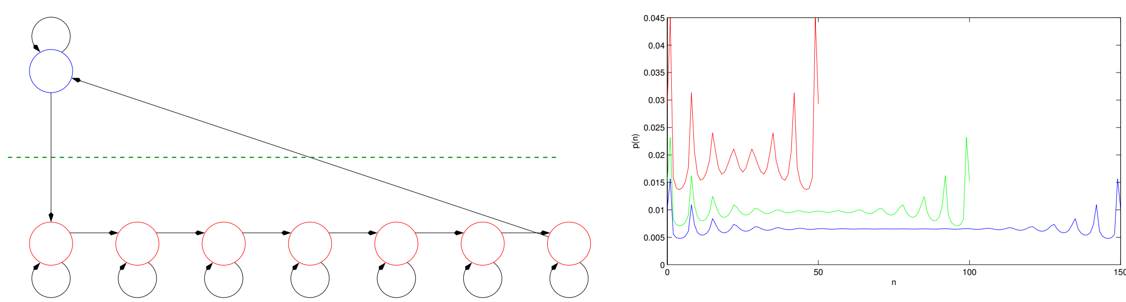

Figure 4 **Example 1**

The transition graph in the left side of Figure 5 is that of the upstream machine of a two-machine line. The transition probabilities from each up state to the next, and from the last up state to the down state, is .9. The transition probability from the down state to the first up state is .09. As a consequence, MTTF=MTTR=11.11 so the isolated efficiency is .5. The downstream machine has a single up state and a single down state, and the probabilities of repair and failure are both .01. It also has an isolated efficiency of .5, but its MTTF=MTTR=100. The buffer size is 100.

The buffer level probability distribution again has the damped sinusoid appearance. In this case, because the first machine has a nearly deterministic up time, it is essentially producing parts in almost-equal-sized batches. Note that the peak probabilities seem to be at close to multiples of 11.11.

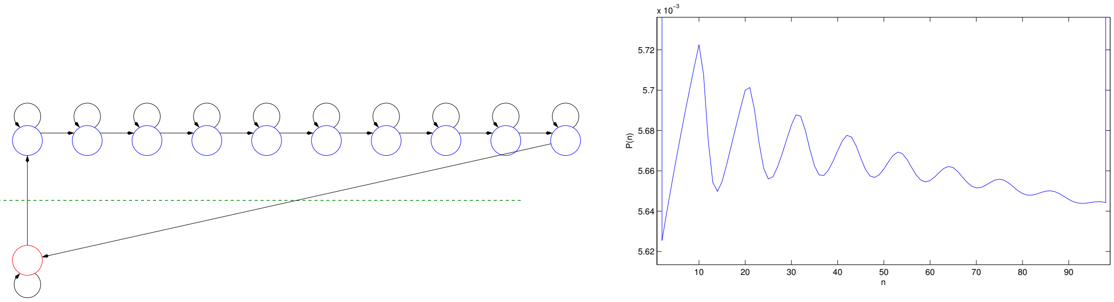

Figure 5 **Example 2**

5. Conclusion

We have briefly summarized a solution method for two-machine lines whose machines' dynamics are described by arbitrary Markov chains. We have demonstrated that the method works for buffers of size 100, even though the line's state space is large and complex.

Acknowledgments

We are grateful for support from the Singapore-MIT Alliance and the General Motors Research and Development Center.

References

Buzacott, John A. 1967. Automatic transfer lines with buffer stocks. *International Journal of Production Research* **5**(3) 183–200.

Colledani, M., Andrea Matta, Tullio Tolio. 2003. Performance evaluation of production lines with finite buffer capacity producing two different products. *Fourth International Conference on Analysis of Manufacturing Systems*.

Colledani, M., Tullio Tolio. 2005. Impact on statistical process control (SPC) on the performance of production systems — part one (small systems). *Fifth International Conference on Analysis of Manufacturing Systems — Production Management*.

Gershwin, S. B. 1994. *Manufacturing Systems Engineering*. Prentice-Hall, Englewood Cliffs, NJ. For corrections, see `http://web.mit.edu/manuf-sys/www/gershwin.errata.html`.

Gershwin, S. B., I. C. Schick. 1983. Modeling and analysis of three stage transfer lines with unreliable machines and finite buffers. *Operations Research* **31**(2) 354–377.

Gershwin, Stanley B., Saeideh Fallah-Fini. 2007. A general model and analysis of a discrete two-machine production line — detailed derivations. In preparation.

Jang, Young Jae. 2007. Mathematical modeling and analysis of flexible production lines. Ph.D. thesis, Massachusetts Institute of Technology.

Kim, Jongyoon, Stanley B. Gershwin. 2005. Integrated quality and quantity modeling of a production line. *OR Spectrum* **27**(2–3) 287–314.

Tan, Baris, Stanley B. Gershwin. 2007. Modeling and analysis of markovian single-buffer fluid flow systems. *Analysis of Manufacturing Systems 2007*.

Tolio, Tullio, Andrea Matta, Stanley B. Gershwin. 2002. Analysis of two-machine lines with multiple failure modes. *IIE Transactions* **34**(1) 51–62.