A General Model and Analysis of a Discrete Two-Machine Production Line

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Problem

Two-Machine Line

$M_1 \rightarrow B \rightarrow M_2$

Examples

- Two-machine production line, where machines are unreliable or have random processing times.
- Queueing system, where the first machine represents the arrival process, the second is the service process.

The buffer is finite.
To allow both machines to have behavior governed by any possible Markov process.

When a machine is in an up state and not blocked or starved, it does an operation. That is, it adds or subtracts one part from the buffer.

When it is in a down state, or blocked or starved, it does not change the buffer level.
Problem

Two-Machine Line

New contribution

\[ M_1 \rightarrow B \rightarrow M_2 \]
Motivation

- In a queueing system, to allow general arrival time and general service time distributions.
- In a production line,
  ★ to allow general processing time or failure time distributions;
  ★ to improve the accuracy of decomposition methods.
Problem

General Independent Arrivals or Service
Machine with Quality Deterioration
Problem

- **Dynamics**
  - discrete time, discrete material \(\leftarrow\) *focus of this talk*
  - continuous time, discrete material
  - continuous time, continuous material

- **Machine behavior**
  - one up state, one down state: Buzacott, Gershwin, etc.
  - one up state, phase-type down: Dallery
  - one up state, multiple down states: Tolio et al.
  - some cases of multiple up and down states: Gershwin et al., Tolio et al. for quantity/quality analysis
  - etc.
Calculate steady-state probability distribution.

Use it to calculate production rate, average inventory, etc.
• Neuts and others have studied *matrix-geometric* methods for solving systems like this.
  ★ The transition matrix is written in block diagonal form, and an analytical solution in terms of matrix exponentials is found.

• We use a *spectral (eigenvalue) analysis* approach.
Problem

Generalization of earlier two-machine line models and solution methods:

1. Form internal (transition or balance) equations, boundary equations, and normalization.
2. Guess a solution (based on elementary methods of ordinary difference or differential equations) of the internal equations.
3. Determine equations for the parameters of the guess.
4. Determine sets of values for the parameters.
5. Extend the guesses to the boundary.
6. Express the solution as a linear combination of the guesses.
7. Determine coefficients.
• State of the system is $s = (n, \alpha_u, \alpha_d)$.
  
  * $n = 0, 1, \ldots, N$.
  * If $\alpha_u = \Upsilon^u_1, \ldots, \Upsilon^u_{I_u}$, $M_u$ is up.
  * If $\alpha_u = \Delta^u_1, \ldots, \Delta^u_{J_u}$, $M_u$ is down.
  * If $\alpha_d = \Upsilon^d_1, \ldots, \Upsilon^d_{I_d}$, $M_d$ is up.
  * If $\alpha_d = \Delta^d_1, \ldots, \Delta^d_{J_d}$, $M_d$ is down.

• **Operation-dependence:** $\alpha_u$ cannot leave an up state if $n = N$. $\alpha_d$ cannot leave an up state if $n = 0$.

\[
\text{prob} \left[ \alpha_u(t + 1) \neq \alpha_u(t) \mid \alpha_u(t) = \Upsilon^u_1, \ldots, \Upsilon^u_{I_u}, n(t) = N \right] = 0
\]

\[
\text{prob} \left[ \alpha_d(t + 1) \neq \alpha_d(t) \mid \alpha_d(t) = \Upsilon^d_1, \ldots, \Upsilon^d_{I_d}, n(t) = 0 \right] = 0,
\]
The dynamics of $\alpha_u$ are given by

$$r_{ji}^u = \text{prob} \left( \alpha_u(t + 1) = \Upsilon_i^u \mid \alpha_u(t) = \Delta_j^u \right)$$

$$p_{ij}^u = \text{prob} \left( \alpha_u(t + 1) = \Delta_j^u \mid \alpha_u(t) = \Upsilon_i^u \text{ and } n(t) < N \right)$$

$$y_{ii'}^u = \text{prob} \left( \alpha_u(t + 1) = \Upsilon_{i'}^u \mid \alpha_u(t) = \Upsilon_i^u \text{ and } n(t) < N \right)$$

$$z_{jj'}^u = \text{prob} \left( \alpha_u(t + 1) = \Delta_{j'}^u \mid \alpha_u(t) = \Delta_j^u \right)$$

where $r$ is a repair probability, $p$ is a failure probability, $y$ is a up state change probability, $z$ is a down state change probability.
Similarly for $\alpha_d$:

$$r_{lk}^d = \text{prob} \left( \alpha_d(t + 1) = \Upsilon_k^d \mid \alpha_d(t) = \Delta_l^d \right)$$

$$p_{kl}^d = \text{prob} \left( \alpha_d(t + 1) = \Delta_l^d \mid \alpha_d(t) = \Upsilon_k^d \text{ and } n(t) > 0 \right)$$

$$y_{kk'}^d = \text{prob} \left( \alpha_d(t + 1) = \Upsilon_{k'}^d \mid \alpha_d(t) = \Upsilon_k^d \text{ and } n(t) > 0 \right)$$

$$z_{ll'}^d = \text{prob} \left( \alpha_d(t + 1) = \Delta_{l'}^d \mid \alpha_d(t) = \Delta_l^d \right)$$
System Definition

Define the matrices

\[ R^u = \{ r^u_{ji} \}, \quad R^d = \{ r^d_{lk} \}, \quad P^u = \{ p^u_{ij} \}, \quad P^d = \{ p^d_{kl} \} \]

Also define the square matrices

\[ Y^u = \{ y^u_{i'i} \}, \quad Y^d = \{ y^d_{kk'} \}, \quad Z^u = \{ z^u_{jj'} \}, \quad Z^d = \{ z^d_{ll'} \} \]

for off-diagonal elements. The diagonal elements are given by

\[
y^u_{ii} = 1 - \sum_{j=1}^{J_u} p^u_{ij} - \sum_{i'=1, i' \neq i}^{I_u} y^u_{ii'} \\
y^d_{kk} = 1 - \sum_{l=1}^{J_d} p^d_{kl} - \sum_{k'=1, k' \neq k}^{I_d} y^d_{kk'} \\
z^u_{jj} = 1 - \sum_{i=1}^{I_u} r^u_{ji} - \sum_{j'=1, j' \neq j}^{J_u} z^u_{jj'} \\
z^d_{jj} = 1 - \sum_{k=1}^{I_d} r^d_{lk} - \sum_{l'=1, l' \neq l}^{J_d} z^d_{ll'}
\]
System Definition

Transition Equations

\[ p(n, \alpha_u, \alpha_d) = \sum_{n', \alpha'_u, \alpha'_d} T(n, \alpha_u, \alpha_d; n', \alpha'_u, \alpha'_d) \ p(n', \alpha'_u, \alpha'_d) \]

where

- \( p(n, \alpha_u, \alpha_d) \) is the steady-state probability distribution
- \( (n', \alpha'_u, \alpha'_d) \) is the state of the system at the previous time step, and
- \( T(n, \alpha_u, \alpha_d; n', \alpha'_u, \alpha'_d) \) is the probability of transition from \( (n', \alpha'_u, \alpha'_d) \) to \( (n, \alpha_u, \alpha_d) \) in one step.
Define $p^{\Upsilon\Upsilon}(n)$ as the matrix of $p(n, \alpha_u, \alpha_d)$ in which $\alpha_u = \Upsilon^u_1, \ldots, \Upsilon^u_{I_u}$ and $\alpha_d = \Upsilon^d_1, \ldots, \Upsilon^d_{I_d}$.

Similarly for $p^{\Upsilon\Delta}(n)$, $p^{\Delta\Upsilon}(n)$, and $p^{\Delta\Delta}(n)$. 
For \( n = 2, \ldots, N - 2 \),

\[
p^{\gamma\gamma}(n) = R_{u}^{T} p^{\Delta\Delta}(n) R^{d} + R_{u}^{T} p^{\Delta\gamma}(n) Y^{d}
\]

\[+ Y_{u}^{T} p^{\gamma\Delta}(n) R^{d} + Y_{u}^{T} p^{\gamma\gamma}(n) Y^{d}\]

\[
p^{\gamma\Delta}(n) = R_{u}^{T} p^{\Delta\Delta}(n - 1) Z^{d} + R_{u}^{T} p^{\Delta\gamma}(n - 1) P^{d}
\]

\[+ Y_{u}^{T} p^{\gamma\Delta}(n - 1) Z^{d} + Y_{u}^{T} p^{\gamma\gamma}(n - 1) P^{d}\]
System Definition

Transition Equations

Internal equations

\[ p^\Delta \gamma (n) = Z^{uT} p^\Delta \Delta (n + 1) R^d + Z^{uT} p^\Delta \gamma (n + 1) Y^d \]

\[ + P^{uT} p^{\gamma \Delta} (n + 1) R^d + P^{uT} p^{\gamma \gamma} (n + 1) Y^d \]

\[ p^\Delta \Delta (n) = Z^{uT} p^\Delta \Delta (n) Z^d + Z^{uT} p^\Delta \gamma (n) P^d \]

\[ + P^{uT} p^{\gamma \Delta} (n) Z^d + P^{uT} p^{\gamma \gamma} (n) P^d \]
Assume that

\[ p(n, \alpha_u, \alpha_d) = \sum_{m=1}^{M} c_m \pi_m(n, \alpha_u, \alpha_d) \]

where, for \( n = 2, \ldots, N - 2 \),

\[ \pi_m(n, \Upsilon_i^u, \Upsilon_k^d) = X_m^n U_{m1i} U_{m2k} \]
\[ \pi_m(n, \Upsilon_i^u, \Delta_l^d) = X_m^n U_{m1i} D_{m2l} \]
\[ \pi_m(n, \Delta_j^u, \Upsilon_k^d) = X_m^n D_{m1j} U_{m2k} \]
\[ \pi_m(n, \Delta_j^u, \Delta_l^d) = X_m^n D_{m1j} D_{m2l} \]

and \( X_m, U_{m1i}, U_{m2k}, D_{m1j}, D_{m2l}, c_m, \) and \( M \) are constants to be determined.

In the following, we suppress the \( m \) subscript until we really need it.
Solution of Internal Equations

Matrix form:

Let

\[ U^u = \{U_{1i}\}, \quad U^d = \{U_{1k}\}, \]
\[ D^u = \{D_{1j}\}, \quad D^d = \{D_{1l}\}. \]

Then

\[ \{U_{1i}U_{2k}\} = U^uU^dT, \]

etc.
Solution of Internal Equations

Then,

\[ \pi^\gamma (n) = \{ \pi(n, \gamma^u_i, \gamma^d_k) \} = X^n U^u U^d T \]

\[ \pi^\Delta (n) = \{ \pi(n, \gamma^u_i, \Delta^d_l) \} = X^n U^u D^d T \]

\[ \pi^\Delta^\gamma (n) = \{ \pi(n, \Delta^u_j, \gamma^d_k) \} = X^n D^u U^d T \]

\[ \pi^\Delta^\Delta (n) = \{ \pi(n, \Delta^u_j, \Delta^d_l) \} = X^n D^u D^d T \]
... and the first internal transition equation becomes

\[ p^{\gamma\gamma}(n) = R^{uT}p^{\Delta\Delta}(n)R^d + R^{uT}p^{\Delta\gamma}(n)Y^d + Y^{uT}p^{\gamma\Delta}(n)R^d + Y^{uT}p^{\gamma\gamma}(n)Y^d \]

\[ X^nU^uU^d^T = R^{uT}X^nD^uD^d^T R^d + R^{uT}X^nD^uU^d^T Y^d + Y^{uT}X^nD^uU^d^T R^d + Y^{uT}X^nU^uU^d^T Y^d \]

or,

\[ U^uU^d^T = R^{uT}D^uD^d^T R^d + R^{uT}D^uU^d^T Y^d + Y^{uT}D^uU^d^T R^d + Y^{uT}U^uU^d^T Y^d \]

This can be factored. Doing the same for the other equations, ...
Solution of Internal Equations

... the internal equations become

\[ U^u U^dT = K^u K^dT \]
\[ X U^u D^dT = K^u L^dT \]
\[ X^{-1} D^u U^dT = L^u K^dT \]
\[ D^u D^dT = L^u L^dT \]

where

\[ K^u = R^{uT} D^u + Y^{uT} U^u \]
\[ L^u = Z^{uT} D^u + P^{uT} U^u \]
\[ K^d = R^{dT} D^d + Y^{dT} U^d \]
\[ L^d = Z^{dT} D^d + P^{dT} U^d \]

**Important:** Each equation is a matrix equation, but the matrices are products of vectors, so each element of each matrix is of the form \( a_i b_j \).

**Observation:** The first equation can be derived from the last three.
The first equation can be written

\[ U_i^u U_k^d = K_i^u K_k^d \]

Therefore, for all \( i \) and \( k \), there must be a scalar \( A \) such that

\[ A U_i^u = K_i^u \]
\[ A^{-1} U_k^d = K_k^d \]

or

\[ A U^u = K^u \]
\[ A^{-1} U^d = K^d \]
Similarly, the last equation can be written

\[ D_j^u D_l^d = L_j^u L_l^d \]

and, similarly, for all \( j \) and \( l \), there must be a scalar \( B \) such that

\[ BD_j^u = L_j^u \]
\[ B^{-1} D_l^d = L_l^d \]

or

\[ BD_j^u = L_j^u \]
\[ B^{-1} D_l^d = L_l^d \]
Of these pairs of equations, there is one that is only about the upstream machine and one that deals only with the downstream machine. The upstream machine equations are

\[ AU^u = K^u \]
\[ BD^u = L^u \]

or

\[ Ru^u Du^u + Y^u U^u - AU^u = 0 \]
\[ Zu^u Du^u + Pu^u U^u - BD^u = 0 \]
Solution of Internal Equations

\[ R^u D^u + Y^u U^u - A U^u = 0 \]
\[ Z^u D^u + P^u U^u - B D^u = 0 \]

Note that \( Y^u \) and \( Z^u \) are square matrices. If \( A \) is such that \( Y^u - A I_{I_u} \) is invertible (where \( I_{I_u} \) is the \( I_u \times I_u \) identity matrix), then

\[ U^u = -(Y^u - A I_{I_u})^{-1} R^u D^u \]

(1)

so

\[(Z^u - P^u (Y^u - A I_{I_u})^{-1} R^u - B I_{J_u}) D^u = 0\]
In order for this equation to have a solution, the coefficient matrix must have a 0 determinant, or

\[
\det(Z^u - P^u(Y^u - AI_{I_u})^{-1}R^u - BI_{I_u}) = 0
\]

This is one equation in the two unknowns \(A\) and \(B\). To put it another way, \(B\) is an eigenvalue of the matrix

\[
Z^u - P^u(Y^u - AI_{I_u})^{-1}R^u. \tag{2}
\]

Then \(D^u\) is an eigenvector of (2), and \(U^u\) is obtained from (1).
Solution of Internal Equations

Analysis of Equations

\[ B = \text{eigval}(Z^u - P^u(Y^u - AI_{Iu})^{-1}R^u) \]

1. This establishes \( B \) as a (multi-valued) function of \( A \). We must do the same thing for the second machine, which will give a similar relationship between \( A \) and \( B \). The simultaneous solutions of both equations determine the internal solutions \((m = 1, 2, ..., M)\).

2. The poles of (2) occur when \( Y^u - AI_{Iu} \) is not invertible; ie when \( A \) is an eigenvalue of \( Y^u \).
3. Asymptotes

- As $A \to$ an eigenvalue of $Y^u$, (2) becomes an infinite matrix and therefore (all?) its eigenvalues become infinite. Consequently, the eigenvalues of $Y^u$ are asymptotes of $A$ as a function of $B$.

- As $B \to$ an eigenvalue of $Z^u$, $P^u(Y^u - AI_{Iu})^{-1}R^u \to 0$. Therefore $Y^u - AI_{Iu} \to \infty$. Consequently, the eigenvalues of $Z^u$ are asymptotes of $B$ as a function of $A$. 
Solution of Internal Equations

Analysis of Equations

\[ B = \text{eigval}(Z^u - P^u(Y^u - AI_{Iu})^{-1} R^u) \]

Similarly, for the downstream machine

\[
\begin{align*}
R^d D^d + Y^d U^d - A^{-1} U^d & = 0 \\
Z^d D^d + P^d U^d - B^{-1} D^d & = 0
\end{align*}
\]

If \( A \) is such that \( Y^d - A^{-1} I_{Id} \) is invertible, then

\[
U^d = -(Y^d - A^{-1} I_{Id})^{-1} R^d D^d
\]

and

\[
(Z^d - P^d(Y^d - A^{-1} I_{Id})^{-1} R^d - B^{-1} I_{Id})D^d = 0
\]

Therefore as before, \( B^{-1} \) is an eigenvalue of

\[
Z^d - P^d(Y^d - A^{-1} I_{Id})^{-1} R^d,
\]

\( D^d \) is an eigenvalue of (4), and \( U^d \) is obtained from (3).
Identical machines with 4 up states, 3 down states, no up-to-up or down-to-down transitions, and

\[ P^u = P^d = \begin{bmatrix} .0011 & .0012 & .0013 \\ .0021 & .0022 & .0023 \\ .0031 & .0032 & .0033 \\ .0041 & .0042 & .0043 \end{bmatrix} \]

\[ (R^u)^T = (R^d)^T = \begin{bmatrix} .011 & .012 & .013 \\ .021 & .022 & .023 \\ .031 & .032 & .033 \\ .041 & .042 & .043 \end{bmatrix} \]
Solution of Internal Equations

Analysis of Equations

Example 1

\[
\begin{align*}
(Z^u - P^u(Y^u - AIu)^{-1}R^u \\
- BIu)D^u &= 0
\end{align*}
\]

\[
\begin{align*}
(Z^d - P^d(Y^d - A^{-1}Id)^{-1}R^d \\
- B^{-1}Id)D^d &= 0
\end{align*}
\]

Red for $M_1$; blue for $M_2$. 

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Observations

- The intersections are where $A$ and $B$ are such that the equations for $X$, $U_{1i}, U_{2k}, D_{1j}$, and $D_{2l}$ are satisfied. Each intersection leads to one set of $X_m, U_{m1i}, U_{m2k}, D_{m1j}$, and $D_{m2l}$.
- At the intersections, $X_m = A_m/B_m$.
- Both sets of graphs go through $(1,1)$ (where $X_m$ is therefore 1). In symmetric cases, there is a double root at $(1,1)$ and the graphs are tangent. In other cases, one root is very close to $(1,1)$, so the curves are almost tangent.
- There are a total of $M = 23$ intersections between the red and blue lines, which means that there are 23 roots (counting $(1,1)$ once). (In general, $I_uJ_d + I_dJ_u - 1$.)
- Don’t be fooled by lines of the same color that seem to intersect. The next graph zooms in on the previous.
$M_1$ same as before; $M_2$ with 2 up states, 2 down states, no up-to-up or down-to-down transitions, and

$$P^d = \begin{bmatrix} .011 & .012 \\ .021 & .022 \end{bmatrix} \quad (R^d)^T = \begin{bmatrix} .11 & .12 \\ .21 & .22 \end{bmatrix}$$
Solution of Internal Equations

Analysis of Equations

Example 2

(The red graph is the same as before.)
The numerical determination of the intersections is not difficult.

★ Each intersection is close to the intersection of asymptotes.
★ We can easily determine the asymptotes.
★ A simple iterative method was developed which appears always to converge, even when roots are complex.
Boundary Equations

Transient States

\[(0, \gamma_i^u, \Delta_i^d), (0, \gamma_i^u, \gamma_k^d), (0, \Delta_j^u, \Delta_i^d), (1, \gamma_i^u, \Delta_i^d),\]

\[(N, \Delta_j^u, \Delta_i^d), (N, \Delta_j^u, \gamma_k^d), (N, \gamma_i^u, \gamma_k^d),\]

\[(N - 1, \Delta_j^u, \gamma_k^d)\]

are all transient.
The boundary \((n = 0, 1, N - 1, N)\) transition equations are messier than the internal equations and do not seem to lead to such (relatively) neat solutions.

All the \(\pi\)s corresponding to transient states are set to 0.

We have \(\pi(1, \Delta_j^u, \Upsilon_k^d) = XD_{1j}U_{2k}\) but we do not have simple expressions for \(\pi(1, \Delta_j^u, \Delta_k^d)\), \(\pi(1, \Upsilon_i^u, \Upsilon_k^d)\), or \(\pi(0, \Delta_j^u, \Upsilon_k^d)\). Similarly for the upper boundary.

A numerical solution of the boundary equations, which also produces the \(c_m\) coefficients, is straightforward. Its complexity depends only on the Markov chains for the machine models, and not the buffer size.
Example 3

Symmetric Line, both machines
Example 3

\[ P = \begin{bmatrix} .1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad Y = \begin{bmatrix} .9 \end{bmatrix}; \]

\[ R = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ .98 \end{bmatrix}; \quad Z = \begin{bmatrix} 0.02 & .98 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.02 & .98 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.02 & .98 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.02 & .98 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.02 & .98 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.02 & .98 \end{bmatrix} \]
Interpretation:

- When a machine is up, it makes a batch of parts, with an average batch size of 10. When it goes down, it stays down for very close to 7 time steps.

- Therefore, when $M_1$ goes down, the buffer loses about 7 parts. When $M_2$ goes down, the buffer gains about 7 parts.

- Therefore, the most likely buffer levels are $0, 7, 14, 21, ..., N, N - 7, N - 14, ...$
$N = 100$, $r$ varied.
More Examples

Example 4

This is the first machine. The second machine has 1 up state, 1 down state.
Interpretation:

- When the first machine is up, it makes a batch of 10 or a few more parts. Each time the first machine is operating, it adds about 10 parts to the buffer.
- The second machine removes one part at a time from the buffer.
- Therefore, the most likely buffer levels are close to multiples of 10.
More Examples

Example 4

\[ P(n) \times 10^{-3} \]

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Conclusions

• We have developed a method for two-machine lines with general machines — essentially, a discrete-time GI/GI/1/\(N\) queue.

• It works for a variety of cases (tested for machines with \(> 20\) states, \(N=100\)).

• Computational effort depends on machine complexity, not buffer size.

• Applications: quality/quantity models, multi-part-type decomposition, machines with non-exponential/geometric failure/repair times, ...
Conclusions

- Play with more cases.
  - Push sizes to limits.
  - Explore singularities with special values of parameters.
- Write efficient code.
- **Conjecture:** Systems with dense machine models (where there are many possible transitions among states) have all real roots; sparse machines lead to complex roots.
Conclusions

- Bernoulli production when a machine is in an upstate.
- Multiple part production when a machine is in an upstate.
- Discrete part, continuous time model.
- Continuous material, continuous time (done).
- Decomposition of large systems.
Conclusions

Discrete state, discrete time:


Continuous Material, continuous time:

Conclusions

Thank you.
System Definition

Transition Equations

Subscript form

\[
p(n, \gamma_i^u, \gamma_k^d) = \sum_{j=1}^{J_u} \sum_{l=1}^{J_d} p(n, \Delta_j^u, \Delta_l^d) r_{ji}^u r_{lk}^d
\]

\[
+ \sum_{j=1}^{J_u} p(n, \Delta_j^u, \gamma_k^d) r_{ji}^u \left( 1 - \sum_{l=1}^{J_d} p_{kl}^d - \sum_{k'=1 \atop k' \neq k}^{I_d} y_{kk'}^d \right) + \sum_{j=1}^{J_u} \sum_{k'=1 \atop k' \neq k}^{I_d} p(n, \Delta_j^u, \gamma_{k'}^d) r_{ji}^u y_{kk'}^d
\]

\[
+ p(n, \gamma_i^u, \gamma_k^d) \left( 1 - \sum_{j=1}^{J_u} p_{ij}^u - \sum_{i'=1 \atop i' \neq i}^{I_u} y_{ii'}^u \right) \left( 1 - \sum_{l=1}^{J_d} p_{kl}^d - \sum_{k'=1 \atop k' \neq k}^{I_d} y_{kk'}^d \right)
\]

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System Definition

Transition Equations

Subscript form

\[
\begin{align*}
&+ \left[ \sum_{k' = 1}^{J_d} p(n, \gamma^u_{i'}, \gamma^d_{k'}) y^d_{k'k} + \sum_{l=1}^{J_d} p(n, \gamma^u_{i'}, \Delta^d_l) r^d_{lk} \right] \left( 1 - \sum_{i=1}^{J_u} p^u_{ii'} - \sum_{i' = 1}^{I_u} y^u_{i'i} \right) \\
&+ \sum_{i' = 1}^{I_u} \sum_{l=1}^{J_d} p(n, \gamma^u_{i'}, \Delta^d_l) y^u_{i'i} r^d_{lk} + \sum_{i' = 1}^{I_u} \sum_{i' = 1}^{I_d} p(n, \gamma^u_{i'}, \gamma^d_{k'}) y^u_{i'i} \\
&+ \sum_{i' = 1}^{I_u} \sum_{k' = 1}^{I_u} \sum_{k' = 1}^{I_d} p(n, \gamma^u_{i'}, \gamma^d_{k'}) y^u_{i'i} y^d_{k'k} \\
&+ \sum_{i' = 1}^{I_u} \sum_{k' = 1}^{I_d} p(n, \gamma^u_{i'}, \gamma^d_{k'}) y^u_{i'i} y^d_{k'k}
\end{align*}
\]
If we plug this into the first internal equation, we get (with \( m \) suppressed)

\[
X^n U_{1i} U_{2k} =
\]

\[
\sum_{j=1}^{J_u} \sum_{l=1}^{J_d} X^n D_{1j} D_{2l} r_{ji}^u r_{lk}^d + \sum_{j=1}^{J_u} X^n D_{1j} U_{2k} r_{ji}^u \left( 1 - \sum_{l=1}^{J_d} p_{kl}^d - \sum_{k' = 1}^{I_d} y_{kk'}^d \right) \]

\[
+ \sum_{j=1}^{J_u} \sum_{k' = 1}^{I_d} \sum_{k' \neq k} X^n D_{1j} U_{2k'} r_{ji}^u y_{k'k}^d + \sum_{l=1}^{J_d} X^n U_{1i} D_{2l} \left( 1 - \sum_{j=1}^{J_u} p_{ij}^u - \sum_{i' = 1}^{I_u} y_{ii'}^u \right) r_{lk}^d \]

\[ \]

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Solution of Internal Equations

\[ + \sum_{k' = 1}^{I_d} X^n U_{1i} U_{2k'} \begin{pmatrix} 1 - \sum_{j=1}^{J_u} p_{ij}^u - \sum_{i' = 1}^{I_u} y_{ii'}^u \end{pmatrix} \begin{pmatrix} y_{k'k}^d \end{pmatrix} \]

\[ + X^n U_{1i} U_{2k} \begin{pmatrix} 1 - \sum_{j=1}^{J_u} p_{ij}^u - \sum_{i' = 1 \atop i' \neq i}^{I_u} y_{ii'}^u \end{pmatrix} \begin{pmatrix} 1 - \sum_{l=1}^{J_d} p_{kl}^d - \sum_{k' = 1 \atop k' \neq k}^{I_d} y_{k'k}^d \end{pmatrix} \]

\[ + \sum_{i' = 1 \atop i' \neq i}^{I_u} \sum_{l=1}^{J_d} X^n U_{1i'} D_{2l} y_{i'i}^u r_{lk}^d \]
Solution of Internal Equations

\[ + \sum_{i' = 1}^{I_u} X^n U_{1i'} U_{2k'} y_{i'i}^u \left( 1 - \sum_{l=1}^{J_d} p_{kl}^d - \sum_{k' = 1}^{I_d} y_{k'k'}^d \right) \]

\[ + \sum_{i' = 1}^{I_u} \sum_{k' = 1}^{I_d} X^n U_{1i'} U_{2k'} y_{i'i}^u y_{k'k}^d \]
This reduces to

\[ U_{1i}U_{2k} = \]

\[
\begin{bmatrix}
J_u 
\sum_{j=1} D_{1j}r_{ji}^u + U_{1i} 
1 - \sum_{j=1}^{J_u} p_{ij}^u - \sum_{i' = 1 \atop i' \neq i}^{I_u} y_{ii'}^u 
\end{bmatrix} + \begin{bmatrix}
I_u 
\sum_{i' = 1 \atop i' \neq i}^{I_u} U_{1i'}y_{i'i'}^u 
\end{bmatrix}
\]

\[
\begin{bmatrix}
J_d
\sum_{l=1}^{J_d} D_{2l}r_{lk}^d + U_{2k} 
1 - \sum_{l=1}^{J_d} p_{kl}^d - \sum_{k' = 1 \atop k' \neq k}^{I_d} y_{kk'}^d 
\end{bmatrix} + \begin{bmatrix}
I_d 
\sum_{k' = 1 \atop k' \neq k}^{I_d} U_{2k'}y_{k'k'}^d 
\end{bmatrix}
\]

which is a set of equations in \( U_{1i}, D_{1j}, U_{2k}, D_{2l} \).
Solution of Internal Equations

Also,

\[ XU_{1i}D_{2l} = \]

\[
\left[ \sum_{j=1}^{J_u} D_{1j} r_{ji}^u + U_{1i} \left( 1 - \sum_{j=1}^{J_u} p_{ij}^u - \sum_{i' = 1 \atop i' \neq i}^{I_u} y_{ii'}^u \right) + \sum_{i' = 1 \atop i' \neq i}^{I_u} U_{1i'i_i'} \right] \]

\[
\left[ D_{2l} \left( 1 - \sum_{k=1}^{I_d} r_{lk}^d - \sum_{l' = 1 \atop l' \neq l}^{J_d} z_{ll'}^d \right) + \sum_{l' = 1 \atop l' \neq l}^{J_d} D_{2l'} z_{ll'}^d + \sum_{k=1}^{I_d} U_{2k} p_{kl}^d \right] \]
Solution of Internal Equations

Guess

Subscript form

\[ X^{-1} D_{1j} U_{2k} = \]

\[
\begin{bmatrix}
D_{1j} \left( 1 - \sum_{i=1}^{I_u} r_{ji}^u - \sum_{j' = 1 \atop j' \neq j}^{J_u} z_{jj'}^u \right) + \sum_{j' = 1 \atop j' \neq j}^{J_u} D_{1j'} z_{jj'}^u + \sum_{i=1}^{I_u} U_{1i} p_{ij}^u \right) \\
\sum_{l=1}^{J_d} D_{2l} r_{lk}^d + U_{2k} \left( 1 - \sum_{l=1}^{J_d} p_{kl}^d - \sum_{k' = 1 \atop k' \neq k}^{I_d} y_{kk'}^d \right) + \sum_{k' = 1 \atop k' \neq k}^{I_d} U_{2k'} y_{kk'}^d 
\end{bmatrix}
\]
Solution of Internal Equations

Guess

Subscript form

\[ D_{1j} D_{2l} = \]

\[
D_{1j} \left( 1 - \sum_{i=1}^{I_u} r_{ji}^u - \sum_{j' = 1, j' \neq j}^{J_u} z_{jj'}^u \right) + \sum_{j' = 1, j' \neq j}^{J_u} D_{1j'} z_{j'j}^u + \sum_{i=1}^{I_u} U_i p_{ij}^u \]

\[
D_{2l} \left( 1 - \sum_{k=1}^{I_d} r_{lk}^d - \sum_{l' = 1, l' \neq l}^{J_d} z_{ll'}^d \right) + \sum_{l' = 1, l' \neq l}^{J_d} D_{2l'} z_{ll'}^d + \sum_{k=1}^{I_d} U_{2k} p_{kl}^d \]
The first equation can be written

\[
1 = \left\{ \sum_{j=1}^{J_u} D_{1j} r_{ji}^u + U_{1i} \left( 1 - \sum_{j=1}^{J_u} p_{ij}^u - \sum_{i' = 1 \atop i' \neq i}^{I_u} y_{ii'}^u \right) + \sum_{i' = 1 \atop i' \neq i}^{I_u} U_{1i'} y_{ii'}^u \right\} \frac{1}{U_{1i}}
\]

\[
\left\{ \sum_{l=1}^{J_d} D_{2l} r_{lk}^d + U_{2k} \left( 1 - \sum_{l=1}^{J_d} p_{kl}^d - \sum_{k' = 1 \atop k' \neq k}^{I_d} y_{kk'}^d \right) + \sum_{k' = 1 \atop k' \neq k}^{I_d} U_{2k'} y_{kk'}^d \right\} \frac{1}{U_{2k}}
\]
Solution of Internal Equations

The first fraction depends on \(i\) and the second depends on \(k\). Therefore, it is reasonable to assume that there exists some \(A\) such that, for all \(i\),

\[
\left\{ \sum_{j=1}^{J_u} D_{1j} r_{ji}^u + U_{1i} \left( 1 - \sum_{j=1}^{J_u} p_{ij}^u - \sum_{i' = 1}^{I_u} y_{i'i}^u \right) + \sum_{i' = 1}^{I_u} U_{1i'} y_{i'i}^u \right\} \frac{1}{U_{1i}} = A
\]

and, for all \(k\),

\[
\left\{ \sum_{l=1}^{J_d} D_{2l} r_{lk}^d + U_{2k} \left( 1 - \sum_{l=1}^{J_d} p_{kl}^d - \sum_{k' = 1 \atop k' \neq k}^{I_d} y_{kk'}^d \right) + \sum_{k' = 1 \atop k' \neq k}^{I_d} U_{2k'} y_{k'k}^d \right\} \frac{1}{U_{2k}} = \frac{1}{A}
\]
Similarly, for all $j$,

$$
\begin{align*}
\left\{ D_{1j} \left( 1 - \sum_{i=1}^{I_u} r_{ji}^u - \sum_{j' = 1 \atop j' \neq j}^{J_u} z_{jj'}^u \right) + \sum_{j' = 1 \atop j' \neq j}^{J_u} D_{1j'} z_{jj'}^u + \sum_{i=1}^{I_u} U_{1i} p_{ij}^u \right\} \frac{1}{D_{1j}} = B
\end{align*}
$$

and, for all $l$,

$$
\begin{align*}
\left\{ D_{2l} \left( 1 - \sum_{k=1}^{I_d} r_{lk}^d - \sum_{l' = 1 \atop l' \neq l}^{J_d} z_{ll'}^d \right) + \sum_{l' = 1 \atop l' \neq l}^{J_d} D_{2l'} z_{ll'}^d + \sum_{k=1}^{I_d} U_{2k} p_{kl}^d \right\} \frac{1}{D_{2l}} = \frac{1}{B}
\end{align*}
$$
The first equations of both pairs can be written

\[
\sum_{j=1}^{J_u} D_{1j} r_{ji}^u + U_{1i} \left( 1 - \sum_{j=1}^{J_u} p_{ij}^u - \sum_{i'=1}^{I_u} y_{ii'}^u \right) + \sum_{i'=1}^{I_u} U_{1i'} y_{ii'}^u = AU_{1i}
\]

\[
D_{1j} \left( 1 - \sum_{i=1}^{I_u} r_{ji}^u - \sum_{j'=1}^{J_u} z_{jj'}^u \right) + \sum_{j'=1}^{J_u} D_{1j'} z_{jj'}^u + \sum_{i=1}^{I_u} U_{1i} p_{ij}^u = BD_{1j}
\]

Given \( A \) and \( B \), these are linear equations in \( U_{1i} \) and \( D_{1j} \).