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# Real-time limit analysis of vaulted masonry buildings

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**Abstract**

This paper presents new structural analysis tools based on limit state analysis for vaulted masonry buildings. Thrust lines are used to visualize the forces within the masonry and to predict possible collapse modes. The models are interactive and parametric to explore relationships between building geometry and possible equilibrium conditions in real time. Collapse analysis due to applied displacements is determined by combining kinematics and statics. The approach is largely two dimensional, though more complex three-dimensional problems are analyzed using the same methods. This paper presents a series of analytical tools that are fast and easy to use in real time, but at the same time rigorous and highly accurate. This work represents a significant improvement over traditional methods of thrust line analysis performed by hand, which are often tedious and time-consuming.


*Keywords:* Thrust line analysis; Limit state analysis; Masonry structures; Collapse mechanisms; Dynamic geometry; Graphic statics; Kinematic analysis

## 1. Introduction

Much of the world's architectural heritage consists of historic buildings in masonry. In addition to their cultural values, such monuments often have important economic value. Though many historic masonry buildings have survived for centuries, there is an acute need for new tools to analyze the stability and the safety of such structures. In particular, there is a need to assess the structural safety of large numbers of buildings quickly and accurately, for example, in the analysis of 100 historic vaulted masonry churches in a particular region. For such an assessment, equilibrium is the most important concept and the engineer should be able to quickly and easily explore the various possible equilibrium conditions for a historic structure. In most cases, historic masonry buildings fail due to instability rather than a lack of material strength because stresses in historical masonry are typically an order of magnitude lower than the crushing capacity of the stone [1]. Therefore, rigid block models with the same proportions as the structures are excellent models to understand their stability. An equilibrium approach is most appropriate for understanding the structural behavior of historical masonry buildings, and limit analysis provides a theoretical framework [2,3]. To apply limit analysis to masonry, three main assumptions can be made: masonry has no tensile strength; it can resist infinite compression; and no sliding will occur within the masonry. Each of these assumptions is a good approximation for historic masonry and can be verified in specific locations of interest. The application of limit analysis to masonry vaulting has been developed in recent years by Huerta and O'Dwyer [4,5].

This paper extends the graphical method for limit analysis using the well known concept of a line of thrust [6,7]. The thrust line represents the path of the resultants of the compressive forces through the masonry [8]. For a pure compression structure to be in equilibrium with the applied loads there must be a line of thrust contained entirely within the section (and within the middle third to avoid tension). The line of thrust can also give information about

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possible collapse mechanisms. When the thrust line touches the boundaries of the structure, a hinge may be created which can allow rotation in one direction and may lead to the formation of a collapse mechanism if sufficient hinges are formed.

This approach can be contrasted directly with elastic analysis as follows. A linear elastic finite element analysis (FEA) shows the stress level predicted in a masonry arch due to its own weight. Consider two semi-circular arches (Fig. 1), one with a thickness to radius ratio of 0.08 (a) and one of 0.16 (b), loaded only by self-weight and supported on rigid foundations. In trying to understand and explain the finite element results, cracking in the crown of the arch may be predicted by the thin zone of compression along the extrados and the tension zone on the intrados. The FEA outputs of the two arches are very similar and it is difficult for the elastic FE analyst to note any significant difference between the two arches of Fig. 1. A simple thrust line analysis immediately reveals the major difference between the two arches. The arch with a thickness to radius ratio of 0.08 is too thin to contain a thrust line and therefore, would not stand under its own weight unless the arch material had some appreciable value of tensile capacity. Therefore, the linear elastic finite element analysis gives an unsafe and deceptive result for the thinner arch by assuming that the material is capable of resisting tension.

The power of the simple thrust line is clearly shown in the previous example. While the FEA shows one possible stress state in the material, a linear elastic analysis does not say anything about the stability or collapse of the arch. The analysis of masonry arches is a well known problem, and this example immediately shows how difficult it is to draw conclusions from stress analysis, even for simple two-dimensional problems. The possible equilibrium configurations illustrated by the thrust line analysis give a clearer understanding of the problem. The thrust line gives more information about collapse than conventional elastic analysis and provides an immediate check for the stability of historic masonry structures.

One must also consider the relative speed of both analysis methods. Crack genesis with large displacements can be simulated using finite element analysis through the use of complex material models and non-linear analysis procedures [9,10]. To achieve more realistic and accurate results with finite element analysis, a great level of detail for the model is necessary, but many of the parameters are highly uncertain in real buildings. For example, historic stone and mortar are inhomogeneous and do not have consistent and predictable mechanical properties. On the contrary, limit analysis makes very simple, but accurate approximations which do not rely on the material properties for the solution procedures. By representing the internal forces with a thrust line, limit analysis can be used to model and understand complex systems and collapse mechanisms. Afterwards the initial assumptions can be verified to ensure that the structure is not at risk of sliding or local compression failure.

Despite the advantages of limit analysis for masonry, existing methods of thrust line analysis are not satisfactory for engineering analysis of masonry buildings. Though finite element methods have advanced rapidly in the last ten years, computational methods for thrust line analysis have lagged behind. Programs such as Archie-M [11] and Ring [12] have been developed for the limit analysis of masonry arch bridges under live loading. This is a specific problem related only to bridges with heavy live loading, whereas the methodology proposed in this paper is intended for vaulted masonry buildings, which are threatened more

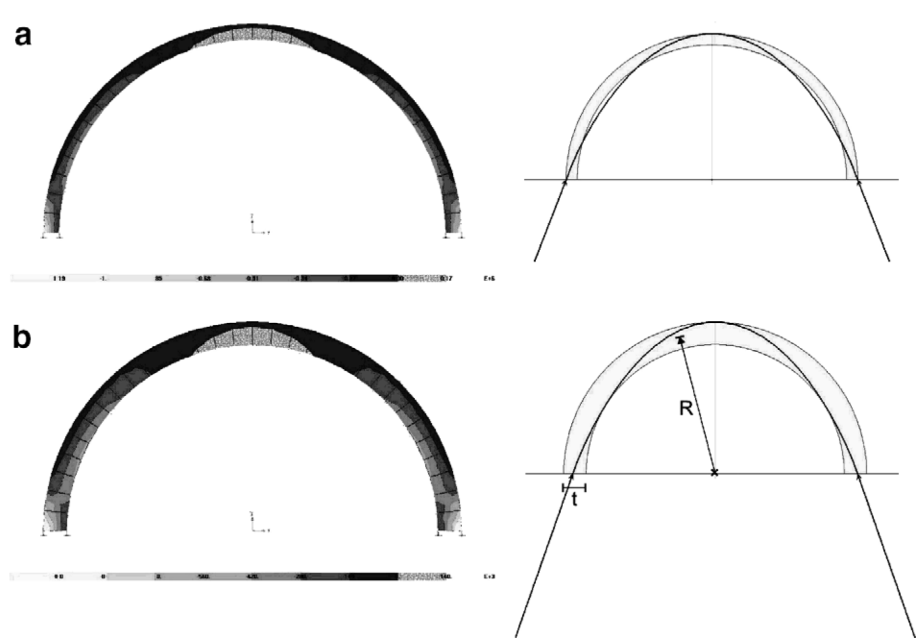


Fig. 1. Elastic finite element analysis (left) versus limit state analysis (right) for arches with $t/R$ ratios of (a) 0.08 and (b) 0.16.

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by the gradual destabilizing effect of large displacements over time rather than live loading from wind or traffic [8].

## 2. Methodology

We have developed new masonry analysis tools using limit analysis to illustrate possible collapse modes and to allow users to clearly visualize the forces within the masonry. The graphical methods provide vastly improved computation tools for thrust line analysis, offering interactivity so that the user can explore various equilibrium states in real time. For most historic structures there are infinite possible load paths and the programs developed can be used to illustrate the range of these potential solutions. The new methods clarify this statically indeterminate character of masonry structures through real-time tools which are available on the web [13,14]. In addition, their parametric setup allows the user to understand the role of geometry in the stability of masonry structures. Collapse mechanisms are explained by combining kinematics and statics, and more complex three-dimensional problems are analyzed with the same methods. This approach brings a rigorous and accurate analysis tool that can also be used as a learning tool for engineers and architects. We explain this new approach by introducing its different components:

- interactive graphic analysis,
- geometrically controlled loads, and
- animated kinematics.

### *2.1. Interactive graphic analysis*

Graphic statics is a powerful method for equilibrium analysis formalized in the 19th century for use in structural engineering [15,16]. By using force polygons to investigate conditions of equilibrium, the graphical method provides rapid solutions for analysis and design. However, the method becomes tedious and unwieldy for the analysis of three-dimensional structures, both because of the complexity of the force polygons and the extensive trial and error required to arrive at acceptable solutions [17]. Recently, colleagues at MIT created interactive graphic statics methods which suggested the possibility of using computerized graphical methods for the analysis of masonry structures [18]. This precedent inspired the current research.

For the methods employed here, all graphic constructions are prepared in advance, so that the user is not hindered by the need to construct the force polygons. Through computation, it is possible to show all of the possible funicular solutions for a certain set of loads without having to redraw the entire construction (Fig. 2). For example, by moving the pole (point O in Fig. 2b), the user discovers a new force polygon (with the pole at O’) and the associated funicular line of compression, shown as the dashed line in Fig. 2a. Thus, the user can explore all possible solutions for the family of funicular polygons in real time. To make the models and applets described here, we use simple two-dimensional drawing programs that allow the construction of fully interactive drawings [20].

### *2.2. Geometrically controlled loads*

The second innovation for this approach is that the geometry of the structure controls the applied loads for the graphical analysis (Fig. 3). This is true for historic masonry buildings in which the dominant loading is due to the self-weight of the structure. Changes in geometry will alter the volume, and therefore the weight, of the blocks.

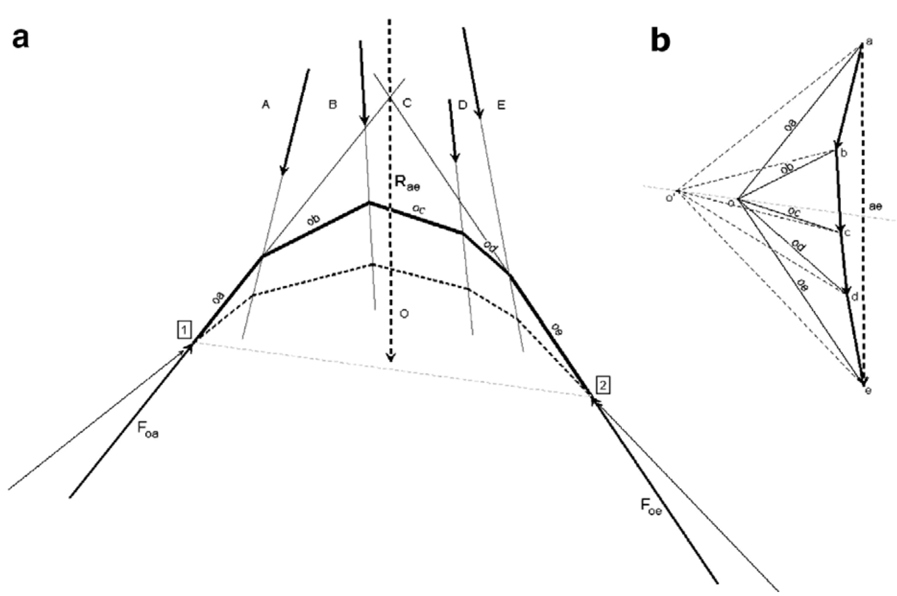


Fig. 2. Graphic statics allows the construction of (a) funicular shapes (only tension or compression) for a given set of loads using Bow’s notation and (b) the corresponding force polygons that give the magnitude of the forces of the segments in the funicular polygon [19].

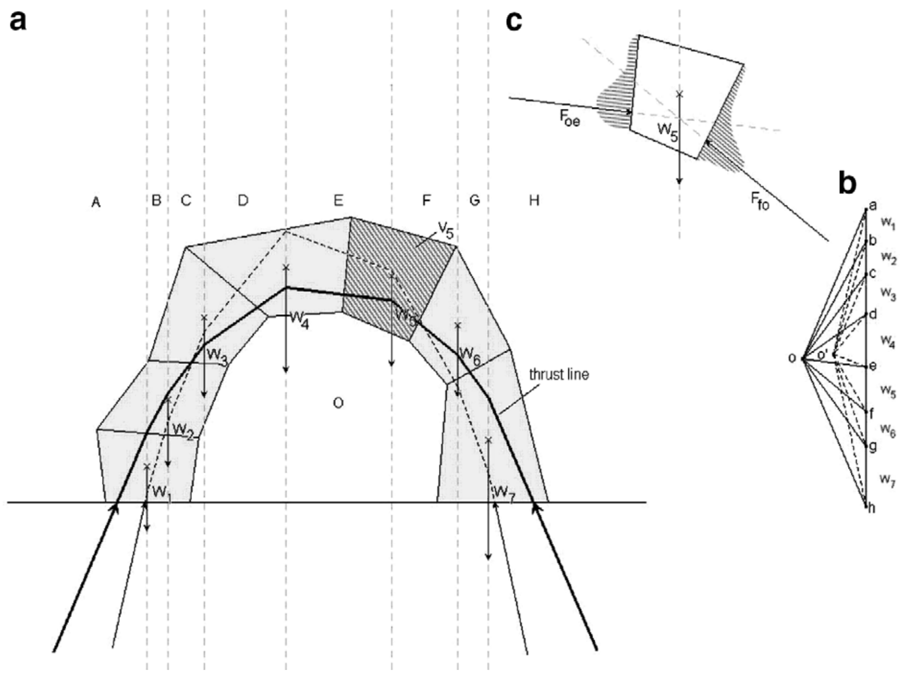


Fig. 3. (a) The self weights of the different blocks are treated as lumped masses applied at their centers of gravity, (b) the force polygon is drawn to scale in order to determine the forces and form of possible thrust lines and (c) the equilibrium of a single voussoir is guaranteed by the thrust line though the exact stress state at the joint of the block is unknown.

This influences the force polygon and hence the state of internal forces as represented by the thrust line. To avoid having to draw a graphic statics construction for every structure being analyzed, the models are generic, representing a family of similar structures, and parametric, so that every characteristic of them can be changed. When a family of related structural elements is included in one model, this provides a tool to compare the relative influence of geometry on the range of possible stable conditions. Because this is possible in real time, the methods developed here have important qualities as learning tools. Immediate structural feedback is given as the user changes the geometry on the screen by actively dragging control points or by inputting numerical values.

Only the thrust lines within the masonry section are possible equilibrium states. A typical masonry arch is statically indeterminate with a range of possible thrust lines [1]. A deeper arch has less horizontal thrust compared to a shallower one, and the force polygon visualizes this clearly. For example in Fig. 3a, the horizontal component of the thrust line, which is represented by the perpendicular distance of the pole ($O$ or $O'$) to the load line, is smaller for the dashed line than for the solid thrust line. For a statically indeterminate arch on rigid supports, both solutions are possible. By exploring the range of possible equilibrium states for a given structure, the user develops an intuitive feel for the relationship between the geometry and the stability of the structure.

### 2.3. Animated kinematics

Large displacements of 300 mm or more are common in historic masonry structures as a result of differential settlements in foundations, defects in construction, and consolidation of materials. The proposed analysis tools do not seek to determine the cause of the displacements but rather aim to understand their importance for the stability of the structure. To do so, a kinematic analysis is necessary in addition to the static analysis. By combining kinematic and static analysis it is possible to understand the range of possible motions and to assess the relative stability of the masonry structure. This allows the user to analyze structures in their actual deformed shape rather than in an idealized undeformed geometry.

The different steps in this process are summarized as follows:

1. The line of thrust in its extreme positions of minimum or maximum thrust provides the possible hinge locations. Anywhere the line touches the boundaries of the structure, a hinge may be created and this suggests a possible kinematic mechanism (Fig. 4a). There is a direct relationship between the support displacements and the state of thrust of a structural element. An arch on spreading supports will act in a state of minimum thrust and the same arch on closing supports will act in a state of maximum thrust [1].
2. This information is used to create a dynamic mode with rigid body movements. The thrust line is constantly updated throughout the movement.
3. Masonry structures often deform as a stable three-hinge mechanism. When the line of thrust can no longer be contained within the masonry, a fourth hinge is created and the structure becomes an unstable collapse mechanism, as in Fig. 4b.

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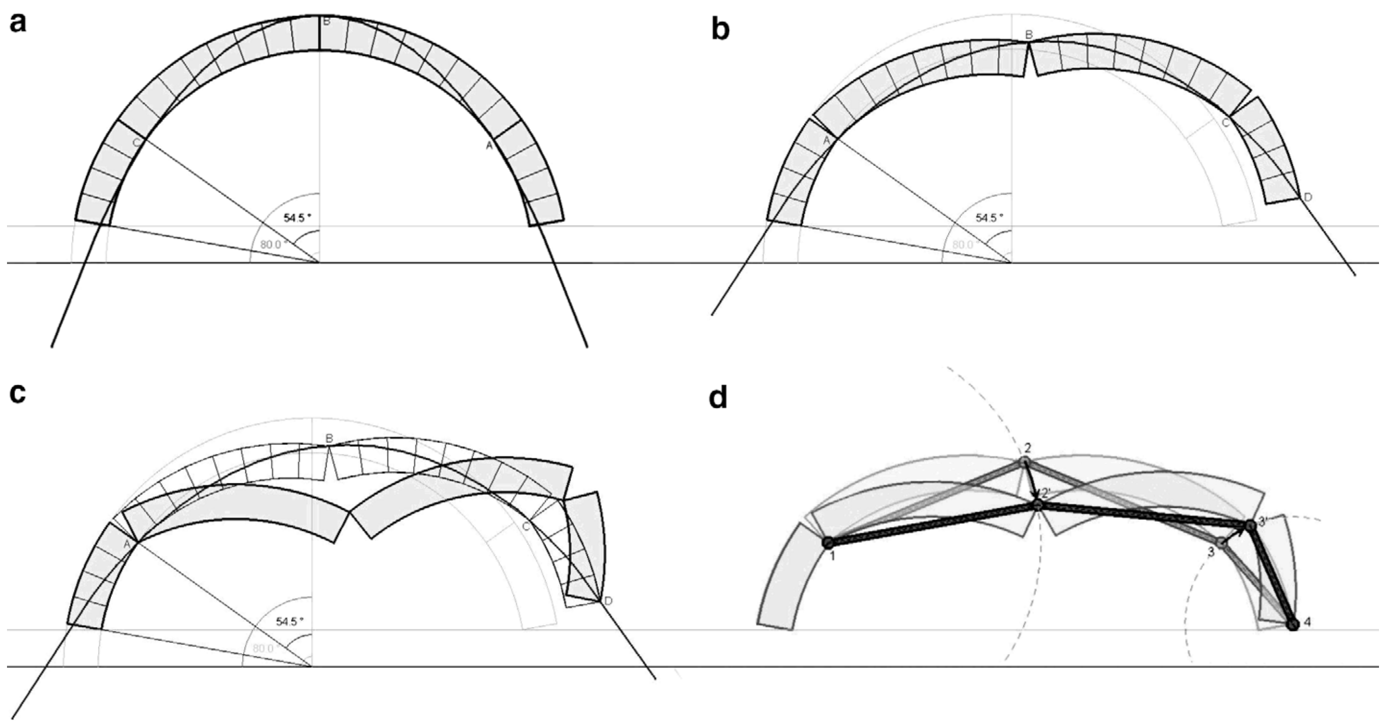


Fig. 4. Possible limit of deformation when the arch in (a) becomes (b) unstable, and (c) a snapshot of animation during collapse. (d) Four hinges define a three-bar-mechanism.

4. Animations show the collapse mechanism for the assumed hinge locations (Fig. 4c).

As shown in Fig. 4d, the behavior of a cracked arch during support movement and collapse can be abstracted and explained using a rigid bar model. The analysis of the structure as an assembly of rigid blocks allows for this abstraction, since all displacements are assumed to be caused by support movements.

This new approach, based on limit analysis, can be used to assess the safety and stability of masonry structures due to large displacements. It uses the calculation power and interactive potential of computers to revive earlier analysis methods. The methodology is summarized by its components of interactive graphic analysis, geometrically controlled loads, and animated kinematics. To apply this method to three-dimensional structures, the structure in question must be divided into slices to allow for the two-dimensional thrust line analyses to be used [1,5].

The following sections present examples of increasing complexity, with each showing a different aspect and the merits of the new approach. The first models illustrate lower bound static analyses, which demonstrate one possible safe equilibrium condition. The next section presents 2-d structural elements, structural systems and finally examples of analyses of three-dimensional vaulted structures using the same methods. The final group of models combines statics and kinematics for real-time analysis of the stability of masonry structures undergoing large displacements.

## 3. Static analysis

This section presents examples of static analysis tools, which can find the limits of equilibrium for a wide range of structural elements. The method can also be used to investigate the importance of geometrical properties on their structural performance and to compare a family of similar elements. First, two-dimensional problems are demonstrated and then examples are shown for more complex three-dimensional structures.

### *3.1. Natural stone arch*

The natural stone arch of Fig. 5a is divided into 32 slices of varying mass, which help to create the corresponding force polygon and one possible resulting line of thrust within the masonry (Fig. 5b). By dragging the nodes of every segment of the arch, the geometry can be changed to create an arch of arbitrary form and the thrust line is updated automatically with the changing applied loads. This allows the user to play with the shape to understand the limits when a virtual arch would not be able to stand. Once an initial geometry is chosen erosion can be simulated by removing material (Fig. 5c). The stone arch will stand as long as at least one thrust line can be found that lies within the section. If the thrust line does not travel through the section, this would require compressive forces to travel through the air, which is only possible if the stone can resist appreciable tension. Of course it is possible to provide an allowable tensile capacity for the stone in order for the

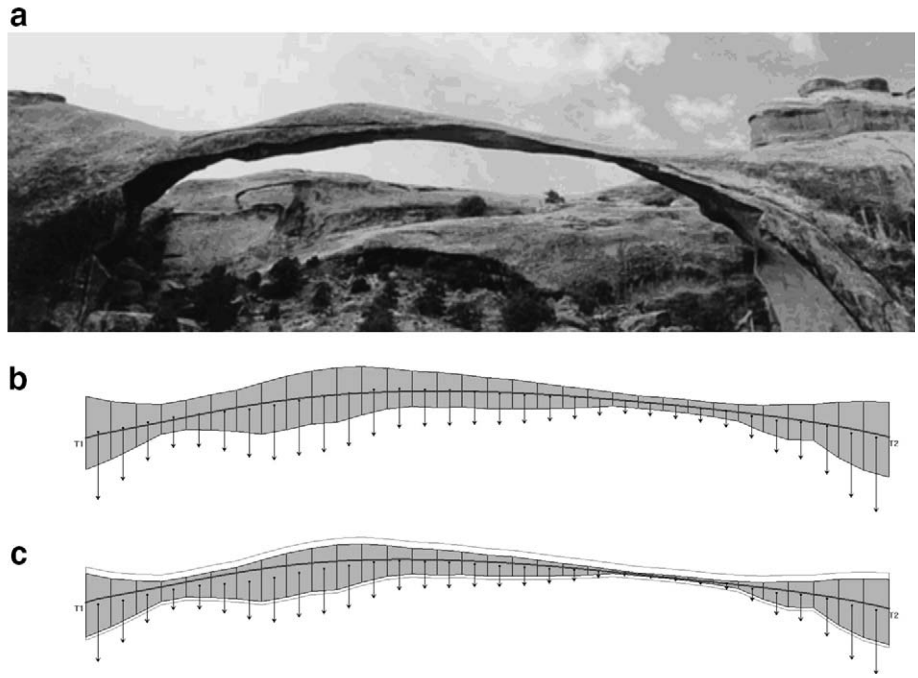


Fig. 5. (a) Landscape Arch in Devil's Garden, Arches National Park, USA, (b) a model showing a possible thrust line in this natural arch currently and (c) after erosion simulation.

thrust line to exit the section, though we make the conservative assumption that masonry has zero tensile capacity. This example shows the power of the line of thrust by explaining the structural stability of the natural arch: if a thrust line can be found within the arch's profile for the given loading, then the arch will stand.

In addition, the method is very powerful for comparing a large number of geometrically related structural elements. A recent study of early French Gothic flying buttresses used this approach to quickly analyze twenty flying buttresses and to make new conclusions on the evolution and performance of flying buttresses [21]. Other aspects such as sliding limits and the influence of the pinnacles on the stability of the flyers were implemented in the same interface.

### 3.2. Romanesque church

The preceding ideas are now extended to an entire structural system, bringing different structural elements together and showing how they interact, which started from collaboration with researchers at Columbia University [22]. To analyze one hundred small Romanesque churches in France, our team developed a computer tool (Fig. 6), which is able to generate all possible church sections for structures with a single nave or with three naves. The user can change the geometry of buttresses and arches, spans and heights of the structure, the thickness of vaults, and the level of fill above the vaults by dragging control points or by inputting numerical values. All parameters are changed in real-time to explore their importance for the stability of the structure. A sectional analysis is very appropriate for a preliminary investigation, because Romanesque buildings are among the most two-dimensional buildings in existence. By adding another sectional analysis which accounts for the church geometry in the other direction (Fig. 6b), the tool provides a complete analysis method for the static equilibrium of the church geometries. Most of these churches have severely deformed arches and leaning buttresses and they are often a collage of elements added over a long period of time. Many of the deformations and historical interventions can be explained quickly using a simple check on the equilibrium conditions of the building geometry. The thrust lines analysis serves as a way to visualize the stability of these structures and to clearly explain their pathologies to an interdisciplinary audience. The interactive analysis tool is extremely useful for comparing a large group of structures, such as 100 churches in a region, to determine which of the structures are in the most precarious state of equilibrium.

It is not trivial to understand the indeterminate character of a masonry vaulted building, in which an infinite number of possible states are possible. An interesting way to represent this is proposed in Fig. 7b, plotting the area containing all admissible combinations of thrust values of the structural system of Fig. 7a, as proposed by Smars originally [23]. The thrust provided by the side arch ($H_{\mathrm{side}}$) is given as a function of the thrust of the main arch ($H_{\mathrm{main}}$). The boundaries of the solution area are the minimum and maximum thrust of both arches and the maximum horizontal thrust capacity of the main and side buttresses ($H_{\mathrm{A}}$, $H_{\mathrm{B}}$). In order to ensure a safe solution in which the entire section of the buttress is acting in compression, it is of interest to include the limits of the middle-third for each

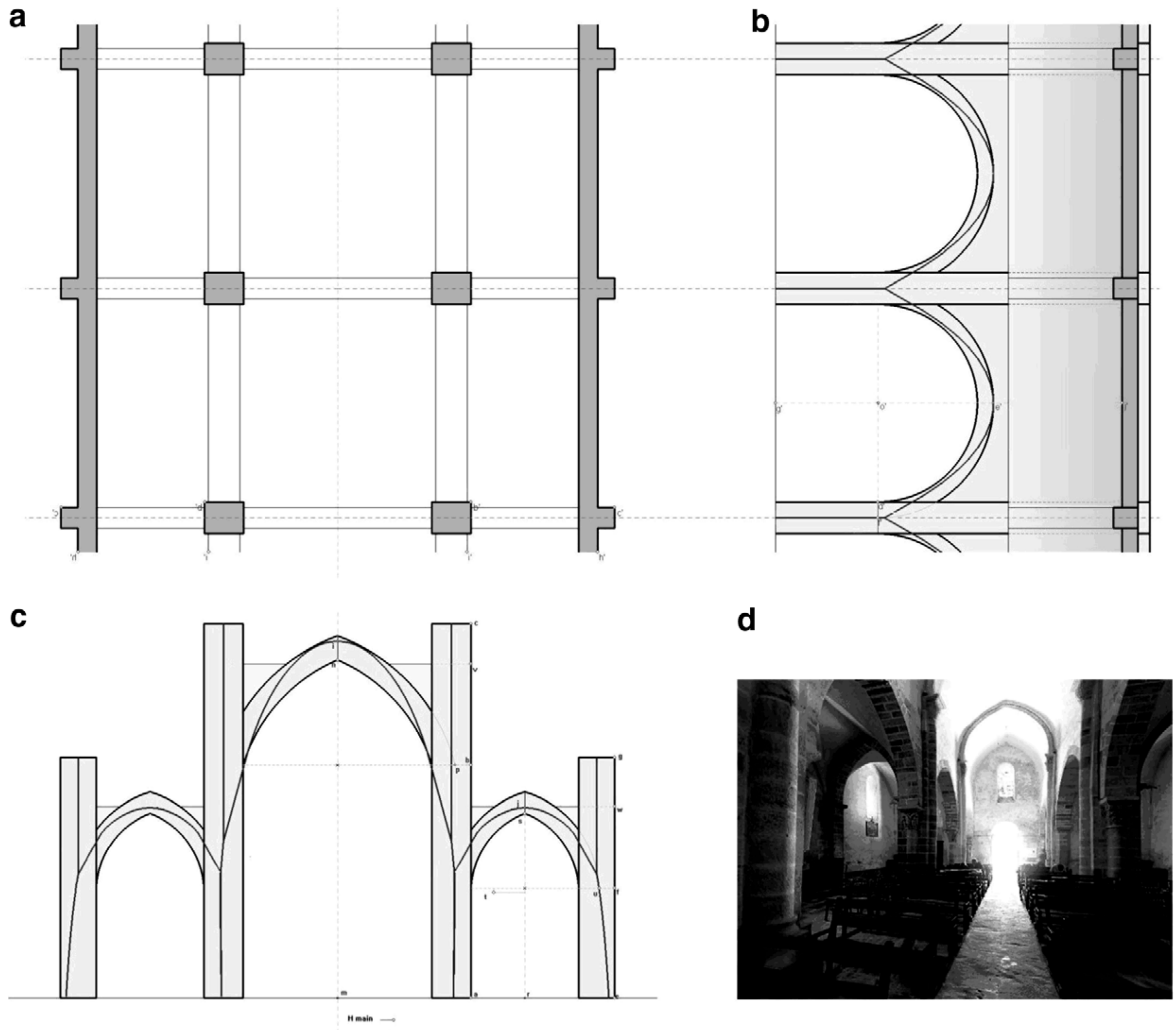


Fig. 6. Interface used to analyze Romanesque churches with a sectional approach (a–c). Here applied to a small 12th century church in Franchesse in the Bourbonnais region of France illustrated in (d). (*Photo source*: Andrew Tallon).

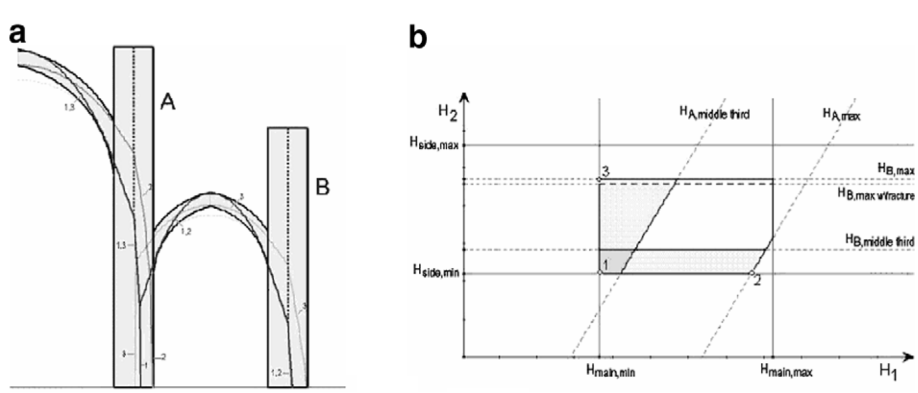


Fig. 7. (a) One half of a generic section of a Romanesque church and (b) a representation of all possible horizontal thrust values for this system.

buttress. Three possible limit cases (1–3) are shown in Fig. 7 as potential internal thrust lines. Note that only case 1 lies within the completely safe area (darker area on graph), where the thrust line in both buttresses stays within the middle third. Case 2 shows a combination of minimum state of thrust in the side isle and a state of thrust limited by the capacity of buttress A in the main isle. Case 3 shows the opposite: minimum thrust for the main isle and a thrust limited by the capacity of buttress B in the side isle. All are possible for this structure. Unless there are obvious cracks showing the actual state of the structure, only the structure knows the true state of internal forces.

### 3.3. Gothic quadripartite vault

The structural behavior of the quadripartite vault has been the focus of debate for centuries [24,25]. We use the assumptions made by Heyman and others that the main ribs bring the forces down to the buttresses, while the web is considered to be a collection of parallel arches spanning between the ribs (Fig. 8). The graphical construction uses the slicing technique and is inspired by the analysis of Wolfe [26]. Because of its fully parametric setup, this model can represent a wide range of different groin and quadripartite vault geometries. This provides an idea of the relative structural behavior of different types of vaults and the influence of changes in geometry to their performance. In Fig. 8c, the range of possible thrust values of the strips can be explored. All of the thrust resultants from the strips influence the position of the thrust line in the main rib (Fig. 8d). Similar to 2-d arches, masonry vaults have a broad range of possible thrust values. Using trigonometry, the range of possible total thrusts for one half of a vault ($H_{\text{vault}}$) can be found easily. Of course the results are dependent on the decision of how to slice the vault, which can be informed by the actual crack patterns in a real vault.

The examples presented until now are static analyses, characterized by an infinite number of possible solutions within a range defined by the minimum and maximum horizontal thrust of that element or system. The possible limit states are difficult to predict for complex vaulting systems where different structural elements interact. The new analysis methods presented here provide a way to visualize the highly indeterminate character of the problem, and a range of lower bound solutions for three-dimensional structures can be produced quickly and easily. This enables the comparison of large families of structures, which is a major improvement in the study of vaulted masonry buildings.

## 4. Combined static and kinematic analysis

The methods presented here could be extended to include the influence of live loading, as in the case of a masonry arch bridge. However, applied displacements are often a greater threat than live loading for historical buildings. To sustain differential support displacements, a rigid block structure must develop cracks. The formation of cracks means that the structure is not as highly indeterminate, and it becomes easier to determine where the forces are acting within the structure. In the case of a simple arch, the cracks signify hinge locations which define the position of the thrust line. The limit of displacement which causes collapse, a unique upper bound solution, can be explored by applying support movements until it is no longer possible to fit a thrust line in the deformed structure. Collapse mechanisms and crack genesis with large displacements are not easy to simulate using existing analytical approaches. The approach demonstrated in this section shows complex displacement analyses using simple but powerful geometrical methods based on graphical analysis. These are possible by adding the static analysis to kinematic analysis illustrating the range of possible displacements.

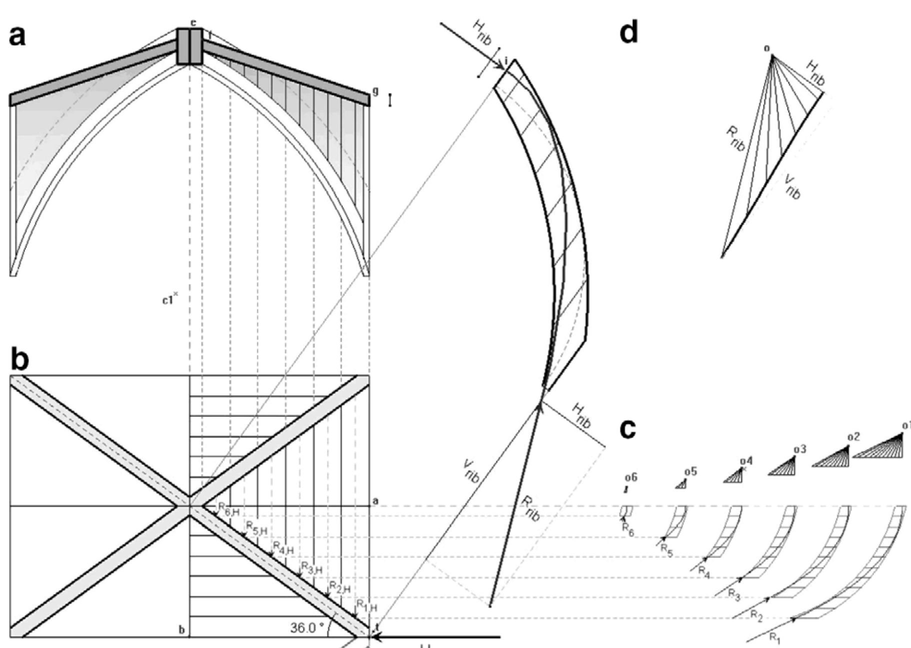


Fig. 8. Interface used to analyze gothic vaults by (a) section, (b) plan to represent the three-dimensional vault, (c) the local analyses of the strips influence and (d) the global analysis of the possible forces in the main ribs.

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### 4.1. Arch on spreading supports

This tool allows the user to explore the effect of support movements on the stability of an arch. The user applies displacements by dragging a control point at a support. It is a major development to be able to navigate between uncracked and collapse states in real time. This demonstrates that some degree of cracking in masonry is not problematic and that most masonry arches have a large capacity to deform before failing. The arch in its minimum thrust state (Fig. 9a) provides information on the hinge locations which open as the supports move apart (Fig. 9b). In theory, support displacements may increase until the four-hinge collapse state shown in Fig. 9c. The purely kinematic or geometrical solution (Fig. 9d) is not the point at which the structure fails, because such a solution is not statically admissible. The unstable collapse mechanism is illustrated as an animation in Fig. 9e. Theoretically, because of symmetry, a fifth hinge would form at the same time but in reality there are typically slight imperfections that make the structure behave asymmetrically. The unique collapse state is found by determining the conditions which are both statically and kinematically admissible for the maximum applied displacement. The applet reports results in this same interactive environment, graphing the relation between the change in horizontal thrust and the change in span (Fig. 9f).

Earlier studies by Ochsendorf [8,27] show that this result may be unsafe since the hinge locations are not fixed, but may move during the support displacements. The graphical analysis tools used here demonstrate this second order effect as well. When moving the supports apart, the thrust line touches the intrados higher than the initial hinge location and may even exit the section. In this applet the user must update the hinge locations manually to account for this effect. We compared our results using these applets to more complex Matlab analyses and confirmed the earlier results. The span increase at failure gave exactly the same result and both models predicted the possible movement of hinges for some arch geometries.

### 4.2. Arch mechanisms

Actual support displacements often combine differential horizontal and vertical movements, rather than purely horizontal as in the previous section. We developed additional tools to investigate the full range of deformations for an arch before collapsing, as shown in Fig. 10. The locus of all kinematically and statically admissible states of this arch is reproduced by tracing the position of a point on the support, shown as the hatched area in Fig. 10b. Instead of generating the contours of this solution zone by an automated process as in Smars [23], the users trace it themselves to explore the limits to the stability of the arch. This interactive control adds to the user's understanding of the arch deformability and the range of possible solutions within a single arch.

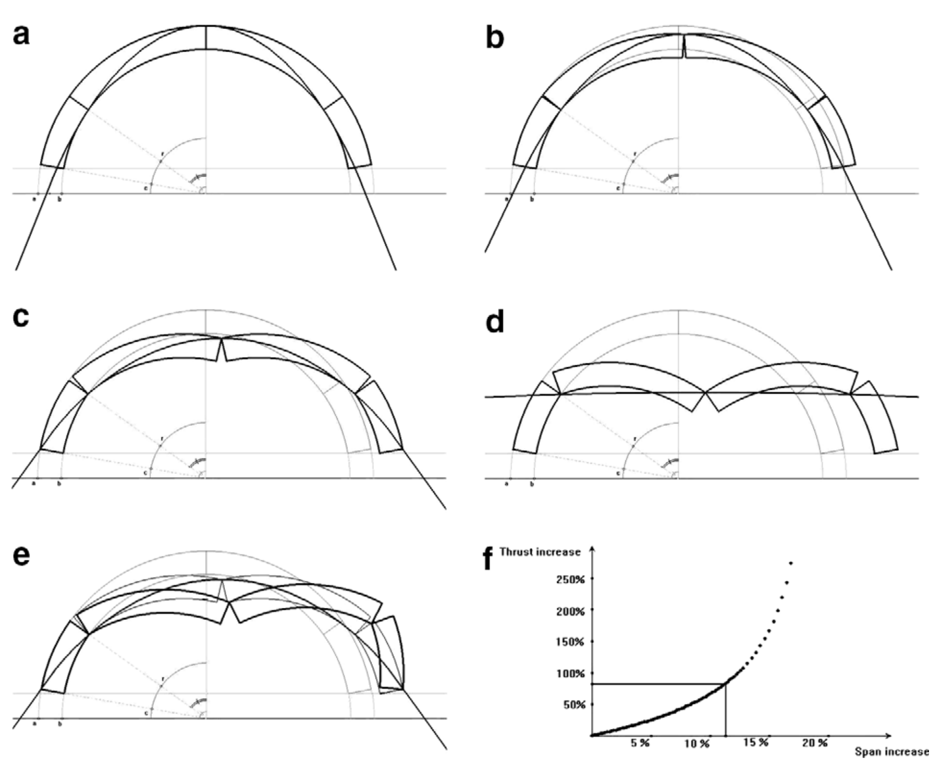


Fig. 9. For an arch with $t/R = 0.15$ and a total angle of embrace of 160°: (a) minimum thrust state with no deformation; (b) an intermediate state of deformation due to a span increase of 4.0%; (c) at collapse due to a span increase of 11.7%; (d) at the kinematic limit for a span increase of 18.9%; (e) snapshot of the animation showing the collapse mechanism at a span increase of 11.7%; and (f) a plot of horizontal thrust increase as a function of the span increase.

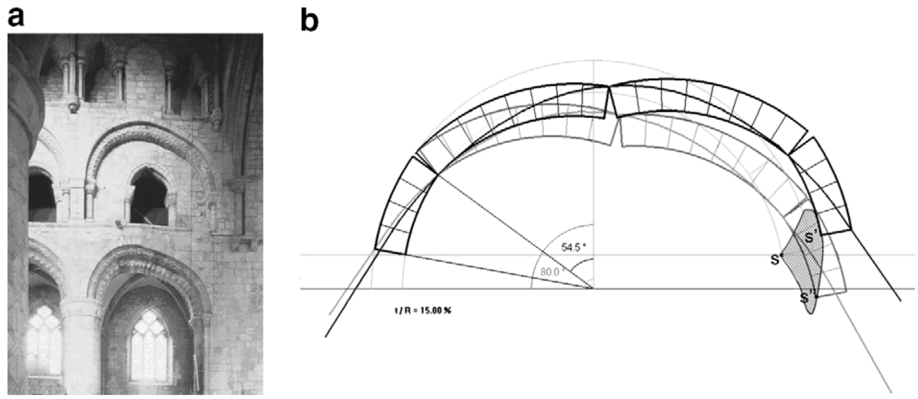


Fig. 10. (a) Example of highly deformed arches in Selby Abbey, England, and (b) two limit states of an arch. The arch in position *s* is in its undeformed state. (*Photo source*: Jacques Heyman).

### 4.3. Arch on leaning buttress

The analysis tool for the Romanesque churches was extended to include the possibility of leaning walls or buttresses, as shown in Fig. 11. The arch may deform due to leaning of one of the buttresses, which is a common pathology noticed in many churches [28]. The structure deforms as a three-hinged arch and failure occurs when a fourth hinge is created on the bottom right at the base of the buttress (Fig. 11c–d). This tool compares the maximum lean for a solid, monolithic buttress and for a fractured buttress with reduced stability by implementing the buttress fracture algorithm proposed by Ochsendorf et al [29]. The buttress of Fig. 11 may reach a maximum lean of 3.0° before collapse, though if buttress fracturing is taken into account, the lean is reduced to 2.5°. Adding second order effects, such as cracking in the buttress, has a significant influence on the results. As in the other examples, the model is fully parametric so that the user can change the thickness of the arch, dimensions of the buttresses, or any other parameter to explore their influence on the stability of the vaulted structure.

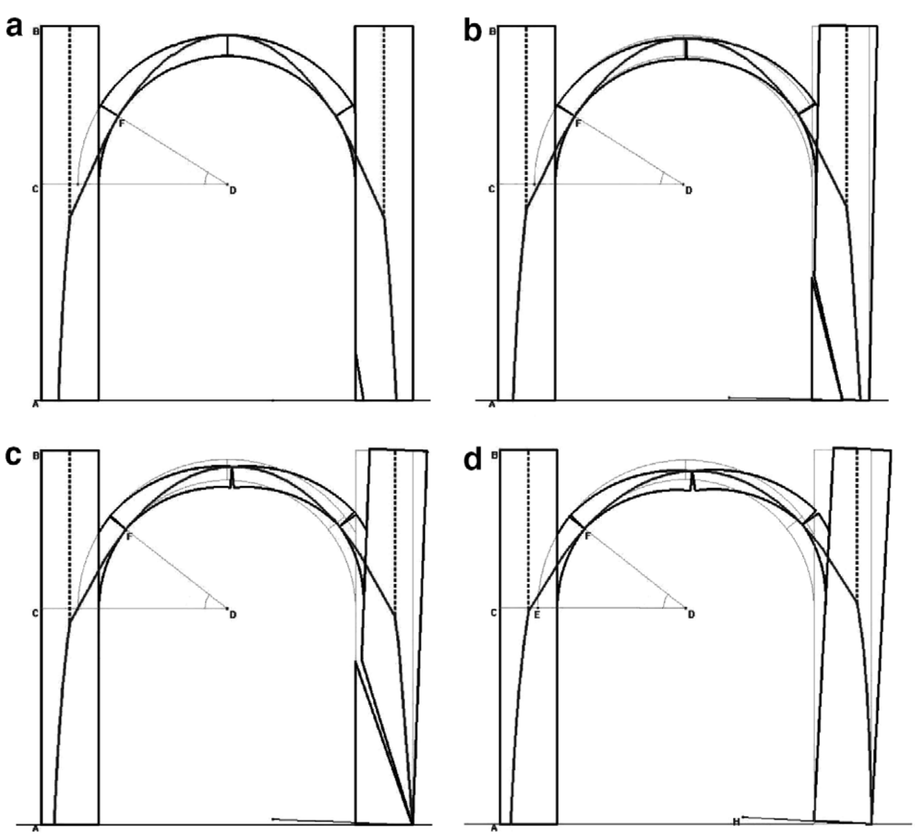


Fig. 11. Arch supported by a leaning buttress with (a) no lean, (b) lean of 1.2°, (c) 2.5° lean at collapse for the case of a fractured buttress, and (d) collapse at 3° for a solid buttress. The assumed fracture state occurs when the thrust line leaves the middle-third of the buttress [29].

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The preceding examples were devoted to displacement analysis for two dimensional masonry structures. The determination of hinge locations, stable deformed states and kinematically admissible limits for these structural elements were determined using interactive thrust line analysis. Support movements due to horizontal and vertical displacements, and combinations of them, such as displacements due to rotating abutments, were considered. Animations of possible collapse mechanisms show the limits to stability and the likely collapse modes. The computation of the crack propagation in the buttress illustrates that more sophisticated models are possible, including non-linear and second order calculations.

## 5. Conclusions

### 5.1. Summary of results

This paper has described new limit analysis tools for masonry structures which are simple yet powerful. Control points allow the user to change different parameters or to apply displacements, and structural feedback is given in real-time while the models are being updated. The methods are rigorous and numerically accurate. Through the use of computer models, the graphical method is inherently accurate because it only describes conditions of static equilibrium, as demonstrated by a closed force polygon. Graphical methods performed by hand will carry small errors, which do not exist in the graphical calculations carried out computationally. Similarly, no approximations are used to model the kinematic behavior of the masonry structures. The rigid body movements are described using pure geometrical constructions. All of the models and results have been verified and compared with earlier numerical studies.

The methods presented are visual and intuitive, providing engineers with a simple approach for examining the stability conditions of masonry buildings and for explaining the results. The methods clearly show the indeterminate character of structural elements as well as the existence of maximum and minimum thrust states, allowing the user to understand the range of stable conditions for a masonry structure. In the displacement analyses, the tools provide information about possible collapse mechanisms, and animations illustrate the collapse modes. Based on our experiences with these tools in the field and in the classroom, it is valuable for users to see the limits at which the structure collapses.

Applying limit analysis assumptions to masonry structures allows for the real-time modeling of statics and kinematics. In particular, the identification of the complete range of kinematically and statically admissible solutions is an improvement on existing methods of analysis. In addition, the versatility of the tools can be used to analyze all possible geometries within a structural type. For example, the model for the Romanesque church can be modified in real-time to compare interventions and to understand a complete family of forms. The simplicity of the tools allows them to run easily over the internet as Java applets, and the methods developed here are web-based and freely available at [13]. Finally, these methods provide a common language that enables engineers to communicate with architects and historians, a necessity for the study of historic structures.

### 5.2. Limitations and future work

The simple two-dimensional drawing software used to create these tools has allowed the rapid production of web-based applets. However, the number of constructions per model is limited and the existing software would not be able to model an entire Gothic cathedral for example. Another limitation of these programs is that they are restricted to the problems that can be described geometrically. In addition, for slightly different problems new models have to be created. For the Romanesque church applet, it is not possible to change the overall form (from three naves to five for example) since these require different geometrical constructions. An open interface programmed directly in C++ or Java would allow user-defined routines to provide more flexibility. This would provide the ability to generate a greater range of possible geometries and to include iterative calculations necessary to simulate second order effects such as the movement of hinges. Furthermore, the tools do not provide a complete analysis of three-dimensional behavior and mechanisms of vaulted structures. The "pseudo 3-d" analysis, using the slicing technique, provides a very conservative possible range of solutions but does not consider possible beneficial effects due to more sophisticated patterns of interaction, as demonstrated by O'Dwyer for example [5]. For these reasons, this work is a first step towards a more robust and more general software package for limit analysis of masonry buildings.

The work described here has considered only the self-weight of masonry buildings, and it should be extended to include live loading. In particular, it would be useful to extend the methodology to equivalent static analysis of horizontal ground acceleration in order to make rapid seismic assessments for vaulted masonry buildings. Graphical analysis can be used to determine the minimum horizontal acceleration required to form a collapse mechanism in a vaulted structure, which is based on the relative proportions of the arches and buttresses. This can be very useful for comparing a large group of buildings in a region to determine which structures require more detailed seismic analysis [8].

There are two additional extensions for the methods proposed here. The three main assumptions introduced by Heyman for limit state analysis to masonry structures are reasonable approximations for the mechanical properties of a masonry structural element [30]. However, a pure rigid-plastic analysis does not perfectly capture the behavior of masonry structures. The models must be refined by incorporating material properties to simulate problems of local crushing at the theoretical rotation points in the hinge.

Although the current applets demonstrate the global mechanisms very well, refined models that incorporate local and second order effects are necessary to assess the safety of historic structures. Finally, the biggest remaining challenge is to extend this interactive and dynamic approach in a fully three-dimensional realm.

This work has shown that there is great potential in using real-time limit analysis for vaulted masonry buildings. It has raised new research questions to be developed and has given possible paths to be considered for further exploration and development. Finally, the paper has shown that graphical computation using thrust lines offers new possibilities for the equilibrium analysis of masonry vaulted buildings, which is essential for conserving architectural heritage in the future.

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