A General Model and Analysis of a Discrete Two-Machine Production Line

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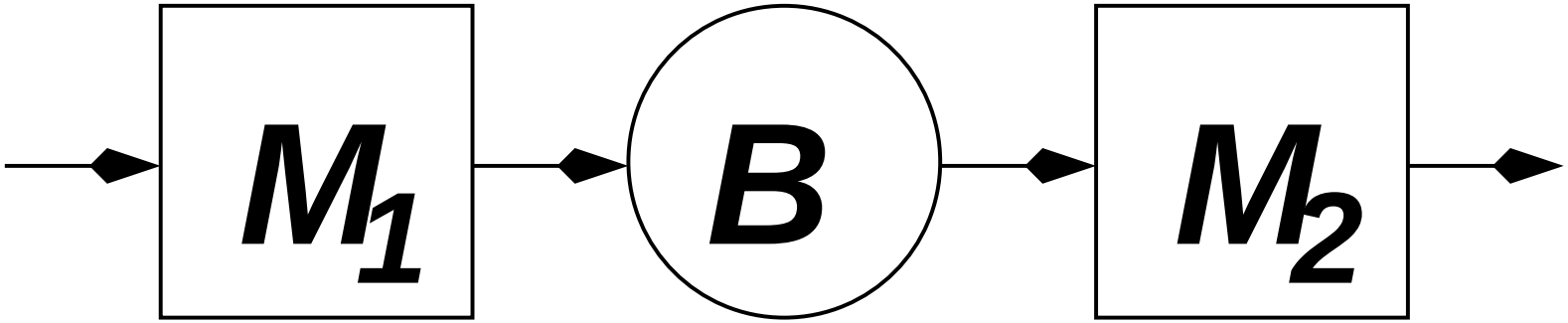
Some errors corrected.

Problem



Examples

- Two-machine production line, where machines are unreliable or have random processing times.
- Queueing system, where the first machine represents the arrival process, the second is the service process.

The buffer is finite.

Problem

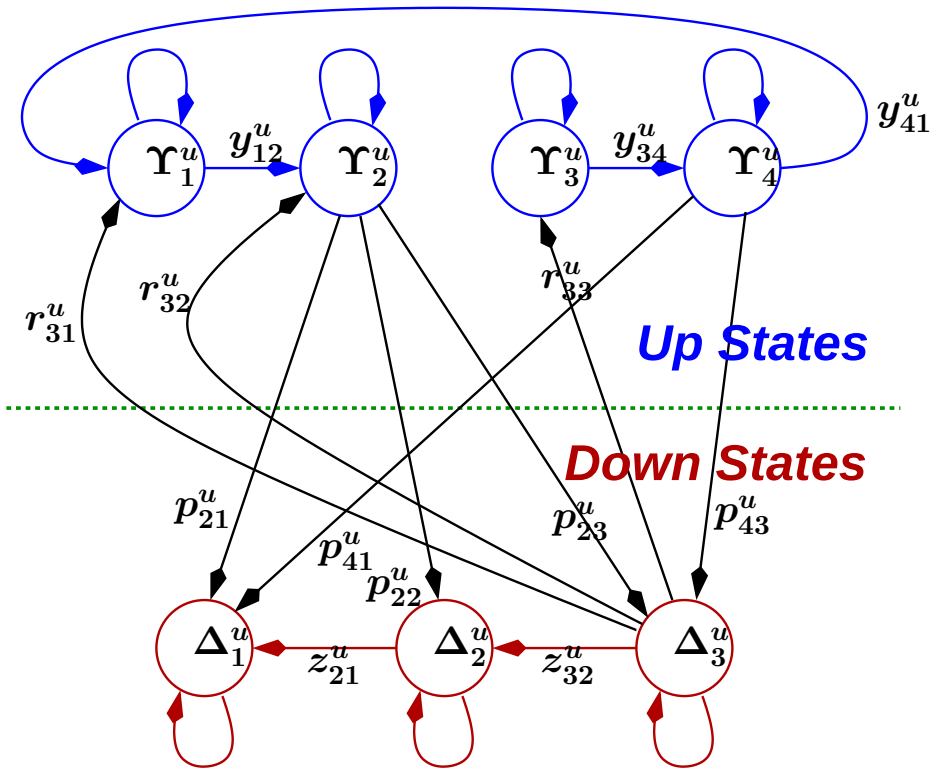
Two-Machine Line

New contribution

To allow both machines to have behavior governed by *any* possible Markov process.

When a machine is in an *up* state and not blocked or starved, it does an operation. That is, it adds or subtracts one part from the buffer.

When it is in a *down* state, or blocked or starved, it does not change the buffer level.

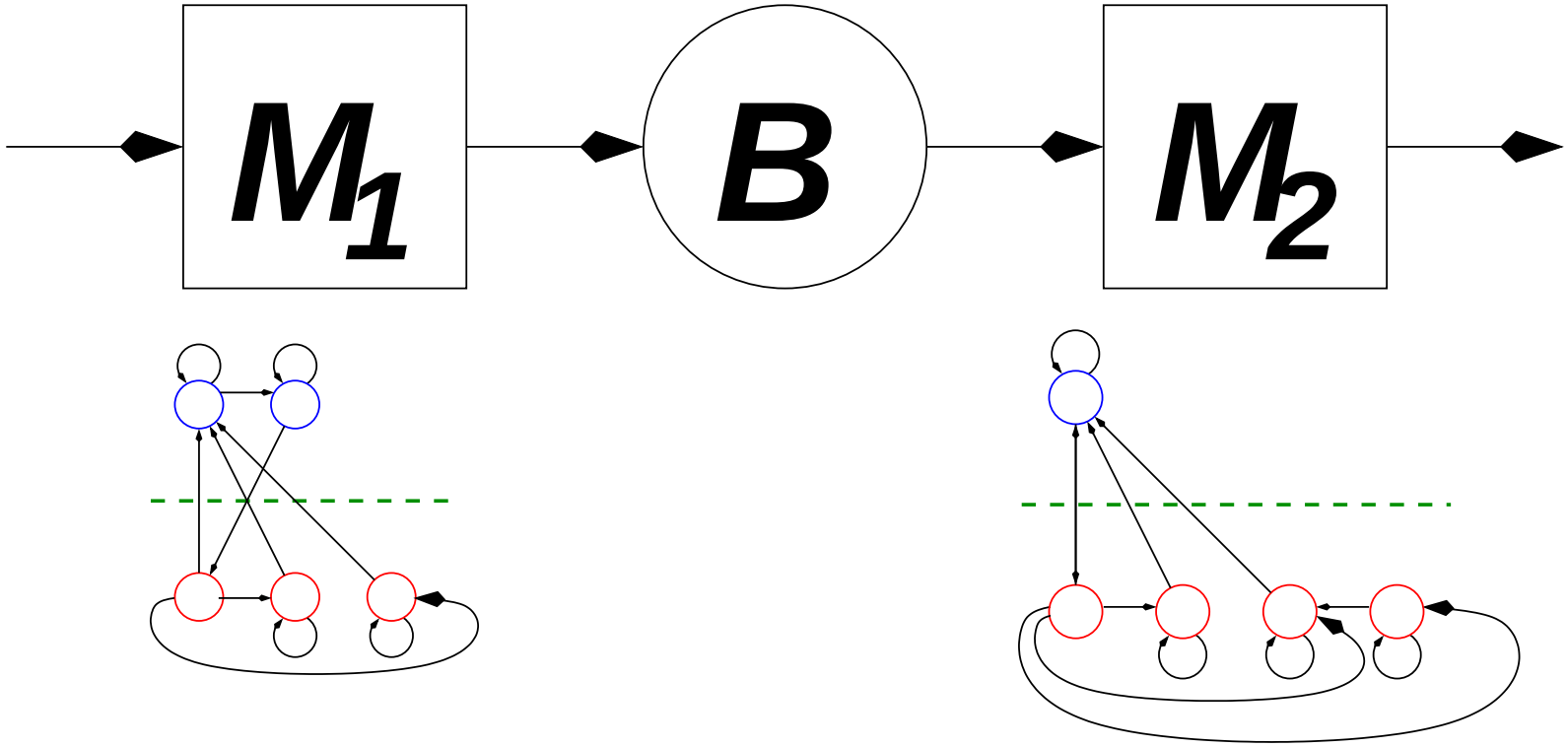


Markov Chain Graph of an Example Machine (Upstream Notation)

Problem

Two-Machine Line

New contribution

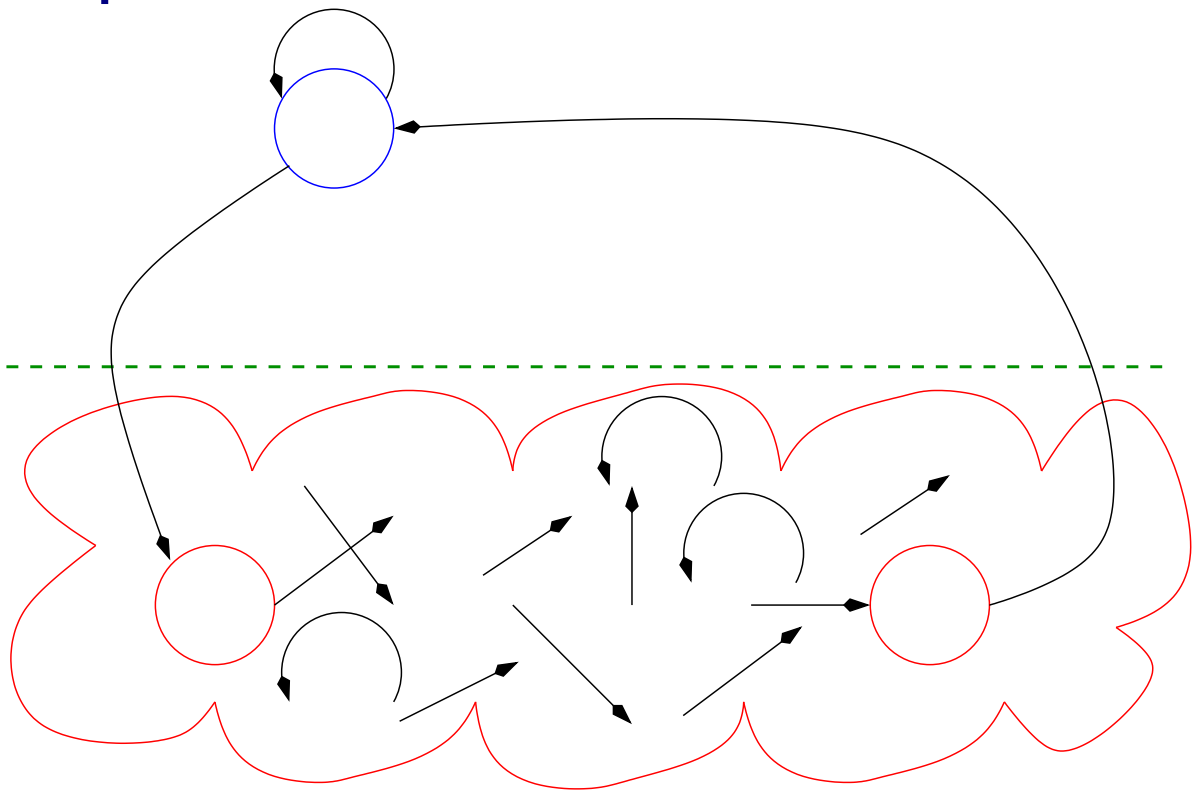


Problem

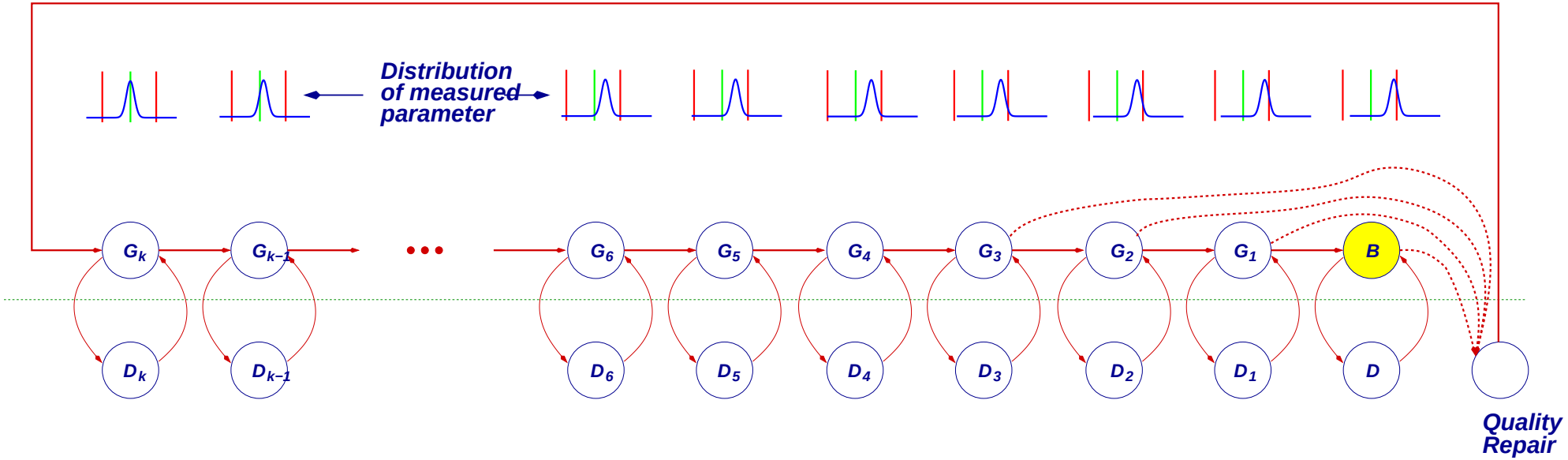
Motivation

- In a queueing system, to allow general arrival time and general service time distributions.
- In a production line,
  - ⋆ to allow general processing time or failure time distributions;
  - ⋆ to improve the accuracy of decomposition methods.

General Independent Arrivals or Service



Machine with Quality Deterioration



Problem

- Dynamics
  - ⋆ discrete time, discrete material ⟵ *focus of this talk*
  - ⋆ continuous time, discrete material
  - ⋆ continuous time, continuous material
- Machine behavior
  - ⋆ one up state, one down state: Buzacott, Gershwin, etc.
  - ⋆ one up state, phase-type down: Dallery
  - ⋆ one up state, multiple down states: Tolio et al.
  - ⋆ some cases of multiple up and down states: Gershwin et al., Tolio et al. for quantity/quality analysis
  - ⋆ etc.

Problem

Research Goal

- Calculate steady-state probability distribution.
- Use it to calculate production rate, average inventory, etc.

Problem

Other related work

- Neuts and others have studied *matrix-geometric* methods for solving systems like this.
  - ⋆ The transition matrix is written in block diagonal form, and an analytical solution in terms of matrix exponentials is found.

- We use a *spectral (eigenvalue) analysis* approach.

Problem

Approach

Generalization of earlier two-machine line models and solution methods:

1. Form internal (transition or balance) equations, boundary equations, and normalization.
2. Guess a solution (based on elementary methods of ordinary difference or differential equations) of the internal equations.
3. Determine equations for the parameters of the guess.
4. Determine sets of values for the parameters.
5. Extend the guesses to the boundary.
6. Express the solution as a linear combination of the guesses.
7. Determine coefficients.

System Definition

States

- State of the system is $s = (n, \alpha_u, \alpha_d)$.
  - ⋆ $n = 0, 1, ..., N$.
  - ⋆ If $\alpha_u = \Upsilon_1^u, ..., \Upsilon_{I_u}^u$, M_u is *up*.
  - ⋆ If $\alpha_u = \Delta_1^u, ..., \Delta_{J_u}^u$, M_u is *down*.
  - ⋆ If $\alpha_d = \Upsilon_1^d, ..., \Upsilon_{I_d}^d$, M_d is *up*.
  - ⋆ If $\alpha_d = \Delta_1^d, ..., \Delta_{J_d}^d$, M_d is *down*.
- *Operation-dependence:* α_u cannot leave an up state if $n = N$. α_d cannot leave an up state if $n = 0$.

$$\text{prob}\ \left[\alpha_u(t+1) \neq \alpha_u(t) \mid \alpha_u(t) = \Upsilon_1^u, ..., \Upsilon_{I_u}^u, n(t) = N\right] = 0$$

$$\text{prob}\ \left[\alpha_d(t+1) \neq \alpha_d(t) \mid \alpha_d(t) = \Upsilon_1^d, ..., \Upsilon_{I_d}^d, n(t) = 0\right] = 0,$$

System Definition

Parameters

The dynamics of α_u are given by

$$r_{ji}^u = \text{prob}\ (\alpha_u(t+1) = \Upsilon_i^u \mid \alpha_u(t) = \Delta_j^u)$$

$$p_{ij}^u = \text{prob}\ (\alpha_u(t+1) = \Delta_j^u \mid \alpha_u(t) = \Upsilon_i^u \text{ and } n(t) < N)$$

$$y_{ii'}^u = \text{prob}\ (\alpha_u(t+1) = \Upsilon_{i'}^u \mid \alpha_u(t) = \Upsilon_i^u) \text{ and } n(t) < N)$$

$$z_{jj'}^u = \text{prob}\ (\alpha_u(t+1) = \Delta_{j'}^u \mid \alpha_u(t) = \Delta_j^u)$$

where r is a *repair probability*, p is a *failure probability*, y is a *up state change probability*, z is a *down state change probability*.

Parameters

Similarly for α_d:

$$r_{lk}^d = \text{prob}\ (\alpha_d(t+1) = \Upsilon_k^d \mid \alpha_d(t) = \Delta_l^d)$$

$$p_{kl}^d = \text{prob}\ (\alpha_d(t+1) = \Delta_l^d \mid \alpha_d(t) = \Upsilon_k^d \text{ and } n(t) > 0)$$

$$y_{kk'}^d = \text{prob}\ (\alpha_d(t+1) = \Upsilon_{k'}^d \mid \alpha_d(t) = \Upsilon_k^d) \text{ and } n(t) > 0)$$

$$z_{ll'}^d = \text{prob}\ (\alpha_d(t+1) = \Delta_{l'}^d \mid \alpha_d(t) = \Delta_l^d)$$

Parameters

Define the matrices

$$R^u = \{r^u_{ji}\},\ R^d = \{r^d_{lk}\},\ P^u = \{p^u_{ij}\},\ P^d = \{p^d_{kl}\}$$

Also define the square matrices

$$Y^u = \{y^u_{i'i}\},\ Y^d = \{y^d_{k'k}\},\ Z^u = \{z^u_{j'j}\},\ Z^d = \{z^d_{l'l}\}$$

for off-diagonal elements. The diagonal elements are given by

$$y^u_{ii} = 1 - \sum_{j=1}^{J_u} p^u_{ij} - \sum_{\substack{i'=1 \\ i' \neq i}}^{I_u} y^u_{ii'} \qquad y^d_{kk} = 1 - \sum_{l=1}^{J_d} p^d_{kl} - \sum_{\substack{k'=1 \\ k' \neq k}}^{I_d} y^d_{kk'}$$

$$z^u_{jj} = 1 - \sum_{i=1}^{I_u} r^u_{ji} - \sum_{\substack{j'=1 \\ j' \neq j}}^{J_u} z^u_{jj'} \qquad z^d_{jj} = 1 - \sum_{k=1}^{I_d} r^d_{lk} - \sum_{\substack{l'=1 \\ l' \neq l}}^{J_d} z^d_{ll'}$$

Transition Equations

$$\mathrm{p}(n, \alpha_u, \alpha_d) =$$

$$\sum_{n', \alpha'_u, \alpha'_d} T(n, \alpha_u, \alpha_d;\ n', \alpha'_u, \alpha'_d)\ \ \mathrm{p}(n', \alpha'_u, \alpha'_d)$$

where

- $\mathrm{p}(n, \alpha_u, \alpha_d)$ is the steady-state probability distribution
- $(n', \alpha'_u, \alpha'_d)$ is the state of the system at the previous time step,and
- $T(n, \alpha_u, \alpha_d; n', \alpha'_u, \alpha'_d)$ is the probability of transition from $(n', \alpha'_u, \alpha'_d)$ to (n, α_u, α_d) in one step.

Transition Equations

Define $\mathbf{p}^{\Upsilon\Upsilon}(n)$ as the matrix of $\mathbf{p}(n, \alpha_u, \alpha_d)$ in which $\alpha_u = \Upsilon_1^u, ..., \Upsilon_{I_u}^u$ and $\alpha_d = \Upsilon_1^d, ..., \Upsilon_{I_d}^d$.

Similarly for $\mathbf{p}^{\Upsilon\Delta}(n)$, $\mathbf{p}^{\Delta\Upsilon}(n)$, and $\mathbf{p}^{\Delta\Delta}(n)$.

For $n = 2, ..., N-2$,

$$\mathbf{p}^{\Upsilon\Upsilon}(n) = \boldsymbol{R}^{uT}\mathbf{p}^{\Delta\Delta}(n)\boldsymbol{R}^{d} + \boldsymbol{R}^{uT}\mathbf{p}^{\Delta\Upsilon}(n)\boldsymbol{Y}^{d}$$

$$+\boldsymbol{Y}^{uT}\mathbf{p}^{\Upsilon\Delta}(n)\boldsymbol{R}^{d} + \boldsymbol{Y}^{uT}\mathbf{p}^{\Upsilon\Upsilon}(n)\boldsymbol{Y}^{d}$$

$$\mathbf{p}^{\Upsilon\Delta}(n) = \boldsymbol{R}^{uT}\mathbf{p}^{\Delta\Delta}(n-1)\boldsymbol{Z}^{d} + \boldsymbol{R}^{uT}\mathbf{p}^{\Delta\Upsilon}(n-1)\boldsymbol{P}^{d}$$

$$+\boldsymbol{Y}^{uT}\mathbf{p}^{\Upsilon\Delta}(n-1)\boldsymbol{Z}^{d} + \boldsymbol{Y}^{uT}\mathbf{p}^{\Upsilon\Upsilon}(n-1)\boldsymbol{P}^{d}$$

Internal equations

$$\mathbf{p}^{\Delta\Upsilon}(n) = \boldsymbol{Z}^{uT}\mathbf{p}^{\Delta\Delta}(n+1)\boldsymbol{R}^{d} + \boldsymbol{Z}^{uT}\mathbf{p}^{\Delta\Upsilon}(n+1)\boldsymbol{Y}^{d}$$

$$+\boldsymbol{P}^{uT}\mathbf{p}^{\Upsilon\Delta}(n+1)\boldsymbol{R}^{d} + \boldsymbol{P}^{uT}\mathbf{p}^{\Upsilon\Upsilon}(n+1)\boldsymbol{Y}^{d}$$

$$\mathbf{p}^{\Delta\Delta}(n) = \boldsymbol{Z}^{uT}\mathbf{p}^{\Delta\Delta}(n)\boldsymbol{Z}^{d} + \boldsymbol{Z}^{uT}\mathbf{p}^{\Delta\Upsilon}(n)\boldsymbol{P}^{d}$$

$$+\boldsymbol{P}^{uT}\mathbf{p}^{\Upsilon\Delta}(n)\boldsymbol{Z}^{d} + \boldsymbol{P}^{uT}\mathbf{p}^{\Upsilon\Upsilon}(n)\boldsymbol{P}^{d}$$

Solution of Internal Equations

Guess

Assume that

$$\mathrm{p}(n, \alpha_u, \alpha_d) = \sum_{m=1}^{M} c_m \pi_m(n, \alpha_u, \alpha_d)$$

where, for $n = 2, \ldots, N-2$,

$$\pi_m(n, \Upsilon_i^u, \Upsilon_k^d) = X_m^n U_{m1i} U_{m2k}$$

$$\pi_m(n, \Upsilon_i^u, \Delta_l^d) = X_m^n U_{m1i} D_{m2l}$$

$$\pi_m(n, \Delta_j^u, \Upsilon_k^d) = X_m^n D_{m1j} U_{m2k}$$

$$\pi_m(n, \Delta_j^u, \Delta_l^d) = X_m^n D_{m1j} D_{m2l}$$

and X_m, U_{m1i}, U_{m2k}, D_{m1j}, D_{m2l}, c_m, and M are constants to be determined.

In the following, we suppress the m subscript until we really need it.

Solution of Internal Equations

Guess

Matrix form:

Let

$$U^u = \{U_{1i}\}, U^d = \{U_{1k}\},$$
$$D^u = \{D_{1j}\}, D^d = \{D_{1l}\}.$$

Then

$$\{U_{1i}U_{2k}\} = U^u U^{d^T},$$

etc.

Solution of Internal Equations

Guess

Then,

$$\pi^{\Upsilon\Upsilon}(n) = \{\pi(n, \Upsilon_i^u, \Upsilon_k^d)\} = X^n U^u U^{d^T}$$

$$\pi^{\Upsilon\Delta}(n) = \{\pi(n, \Upsilon_i^u, \Delta_l^d)\} = X^n U^u D^{d^T}$$

$$\pi^{\Delta\Upsilon}(n) = \{\pi(n, \Delta_j^u, \Upsilon_k^d)\} = X^n D^u U^{d^T}$$

$$\pi^{\Delta\Delta}(n) = \{\pi(n, \Delta_j^u, \Delta_l^d)\} = X^n D^u D^{d^T}$$

Solution of Internal Equations

Guess

... and the first internal transition equation becomes

$$\mathbf{p}^{\Upsilon\Upsilon}(n) = \boldsymbol{R}^{u\boldsymbol{T}}\mathbf{p}^{\Delta\Delta}(n)\boldsymbol{R}^{\boldsymbol{d}} + \boldsymbol{R}^{u\boldsymbol{T}}\mathbf{p}^{\Delta\Upsilon}(n)\boldsymbol{Y}^{\boldsymbol{d}}$$
$$+\boldsymbol{Y}^{u\boldsymbol{T}}\mathbf{p}^{\Upsilon\Delta}(n)\boldsymbol{R}^{\boldsymbol{d}} + \boldsymbol{Y}^{u\boldsymbol{T}}\mathbf{p}^{\Upsilon\Upsilon}(n)\boldsymbol{Y}^{\boldsymbol{d}}$$

$$\boldsymbol{X}^{\boldsymbol{n}}\boldsymbol{U}^{\boldsymbol{u}}\boldsymbol{U}^{\boldsymbol{d}\boldsymbol{T}} = \boldsymbol{R}^{u\boldsymbol{T}}\boldsymbol{X}^{\boldsymbol{n}}\boldsymbol{D}^{\boldsymbol{u}}\boldsymbol{D}^{\boldsymbol{d}\boldsymbol{T}}\boldsymbol{R}^{\boldsymbol{d}} + \boldsymbol{R}^{u\boldsymbol{T}}\boldsymbol{X}^{\boldsymbol{n}}\boldsymbol{D}^{\boldsymbol{u}}\boldsymbol{U}^{\boldsymbol{d}\boldsymbol{T}}\boldsymbol{Y}^{\boldsymbol{d}}$$
$$+\boldsymbol{Y}^{u\boldsymbol{T}}\boldsymbol{X}^{\boldsymbol{n}}\boldsymbol{D}^{\boldsymbol{u}}\boldsymbol{U}^{\boldsymbol{d}\boldsymbol{T}}\boldsymbol{R}^{\boldsymbol{d}} + \boldsymbol{Y}^{u\boldsymbol{T}}\boldsymbol{X}^{\boldsymbol{n}}\boldsymbol{U}^{\boldsymbol{u}}\boldsymbol{U}^{\boldsymbol{d}\boldsymbol{T}}\boldsymbol{Y}^{\boldsymbol{d}}$$

or,

$$\boldsymbol{U}^{\boldsymbol{u}}\boldsymbol{U}^{\boldsymbol{d}\boldsymbol{T}} = \boldsymbol{R}^{u\boldsymbol{T}}\boldsymbol{D}^{\boldsymbol{u}}\boldsymbol{D}^{\boldsymbol{d}\boldsymbol{T}}\boldsymbol{R}^{\boldsymbol{d}} + \boldsymbol{R}^{u\boldsymbol{T}}\boldsymbol{D}^{\boldsymbol{u}}\boldsymbol{U}^{\boldsymbol{d}\boldsymbol{T}}\boldsymbol{Y}^{\boldsymbol{d}}$$
$$+\boldsymbol{Y}^{u\boldsymbol{T}}\boldsymbol{D}^{\boldsymbol{u}}\boldsymbol{U}^{\boldsymbol{d}\boldsymbol{T}}\boldsymbol{R}^{\boldsymbol{d}} + \boldsymbol{Y}^{u\boldsymbol{T}}\boldsymbol{U}^{\boldsymbol{u}}\boldsymbol{U}^{\boldsymbol{d}\boldsymbol{T}}\boldsymbol{Y}^{\boldsymbol{d}}$$

This can be factored. Doing the same for the other equations, ...

Solution of Internal Equations

Guess

... the internal equations become

$$U^u U^{d^T} = K^u K^{d^T}$$
$$X U^u D^{d^T} = K^u L^{d^T}$$
$$X^{-1} D^u U^{d^T} = L^u K^{d^T}$$
$$D^u D^{d^T} = L^u L^{d^T}$$

where

$$K^u = R^{u^T} D^u + Y^{u^T} U^u$$
$$L^u = Z^{u^T} D^u + P^{u^T} U^u$$
$$K^d = R^{d^T} D^d + Y^{d^T} U^d$$
$$L^d = Z^{d^T} D^d + P^{d^T} U^d$$

Important: Each equation is a matrix equation, but the matrices are products of vectors, so each element of each matrix is of the form $a_i b_j$.

Observation: The first equation can be derived from the last three.

Solution of Internal Equations

The first equation can be written

$$U_i^u U_k^d = K_i^u K_k^d$$

Therefore, for all i and k, there must be a scalar A such that

$$AU_i^u = K_i^u$$
$$A^{-1}U_k^d = K_k^d$$

or

$$AU^u = K^u$$
$$A^{-1}U^d = K^d$$

Solution of Internal Equations

Similarly, the last equation can be written

$$D_j^u D_l^d = L_j^u L_l^d$$

and, similarly, for all j and l, there must be a scalar B such that

$$BD_j^u = L_j^u$$
$$B^{-1}D_l^d = L_l^d$$

or

$$BD^u = L^u$$
$$B^{-1}D^d = L^d$$

Solution of Internal Equations

Of these pairs of equations, there is one that is only about the upstream machine and one that deals only with the downstream machine. The upstream machine equations are

$$\boldsymbol{A}\boldsymbol{U}^u = \boldsymbol{K}^u$$
$$\boldsymbol{B}\boldsymbol{D}^u = \boldsymbol{L}^u$$

or

$$\boldsymbol{R}^u\boldsymbol{D}^u + \boldsymbol{Y}^u\boldsymbol{U}^u - \boldsymbol{A}\boldsymbol{U}^u = \boldsymbol{0}$$
$$\boldsymbol{Z}^u\boldsymbol{D}^u + \boldsymbol{P}^u\boldsymbol{U}^u - \boldsymbol{B}\boldsymbol{D}^u = \boldsymbol{0}$$

Solution of Internal Equations

$$R^u D^u + Y^u U^u - A U^u = 0$$
$$Z^u D^u + P^u U^u - B D^u = 0$$

Note that Y^u and Z^u are square matrices. If A is such that $Y^u - A I_{I_u}$ is invertible (where I_{I_u} is the $I_u \times I_u$ identity matrix), then

$$U^u = -(Y^u - A I_{I_u})^{-1} R^u D^u \tag{1}$$

so

$$(Z^u - P^u (Y^u - A I_{I_u})^{-1} R^u - B I_{J_u}) D^u = 0$$

Solution of Internal Equations

In order for this equation to have a solution, the coefficient matrix must have a 0 determinant, or

$$\det(\boldsymbol{Z^u} - \boldsymbol{P^u}(\boldsymbol{Y^u} - \boldsymbol{A}\boldsymbol{I_{I_u}})^{-1}\boldsymbol{R^u} - \boldsymbol{B}\boldsymbol{I_{J_u}}) = \boldsymbol{0}$$

This is one equation in the two unknowns $\boldsymbol{A}$ and $\boldsymbol{B}$. To put it another way, $\boldsymbol{B}$ is an eigenvalue of the matrix

$$\boldsymbol{Z^u} - \boldsymbol{P^u}(\boldsymbol{Y^u} - \boldsymbol{A}\boldsymbol{I_{I_u}})^{-1}\boldsymbol{R^u}. \tag{2}$$

Then $\boldsymbol{D^u}$ is an eigenvector of (2), and $\boldsymbol{U^u}$ is obtained from (1).

Solution of Internal Equations

Analysis of Equations

$$B = \text{eigval}(Z^u - P^u(Y^u - AI_{I_u})^{-1}R^u)$$

1. This establishes B as a (multi-valued) function of A. We must do the same thing for the second machine, which will give a similar relationship between A and B. The simultaneous solutions of both equations determine the internal solutions $(m = 1, 2, ..., M)$.

2. The poles of (2) occur when $Y^u - AI_{I_u}$ is not invertible; ie when A is an eigenvalue of Y^u.

Solution of Internal Equations

Analysis of Equations

$B = \text{eigval}(Z^u - P^u(Y^u - AI_{I_u})^{-1}R^u)$

3. **Asymptotes**

- As $A \rightarrow$ an eigenvalue of Y^u, (2) becomes an infinite matrix and therefore (all?) its eigenvalues become infinite. Consequently, the eigenvalues of Y^u are asymptotes of A as a function of B.
- As $B \rightarrow$ an eigenvalue of Z^u, $P^u(Y^u - AI_{I_u})^{-1}R^u \rightarrow 0$. Therefore $Y^u - AI_{I_u} \rightarrow \infty$. Consequently, the eigenvalues of Z^u are asymptotes of B as a function of A.

Solution of Internal Equations

Analysis of Equations

$B = \text{eigval}(Z^u - P^u(Y^u - AI_{I_u})^{-1}R^u)$

Similarly, for the downstream machine

$$R^d D^d + Y^d U^d - A^{-1} U^d = 0$$
$$Z^d D^d + P^d U^d - B^{-1} D^d = 0$$

If A is such that $Y^d - A^{-1} I_{I_d}$ is invertible, then

$$U^d = -(Y^d - A^{-1} I_{I_d})^{-1} R^d D^d \tag{3}$$

and

$$(Z^d - P^d(Y^d - A^{-1} I_{I_d})^{-1} R^d - B^{-1} I_{I_d}) D^d = 0$$

Therefore as before, B^{-1} is an eigenvalue of

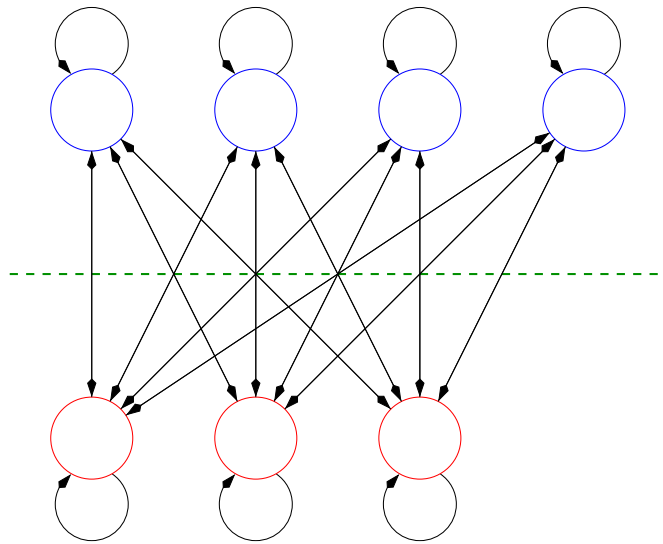
$$Z^d - P^d(Y^d - A^{-1} I_{I_d})^{-1} R^d, \tag{4}$$

D^d is an eigenvalue of (4), and U^d is obtained from (3).

Solution of Internal Equations

Analysis of Equations

Example 1



Identical machines with 4 up states, 3 down states, no up-to-up or down-to-down transitions, and

$$P^u = P^d = \begin{bmatrix} .0011 & .0012 & .0013 \\ .0021 & .0022 & .0023 \\ .0031 & .0032 & .0033 \\ .0041 & .0042 & .0043 \end{bmatrix} \qquad (R^u)^T = (R^d)^T = \begin{bmatrix} .011 & .012 & .013 \\ .021 & .022 & .023 \\ .031 & .032 & .033 \\ .041 & .042 & .043 \end{bmatrix}$$

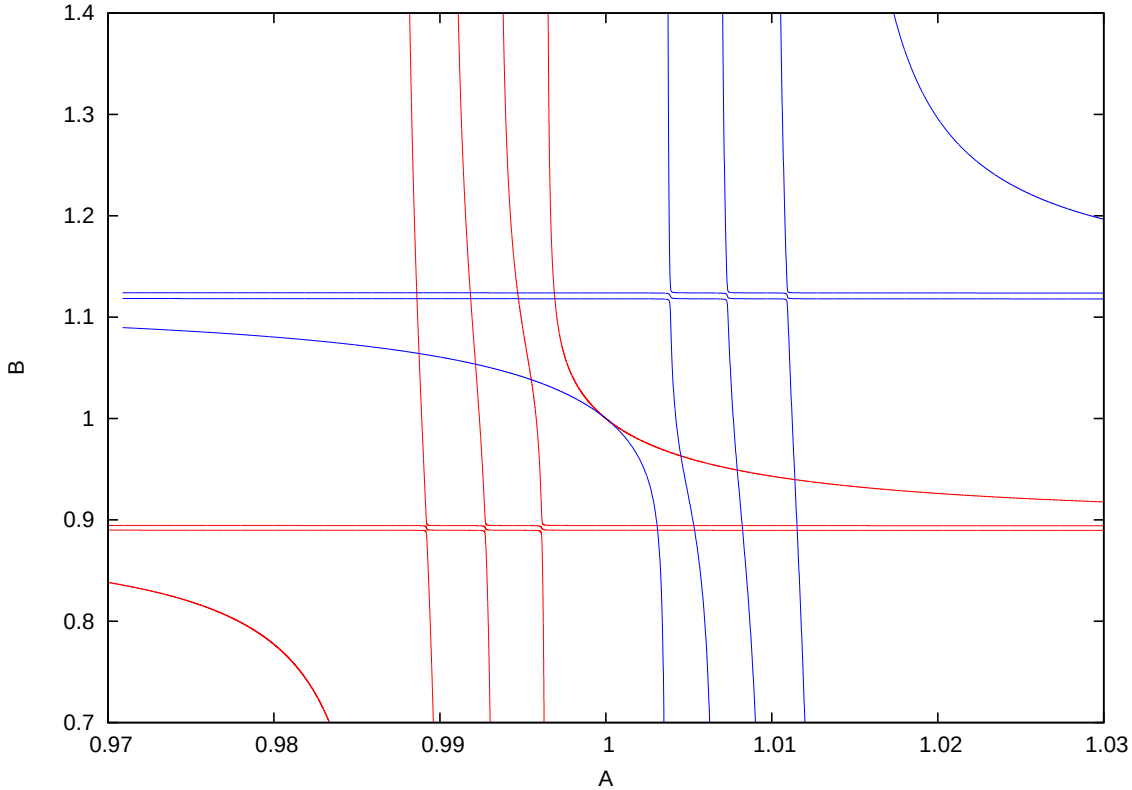
Solution of Internal Equations

Analysis of Equations

Example 1

$$(Z^u - P^u(Y^u - AI_{I_u})^{-1}R^u - BI_{J_u})D^u = 0$$

$$(Z^d - P^d(Y^d - A^{-1}I_{I_d})^{-1}R^d - B^{-1}I_{I_d})D^d = 0$$



Red for M_1; blue for M_2.

Solution of Internal Equations

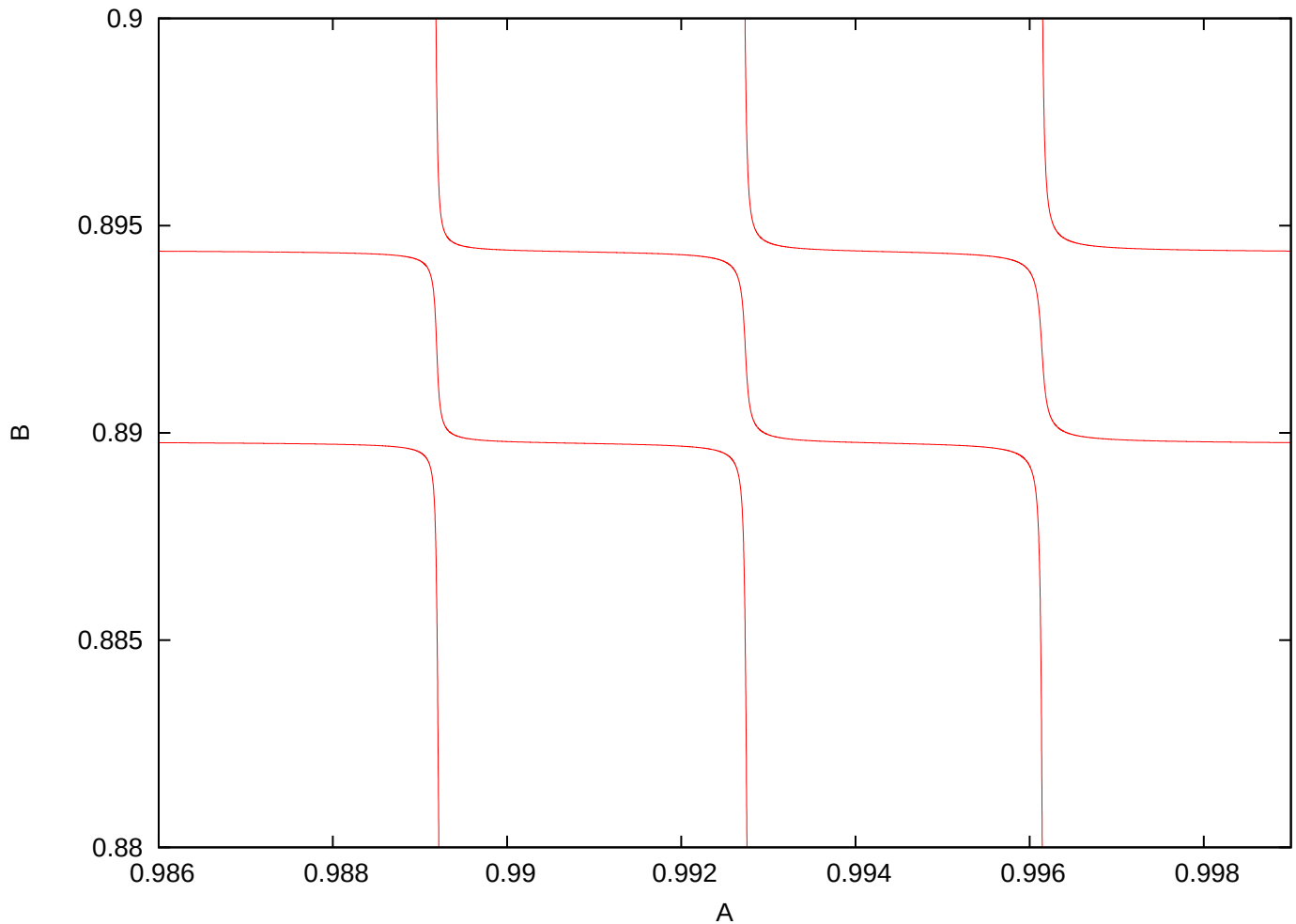
Observations

- The intersections are where A and B are such that the equations for X, U_{1i}, U_{2k}, D_{1j}, and D_{2l} are satisfied. Each intersection leads to one set of X_m, U_{m1i}, U_{m2k}, D_{m1j}, and D_{m2l}.
- At the intersections, $X_m = A_m/B_m$.
- Both sets of graphs go through (1,1) (where X_m is therefore 1). In symmetric cases, there is a double root at (1,1) and the graphs are tangent. In other cases, one root is very close to (1,1), so the curves are almost tangent.
- There are a total of $M = 23$ intersections between the red and blue lines, which means that there are 23 roots (counting (1,1) once). (In general, $I_u J_d + I_d J_u - 1$.)
- Don't be fooled by lines of the same color that seem to intersect. The next graph zooms in on the previous.

Solution of Internal Equations

Analysis of Equations

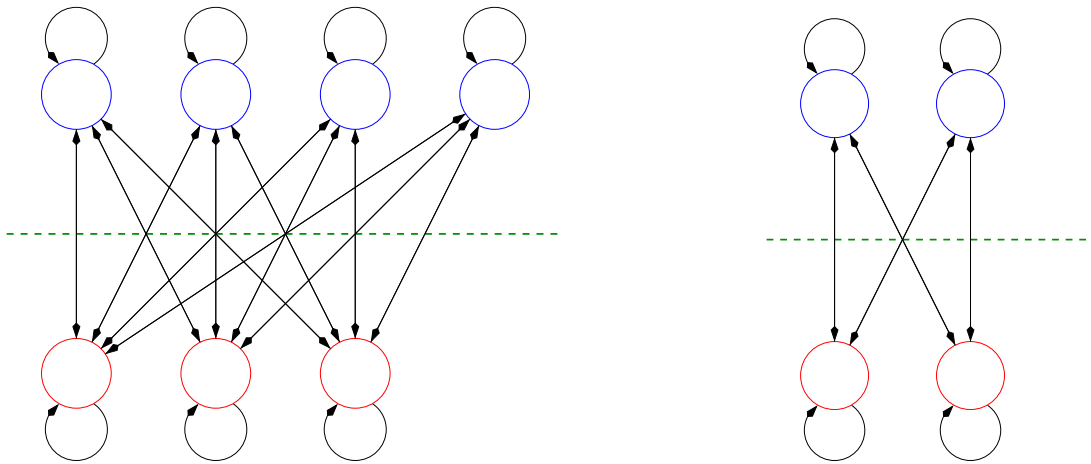
Example 1



Solution of Internal Equations

Analysis of Equations

Example 2



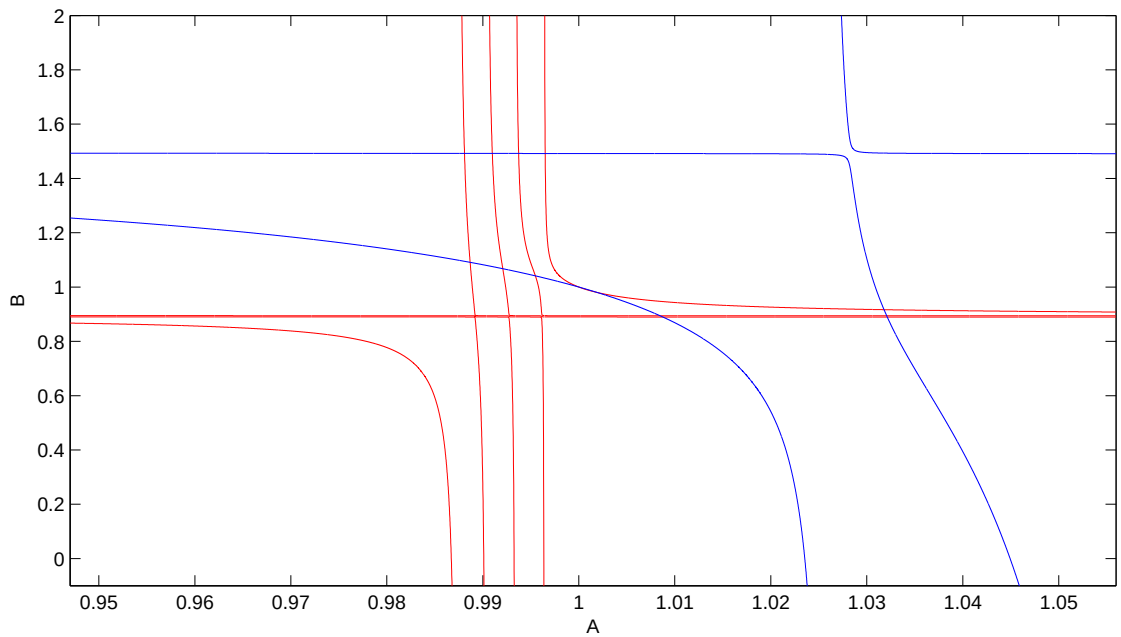
M_1 same as before; M_2 with 2 up states, 2 down states, no up-to-up or down-to-down transitions, and

$$P^d = \begin{bmatrix} .011 & .012 \\ .021 & .022 \end{bmatrix} \qquad (R^d)^T = \begin{bmatrix} .11 & .12 \\ .21 & .22 \end{bmatrix}$$

Solution of Internal Equations

Analysis of Equations

Example 2



(The red graph is the same as before.)

Solution of Internal Equations

- The numerical determination of the intersections is not difficult.
  - ⋆ Each intersection is close to the intersection of asymptotes.
  - ⋆ We can easily determine the asymptotes.
  - ⋆ A simple iterative method was developed which appears always to converge, *even when roots are complex.*

Boundary Equations

Transient States

$$(0, \Upsilon_i^u, \Delta_l^d), (0, \Upsilon_i^u, \Upsilon_k^d), (0, \Delta_j^u, \Delta_l^d), (1, \Upsilon_i^u, \Delta_l^d),$$

$$(N, \Delta_j^u, \Delta_l^d), (N, \Delta_j^u, \Upsilon_k^d), (N, \Upsilon_i^u, \Upsilon_k^d),$$
$$(N-1, \Delta_j^u, \Upsilon_k^d)$$

are all transient.

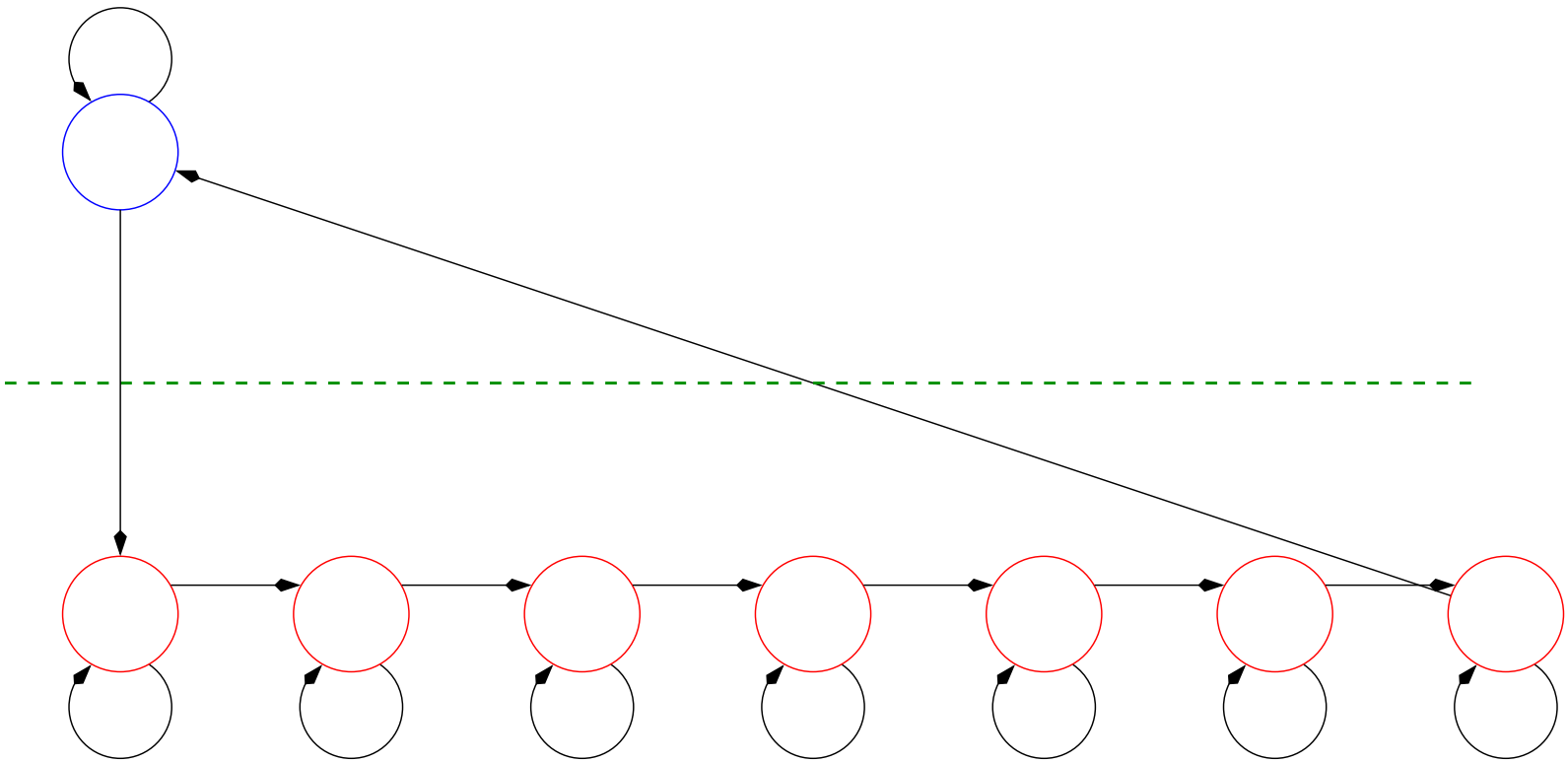
Boundary Equations

Boundary Equations

- The boundary $(n = 0, 1, N-1, N)$ transition equations are messier than the internal equations and do not seem to lead to such (relatively) neat solutions.
- All the πs corresponding to transient states are set to 0.
- We have $\pi(1, \Delta_j^u, \Upsilon_k^d) = XD_{1j}U_{2k}$ but we do not have simple expressions for $\pi(1, \Delta_j^u, \Delta_l^d)$, $\pi(1, \Upsilon_i^u, \Upsilon_k^d)$, or $\pi(0, \Delta_j^u, \Upsilon_k^d)$. Similarly for the upper boundary.
- A numerical solution of the boundary equations, which also produces the c_m coefficients, is straightforward. Its complexity depends only on the Markov chains for the machine models, and not the buffer size.

More Examples

Example 3



Symmetric Line, both machines

More Examples

Example 3

$$P = [.1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]; \qquad Y = [.9];$$

$$R = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ .98 \end{bmatrix} \qquad Z = \begin{bmatrix} 0.02 & .98 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.02 & .98 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.02 & .98 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.02 & .98 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.02 & .98 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.02 & .98 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.02 \end{bmatrix}$$

More Examples

Example 3

Interpretation:

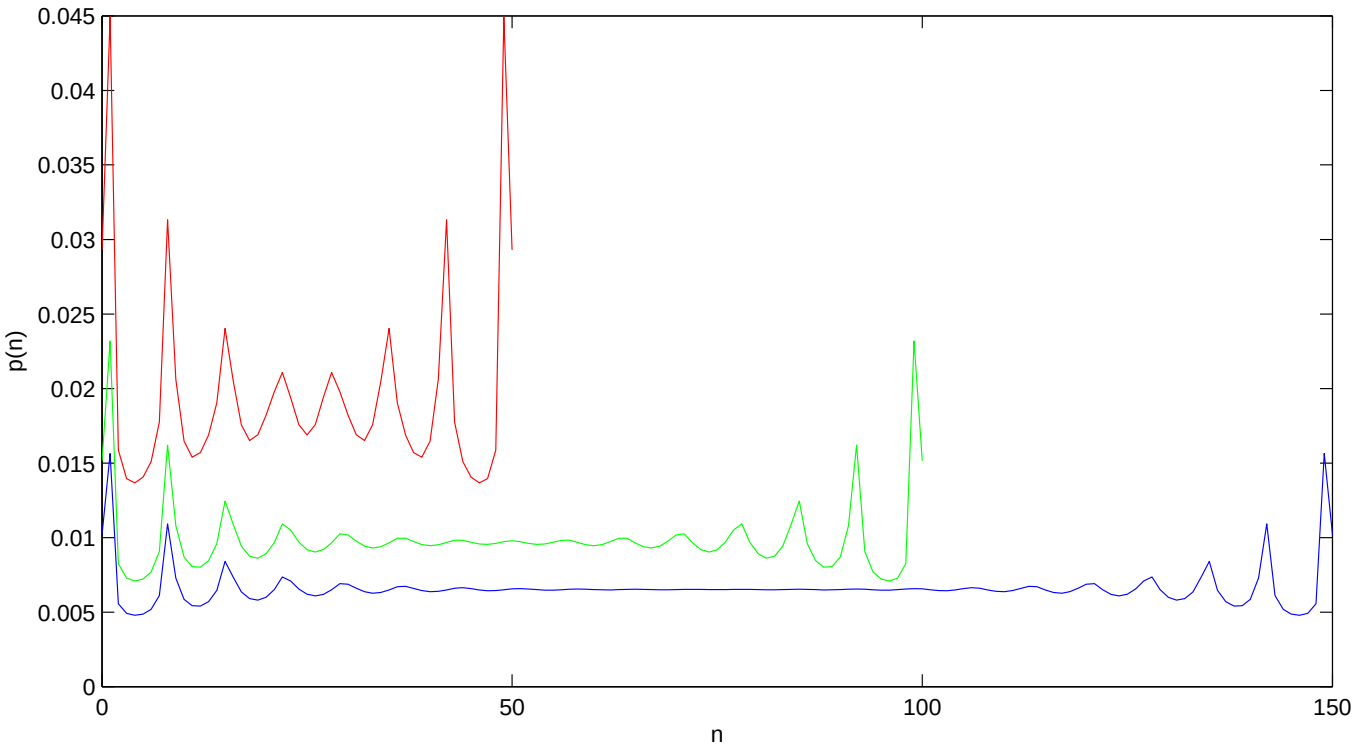
- When a machine is up, it makes a batch of parts, with an average batch size of 10. When it goes down, it stays down for very close to 7 time steps.

- Therefore, when M_1 goes down, the buffer loses about 7 parts. When M_2 goes down, the buffer gains about 7 parts.

- Therefore, the most likely buffer levels are 0, 7, 14, 21, ..., N, $N-7$, $N-14$, ...

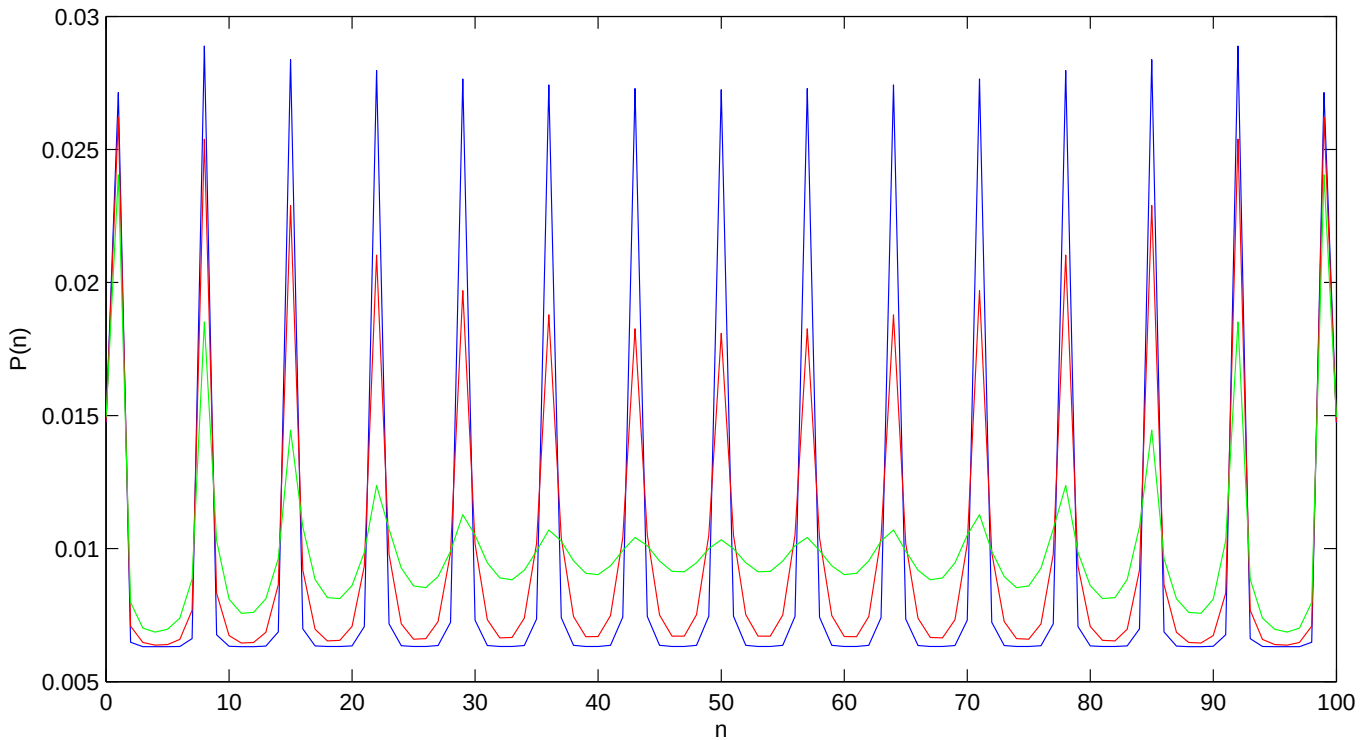
More Examples

Example 3



More Examples

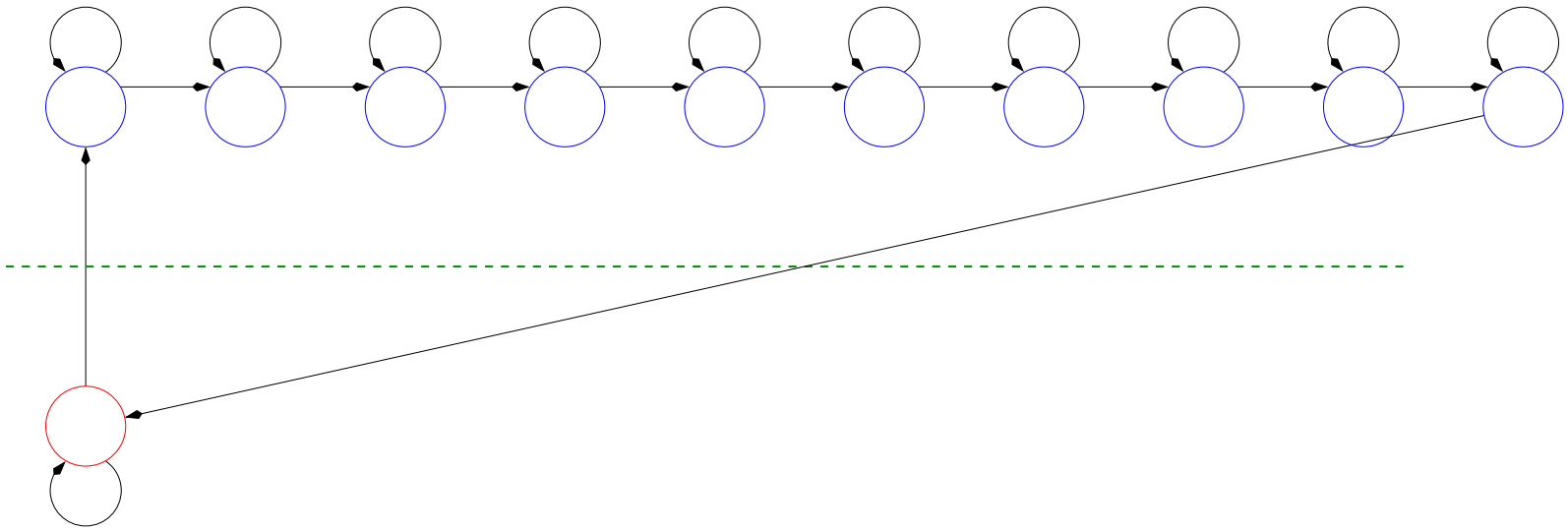
Example 3



$N = 100$, r varied.

More Examples

Example 4



This is the first machine. The second machine has 1 up state, 1 down state.

More Examples

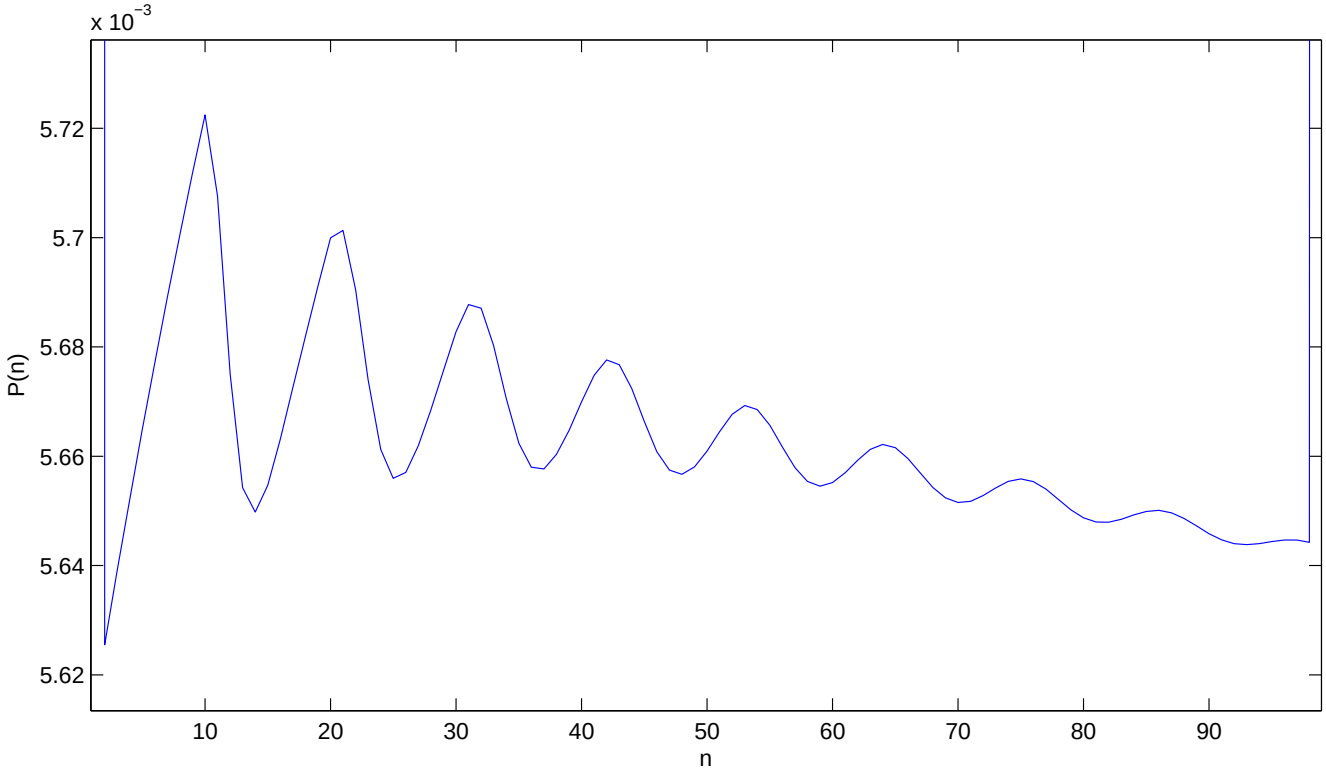
Example 4

Interpretation:

- When the first machine is up, it makes a batch of 10 or a few more parts. Each time the first machine is operating, it adds about 10 parts to the buffer.
- The second machine removes one part at a time from the buffer.
- Therefore, the most likely buffer levels are close to multiples of 10.

More Examples

Example 4



Conclusions

- We have developed a method for two-machine lines with general machines — essentially, a discrete-time GI/GI/1/N queue.
- It works for a variety of cases (tested for machines with $>$ 20 states, N=100).
- Computational effort depends on machine complexity, not buffer size.
- Applications: quality/quantity models, multi-part-type decomposition, machines with non-exponential/geometric failure/repair times, ...

Conclusions

Further work

- Play with more cases.
  - ⋆ Push sizes to limits.
  - ⋆ Explore singularities with special values of parameters.
- Write efficient code.
- *Conjecture:* Systems with dense machine models (where there are many possible transitions among states) have all real roots; sparse machines lead to complex roots.

Conclusions

- Bernoulli production when a machine is in an up state.
- Multiple part production when a machine is in an up state.
- Discrete part, continuous time model.
- Continuous material, continuous time *(done)* .
- Decomposition of large systems.

Conclusions

References

Discrete state, discrete time:

- Stanley B. Gershwin and Saeideh Fallah-Fini, “A General Model and Analysis of a Discrete Two-Machine Production Line,” Sixth Conference on the Analysis of Manufacturing Systems, Lunteren, The Netherlands, 11-16 May 2007.
- Stanley B. Gershwin and Saeideh Fallah-Fini, MIT report in preparation.

Continuous Material, continuous time:

- Barış Tan and Stanley B. Gershwin, “Modeling and Analysis of Markovian Continuous Flow Production Systems with a Finite Buffer: A General Methodology and Applications,” MIT Operations Research Center Working Paper OR 381-07, March, 2007.
- Barış Tan and Stanley B. Gershwin, “Modelling and Analysis of Markovian Single-Buffer Fluid Flow Systems,” Sixth Conference on the Analysis of Manufacturing Systems, Lunteren, The Netherlands, 11-16 May 2007.

Conclusions

Thank you.

Transition Equations

Subscript form

$$\mathrm{p}(n, \Upsilon_i^u, \Upsilon_k^d) = \sum_{j=1}^{J_u} \sum_{l=1}^{J_d} \mathrm{p}(n, \Delta_j^u, \Delta_l^d) r_{ji}^u r_{lk}^d$$

$$+ \sum_{j=1}^{J_u} \mathrm{p}(n, \Delta_j^u, \Upsilon_k^d) r_{ji}^u \left(1 - \sum_{l=1}^{J_d} p_{kl}^d - \sum_{\substack{k' = 1 \\ k' \neq k}}^{I_d} y_{kk'}^d \right) + \sum_{j=1}^{J_u} \sum_{\substack{k' = 1 \\ k' \neq k}}^{I_d} \mathrm{p}(n, \Delta_j^u, \Upsilon_{k'}^d) r_{ji}^u y_{k'k}^d$$

$$+ \mathrm{p}(n, \Upsilon_i^u, \Upsilon_k^d) \left(1 - \sum_{j=1}^{J_u} p_{ij}^u - \sum_{\substack{i' = 1 \\ i' \neq i}}^{I_u} y_{ii'}^u \right) \left(1 - \sum_{l=1}^{J_d} p_{kl}^d - \sum_{\substack{k' = 1 \\ k' \neq k}}^{I_d} y_{kk'}^d \right)$$

Transition Equations

Subscript form

$$+\left[\sum_{\substack{k'=1\\k'\neq k}}^{I_d} \mathrm{p}(n,\Upsilon_i^u,\Upsilon_{k'}^d)y_{k'k}^d+\sum_{l=1}^{J_d}\mathrm{p}(n,\Upsilon_i^u,\Delta_l^d)r_{lk}^d\right]\left(1-\sum_{j=1}^{J_u}p_{ij}^u-\sum_{\substack{i'=1\\i'\neq i}}^{I_u} y_{ii'}^u\right)$$

$$+\sum_{\substack{i'=1\\i'\neq i}}^{I_u}\sum_{l=1}^{J_d}\mathrm{p}(n,\Upsilon_{i'}^u,\Delta_l^d)y_{i'i}^u r_{lk}^d+\sum_{\substack{i'=1\\i'\neq i}}^{I_u}\mathrm{p}(n,\Upsilon_{i'}^u,\Upsilon_k^d)y_{i'i}^u\left(1-\sum_{l=1}^{J_d}p_{kl}^d-\sum_{\substack{k'=1\\k'\neq k}}^{I_d} y_{kk'}^d\right)$$

$$+\sum_{\substack{i'=1\\i'\neq i}}^{I_u}\sum_{\substack{k'=1\\k'\neq k}}^{I_d}\mathrm{p}(n,\Upsilon_{i'}^u,\Upsilon_{k'}^d)y_{i'i}^u y_{k'k}^d$$

Solution of Internal Equations

Guess

Subscript form

If we plug this into the first internal equation, we get (with m suppressed)

$$X^n U_{1i} U_{2k} = \tag{5}$$

$$\sum_{j=1}^{J_u} \sum_{l=1}^{J_d} X^n D_{1j} D_{2l} r_{ji}^u r_{lk}^d + \sum_{j=1}^{J_u} X^n D_{1j} U_{2k} r_{ji}^u \left(1 - \sum_{l=1}^{J_d} p_{kl}^d - \sum_{\substack{k' = 1 \\ k' \neq k}}^{I_d} y_{kk'}^d \right)$$

$$+ \sum_{j=1}^{J_u} \sum_{\substack{k' = 1 \\ k' \neq k}}^{I_d} X^n D_{1j} U_{2k'} r_{ji}^u y_{k'k}^d + \sum_{l=1}^{J_d} X^n U_{1i} D_{2l} \left(1 - \sum_{j=1}^{J_u} p_{ij}^u - \sum_{\substack{i' = 1 \\ i' \neq i}}^{I_u} y_{ii'}^u \right) r_{lk}^d$$

Solution of Internal Equations

Guess

Subscript form

$$+\sum_{\substack{k'=1\\k'\neq k}}^{I_d} X^n U_{1i} U_{2k'} \left(1-\sum_{j=1}^{J_u} p_{ij}^u - \sum_{\substack{i'=1\\i'\neq i}}^{I_u} y_{ii'}^u\right) y_{k'k}^d$$

$$+X^n U_{1i} U_{2k} \left(1-\sum_{j=1}^{J_u} p_{ij}^u - \sum_{\substack{i'=1\\i'\neq i}}^{I_u} y_{ii'}^u\right)\left(1-\sum_{l=1}^{J_d} p_{kl}^d - \sum_{\substack{k'=1\\k'\neq k}}^{I_d} y_{kk'}^d\right)$$

$$+\sum_{\substack{i'=1\\i'\neq i}}^{I_u} \sum_{l=1}^{J_d} X^n U_{1i'} D_{2l} y_{i'i}^u r_{lk}^d$$

Solution of Internal Equations

Guess

Subscript form

$$+ \sum_{\substack{i' = 1 \\ i' \neq i}}^{I_u} X^n U_{1i'} U_{2k} y^u_{i'i} \left(1 - \sum_{l=1}^{J_d} p^d_{kl} - \sum_{\substack{k' = 1 \\ k' \neq k}}^{I_d} y^d_{kk'} \right)$$

$$+ \sum_{\substack{i' = 1 \\ i' \neq i}}^{I_u} \sum_{\substack{k' = 1 \\ k' \neq k}}^{I_d} X^n U_{1i'} U_{2k'} y^u_{i'i} y^d_{k'k}$$

Solution of Internal Equations

Guess

Subscript form

This reduces to

$$U_{1i}U_{2k} =$$

$$\left[\sum_{j=1}^{J_u} D_{1j} r_{ji}^u + U_{1i}\left(1 - \sum_{j=1}^{J_u} p_{ij}^u - \sum_{\substack{i'=1\\ i'\neq i}}^{I_u} y_{ii'}^u\right) + \sum_{\substack{i'=1\\ i'\neq i}}^{I_u} U_{1i'} y_{i'i}^u\right] \bullet$$

$$\left[\sum_{l=1}^{J_d} D_{2l} r_{lk}^d + U_{2k}\left(1 - \sum_{l=1}^{J_d} p_{kl}^d - \sum_{\substack{k'=1\\ k'\neq k}}^{I_d} y_{kk'}^d\right) + \sum_{\substack{k'=1\\ k'\neq k}}^{I_d} U_{2k'} y_{k'k}^d\right]$$

which is a set of equations in U_{1i}, D_{1j}, U_{2k}, D_{2l}.

Solution of Internal Equations

Guess

Subscript form

Also,

$$XU_{1i}D_{2l} =$$

$$\left[\sum_{j=1}^{J_u} D_{1j} r_{ji}^u + U_{1i}\left(1 - \sum_{j=1}^{J_u} p_{ij}^u - \sum_{\substack{i'=1 \\ i' \neq i}}^{I_u} y_{ii'}^u\right) + \sum_{\substack{i'=1 \\ i' \neq i}}^{I_u} U_{1i'} y_{i'i}^u\right] \bullet$$

$$\left[D_{2l}\left(1 - \sum_{k=1}^{I_d} r_{lk}^d - \sum_{\substack{l'=1 \\ l' \neq l}}^{J_d} z_{ll'}^d\right) + \sum_{\substack{l'=1 \\ l' \neq l}}^{J_d} D_{2l'} z_{l'l}^d + \sum_{k=1}^{I_d} U_{2k} p_{kl}^d\right]$$

Solution of Internal Equations

Guess

Subscript form

$$X^{-1} D_{1j} U_{2k} =$$

$$\left[D_{1j} \left(1 - \sum_{i=1}^{I_u} r_{ji}^u - \sum_{\substack{j' = 1 \\ j' \neq j}}^{J_u} z_{jj'}^u \right) + \sum_{\substack{j' = 1 \\ j' \neq j}}^{J_u} D_{1j'} z_{j'j}^u + \sum_{i=1}^{I_u} U_{1i} p_{ij}^u \right] \bullet$$

$$\left[\sum_{l=1}^{J_d} D_{2l} r_{lk}^d + U_{2k} \left(1 - \sum_{l=1}^{J_d} p_{kl}^d - \sum_{\substack{k' = 1 \\ k' \neq k}}^{I_d} y_{kk'}^d \right) + \sum_{\substack{k' = 1 \\ k' \neq k}}^{I_d} U_{2k'} y_{k'k}^d \right]$$

Solution of Internal Equations

Guess

Subscript form

$$D_{1j} D_{2l} =$$

$$\left[D_{1j} \left(1 - \sum_{i=1}^{I_u} r_{ji}^{u} - \sum_{\substack{j' = 1 \\ j' \neq j}}^{J_u} z_{jj'}^{u} \right) + \sum_{\substack{j' = 1 \\ j' \neq j}}^{J_u} D_{1j'} z_{j'j}^{u} + \sum_{i=1}^{I_u} U_{1i} p_{ij}^{u} \right] \bullet$$

$$\left[D_{2l} \left(1 - \sum_{k=1}^{I_d} r_{lk}^{d} - \sum_{\substack{l' = 1 \\ l' \neq l}}^{J_d} z_{ll'}^{d} \right) + \sum_{\substack{l' = 1 \\ l' \neq l}}^{J_d} D_{2l'} z_{l'l}^{d} + \sum_{k=1}^{I_d} U_{2k} p_{kl}^{d} \right]$$

Solution of Internal Equations

The first equation can be written

$$1 = \left\{ \sum_{j=1}^{J_u} D_{1j} r_{ji}^{u} + U_{1i} \left(1 - \sum_{j=1}^{J_u} p_{ij}^{u} - \sum_{\substack{i' = 1 \\ i' \neq i}}^{I_u} y_{ii'}^{u} \right) + \sum_{\substack{i' = 1 \\ i' \neq i}}^{I_u} U_{1i'} y_{i'i}^{u} \right\} \frac{1}{U_{1i}} \bullet$$

$$\left\{ \sum_{l=1}^{J_d} D_{2l} r_{lk}^{d} + U_{2k} \left(1 - \sum_{l=1}^{J_d} p_{kl}^{d} - \sum_{\substack{k' = 1 \\ k' \neq k}}^{I_d} y_{kk'}^{d} \right) + \sum_{\substack{k' = 1 \\ k' \neq k}}^{I_d} U_{2k'} y_{k'k}^{d} \right\} \frac{1}{U_{2k}}$$

Solution of Internal Equations

Analysis of Equations

The first fraction depends on i and the second depends on k. Therefore, it is reasonable to assume that there exists some A such that, for all i,

$$\left\{ \sum_{j=1}^{J_u} D_{1j} r_{ji}^u + U_{1i} \left(1 - \sum_{j=1}^{J_u} p_{ij}^u - \sum_{\substack{i' = 1 \\ i' \neq i}}^{I_u} y_{ii'}^u \right) + \sum_{\substack{i' = 1 \\ i' \neq i}}^{I_u} U_{1i'} y_{i'i}^u \right\} \frac{1}{U_{1i}} = A$$

and, for all k,

$$\left\{ \sum_{l=1}^{J_d} D_{2l} r_{lk}^d + U_{2k} \left(1 - \sum_{l=1}^{J_d} p_{kl}^d - \sum_{\substack{k' = 1 \\ k' \neq k}}^{I_d} y_{kk'}^d \right) + \sum_{\substack{k' = 1 \\ k' \neq k}}^{I_d} U_{2k'} y_{k'k}^d \right\} \frac{1}{U_{2k}} = \frac{1}{A}$$

Solution of Internal Equations

Similarly, for all j,

$$\left\{ D_{1j} \left(1 - \sum_{i=1}^{I_u} r_{ji}^u - \sum_{\substack{j' = 1 \\ j' \neq j}}^{J_u} z_{jj'}^u \right) + \sum_{\substack{j' = 1 \\ j' \neq j}}^{J_u} D_{1j'} z_{j'j}^u + \sum_{i=1}^{I_u} U_{1i} p_{ij}^u \right\} \frac{1}{D_{1j}} = B$$

and, for all l,

$$\left\{ D_{2l} \left(1 - \sum_{k=1}^{I_d} r_{lk}^d - \sum_{\substack{l' = 1 \\ l' \neq l}}^{J_d} z_{ll'}^d \right) + \sum_{\substack{l' = 1 \\ l' \neq l}}^{J_d} D_{2l'} z_{l'l}^d + \sum_{k=1}^{I_d} U_{2k} p_{kl}^d \right\} \frac{1}{D_{2l}} = \frac{1}{B}$$

Solution of Internal Equations

The first equations of both pairs can be written

$$\sum_{j=1}^{J_u} D_{1j} r_{ji}^u + U_{1i} \left(1 - \sum_{j=1}^{J_u} p_{ij}^u - \sum_{\substack{i' = 1 \\ i' \neq i}}^{I_u} y_{ii'}^u \right) + \sum_{\substack{i' = 1 \\ i' \neq i}}^{I_u} U_{1i'} y_{i'i}^u = A U_{1i}$$

$$D_{1j} \left(1 - \sum_{i=1}^{I_u} r_{ji}^u - \sum_{\substack{j' = 1 \\ j' \neq j}}^{J_u} z_{jj'}^u \right) + \sum_{\substack{j' = 1 \\ j' \neq j}}^{J_u} D_{1j'} z_{j'j}^u + \sum_{i=1}^{I_u} U_{1i} p_{ij}^u = B D_{1j}$$

Given A and B, these are *linear* equations in U_{1i} and D_{1j}.