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## *As Hangs the Flexible Line: Equilibrium of Masonry Arches*

**Abstract.** In 1675, English scientist Robert Hooke discovered “the true... ..manner of arches for building,” which he summarized with a single phrase: “As hangs the flexible line, so but inverted will stand the rigid arch.” In the centuries that followed, Hooke’s simple idea has been used to understand and design numerous important works. Recent research at MIT on the interactive analysis of structural forces provides new graphical tools for the understanding of arch behavior, which are useful for relating the forces and geometry of masonry structures. The key mathematical principle is the use of graphical analysis to determine possible equilibrium states.

### *Introduction*

**Robert Hooke’s hanging chain.** Robert Hooke (1635-1703) described the relationship between a hanging chain, which forms a catenary in tension under its own weight, and an arch, which stands in compression (fig. 1a).

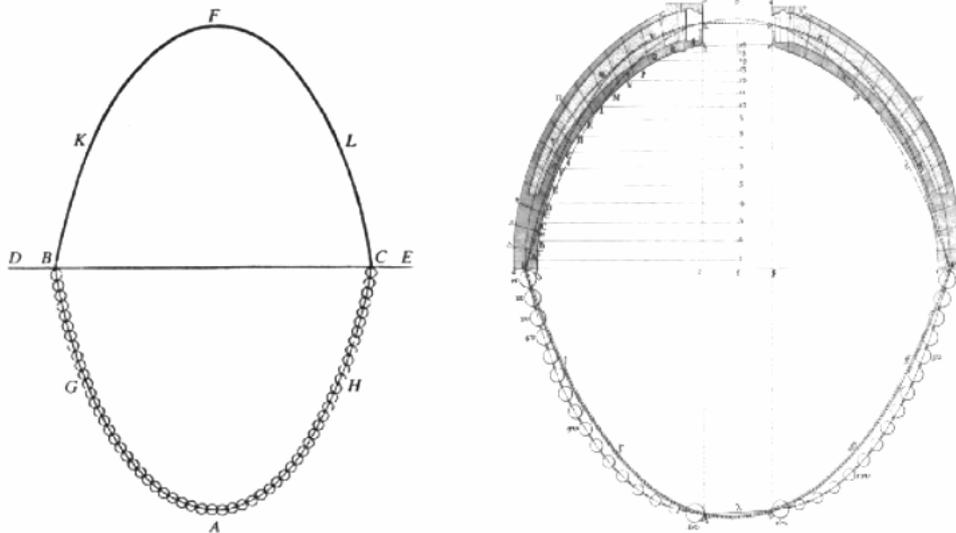


Fig. 1. (a) Poleni’s drawing of Hooke’s analogy between an arch and a hanging chain, and (b) his analysis of the Dome of St.-Peter’s in Rome [1748]

Though he could not derive the equation of a catenary, Hooke knew that his intuition was right and therefore wrote his finding as an anagram in Latin in the margin of another book [Hooke 1675]. Once descrambled, the anagram reads: *ut pendet continuum flexile, sic stabit contiguum rigidum inversum*, and translates to “as hangs the flexible line, so but inverted will stand the rigid arch” [Heyman 1998]. Both the hanging chain and the arch must be in equilibrium, and the forces are simply reversed. The chain can support only tension, and the masonry arch acts in compression. Generalized, this idea signifies that the shape a string takes under a set of loads, if rigidified and inverted, illustrates a path of compressive forces for an arched structure to support the same set of loads. This shape of the string and the inverted arch is called a *funicular* shape for these loads.

In 1748, Poleni analyzed a real structure using Hooke’s idea to assess the safety of the cracked dome of St. Peter’s in Rome. Poleni showed that the dome was safe by employing the hanging chain principle. For this, he divided the dome in slices and hung 32 unequal weights proportional to the weight of corresponding sections of that “arch” wedge, and then showed that the hanging chain could fit within the section of the arch (fig. 1b). If a line of force can be found that lies everywhere within the masonry, then the structure can be shown to be safe for that set of loads [Heyman 1966].

**Graphic Statics.** From the introduction of Simon Stevin’s (1548-1602) parallelogram rule, equilibrium could be described graphically using force vectors and closed force polygons [Stevin 1586]. This was the start of equilibrium analysis of structural systems, and also the start of graphical methods. It was now possible to explain experimental results such as weights hanging from a string and to “calculate” the forces in the string using these new graphical methods (fig. 2).

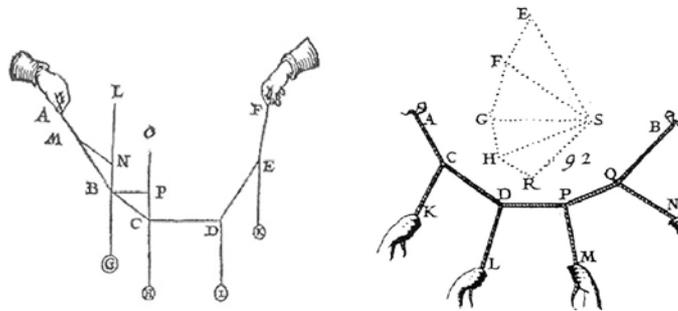


Fig. 2. (a) One of Stevin’s drawings of force equilibrium of hanging weights on a string [1586], and (b) an illustration by Varignon showing a graphical analysis of a funicular shape [1725]

Culmann [1866] was the first to formalize graphical analysis as a powerful method for equilibrium analysis in structural engineering. His *Die graphische Statik* had a strong theoretical foundation in mathematics, specifically in projected geometry. Graphical analysis provides a rigorous analysis method for trusses, arches, cables, and other structural systems. At the end of the nineteenth and the beginning of the twentieth century, graphic statics was the most common method to determine equilibrium for structures. Many trusses and masonry arch bridges were calculated using graphical methods, which still stand without any significant failures [Lévy 1888]. Handbooks from the beginning of the twentieth century, such as Wolfe [1921], use graphical constructions to solve advanced structural problems that would demand higher order differential equations when solved numerically. Maurer [1998] provides an excellent historical overview of the development and evolution of Culmann’s method.

Even though graphic statics was successfully used in engineering practice, its popularity did not last long. By 1920, graphical methods were largely replaced by the theory of elasticity, which provided elegant closed-form analytical solutions that did not require the same drawing skills. As Boothby [2001] pointed out, graphical methods give good but conservative results, though the process and analysis can become very tedious. Few engineers or architects today have the specific knowledge and patience to make these complex and advanced graphic constructions. In recent years, Zalewski and Allen [1998] have published a new textbook extolling the virtues of graphic statics for design, which suggests that there are new applications for this historical analysis method.

**Analysis of Masonry Arches.** There are only three types of equations that can be used for structural analysis: equilibrium (statics); geometrical (compatibility), and materials (stresses). For historical masonry structures, the first two types of equations are most important, since stresses are typically an order of magnitude below the failure stress of the masonry. A stability or equilibrium approach will therefore be most valuable to assess the safety of masonry structures, and limit analysis provides a theoretical framework. To apply limit analysis to masonry, Heyman [1966] demonstrated that it is necessary to make three main assumptions: masonry has no tensile strength; it can resist infinite compression; and no sliding will occur within the masonry. Heyman [1982, 1995] and Huerta [2004] provide additional background on limit analysis for masonry arches.

Equilibrium in a masonry arch can be visualized using a *line of thrust*. This is a theoretical line, which represents the path of the resultants of the compressive forces through the stone structure. This line is the inverted catenary discussed above. For a pure compression structure to be in equilibrium with the applied loads there must be a line of thrust that lies entirely within the masonry section. The concept was first rigorously formulated by Moseley [1833] and an excellent mathematical treatment was offered by Milankovitch [1907]. The analytical solution for this concept, has been defined more precisely as the *locus of pressure points* by Ochsendorf [2002].

#### ***New interactive equilibrium tools***

Recent research at MIT has produced new computer methods for graphic statics. Greenwold and Allen [2003] developed *Active Statics*, a series of interactive online tutorials that implement graphic statics for a series of chosen problems. As a continuation, new methods for exploring the equilibrium and compatibility of masonry structures have been developed by Block et al [2005] which can be applied in real-time. In the tradition of limit analysis developed by Heyman, thrust line analysis is used to provide a clear means for understanding the behavior and safety of traditional masonry structures. Limit analysis using thrust lines can establish the relative stability of the structures and as well as possible collapse mechanisms.

This paper presents a series of models and tools demonstrating the possibilities of this approach for analysis, design and teaching. This new approach brings back graphical analysis, greatly enhanced by the use of computers, which allow for parametric models of structural elements or systems. Their geometry is linked to the graphical construction and controls the loads of the analysis. A rigid block model is used to provide displacement or kinematic analysis and animations illustrate collapse. The models are interactive and parametric, allowing the user to change all parameters (such as thickness, height, span etc.) in order to explore an entire family of structural shapes and to understand the relation between geometrical changes and stability. At the same time, the methods are numerically accurate and rigorous.

The three new ideas of this approach are: interactive graphic statics, geometry controlled loads, and animated kinematics. Block et al. [2005] provides a detailed explanation of these steps.

1. Interactive graphic statics:  
The models are made using simple two-dimensional drawing packages, such as Cabri Geometry [2006]. They allow the creation of dynamic geometric constructions, and to make these easily available over the internet. Computer aided drawing programs overcome the inconveniences and drawbacks of graphic statics by guaranteeing accuracy. Furthermore, parametric modeling prevents the user from having to redo the graphical constructions for every analysis.
2. Loads controlled by the geometry:  
The geometry of the structural element is linked to the graphical construction. Altering the section on the screen will influence the self-weight and this is updated instantly thereby changing the applied loads. This results in an interactive tool that provides real-time structural feedback.
3. Animated kinematics:  
The assumptions necessary to apply limit analysis to masonry structures allow displacement analysis to be performed in real time. The blocks in the models stay rigid during the kinematic movement and their movement can be described using geometry.

### **Examples**

This section presents examples of tools made possible with this new approach. The project website at <http://web.mit.edu/masonry/interactiveThrust> contains more in-depth analysis.

**Thrust-line analysis of a random arch.** Fig. 3 uses (a) Bow's notation and (b) a force polygon to give the magnitude of the forces of the segments in the funicular polygon for a random arch. This force polygon is drawn to its own scale and represents and visualizes the equilibrium of the system. The funicular construction and its polygon are related by geometry. The mathematical foundations for this reciprocal relationship are clearly summarized by Scholtz [1989]. The horizontal distance from the pole  $o$  to the load line  $ah$  gives the horizontal thrust in the system. This is the amount the arch pushes (thrusts) outwards and analogously the amount the hanging string pulls inwards. Looking more closely at the fan shape of the force polygon, we can isolate the different closed vector triangles (bold lines) showing the equilibrium of each block in the random arch (Fig. 3c, d).

First, the structure has to be divided into discrete parts. The actions of the different blocks are treated as lumped masses applied at their center of gravity (Fig. 3a). The magnitudes of the forces are proportional to their weight and transferred to the force polygon (b). If the user drags a corner point, this would influence the area, hence the weight, of the two adjacent blocks. This will then also alter the associated vector in the force polygon.

It can also be seen that there is not only one solution to this problem since a different horizontal thrust in the system results in a deeper (for decreased thrust) or a more shallow (for increased thrust) section. Thanks to the interactive setup the user can explore this solution easily by controlling the unknown in the "equations", for example the amount of horizontal thrust in the system. The notion that there is no one answer to this problem will be expanded in the following section.

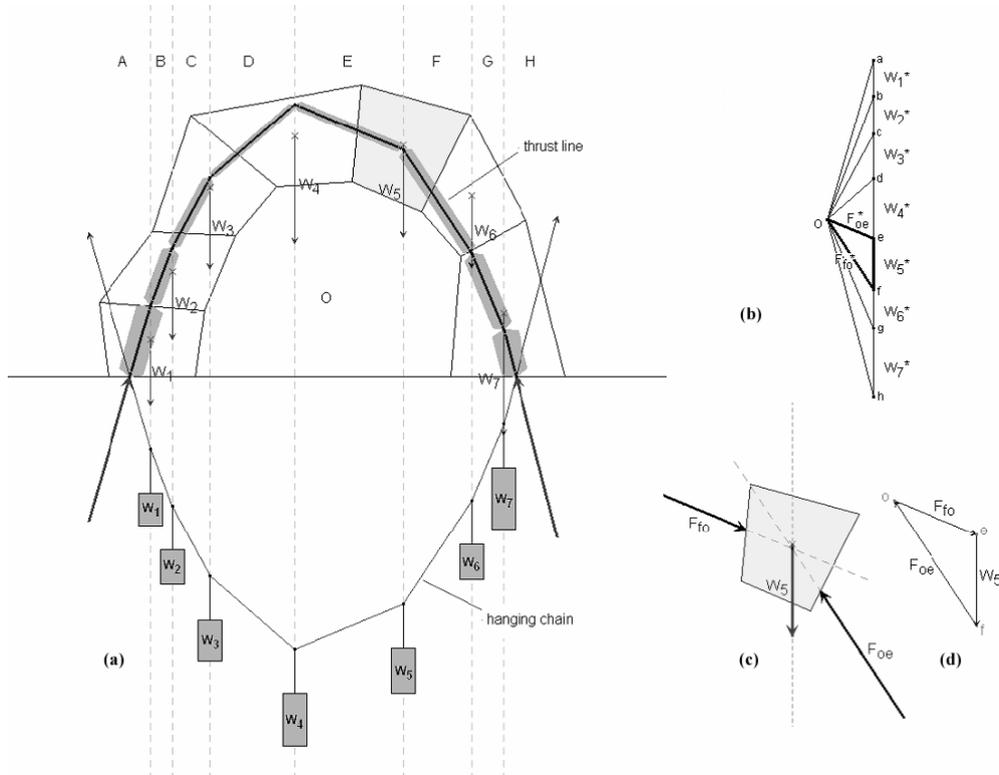


Fig. 3. For a random arched structure, (a) a possible thrust line and its equivalent hanging chain are constructed using graphic statics; (b) the force equilibrium of the system is represented in the funicular polygon; (c) the equilibrium of one of the *voussoirs*; and, (d) the vectors representing the forces in and on the block

**Semi-circular versus pointed arch.** For every masonry structural element, there is an infinite amount of valid thrust lines that fit within its section, all lying between a maximum and minimum value (fig.4).

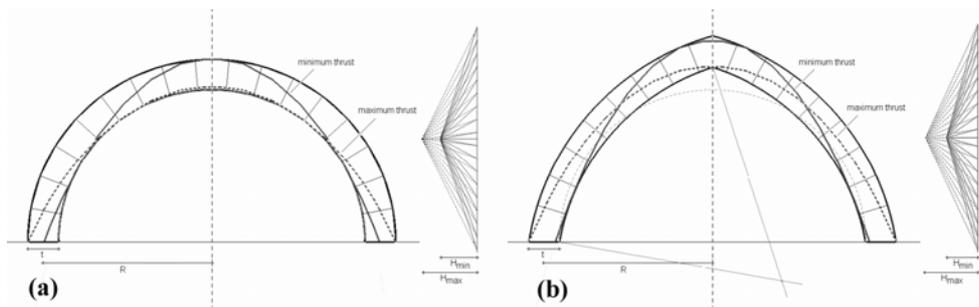


Fig. 4. This image compares (a) a semi-circular arch with (b) a pointed arch with the same  $t/R$  ratio. The minimum and maximum thrust of each arch is shown

The minimum thrust of a structural element is the minimum amount of force with which this element pushes against its neighbors or abutments. The maximum thrust or active state of that element is then the maximum amount of horizontal force it can transfer or provide. This value can become very large and therefore the maximum thrust of an element will be limited by the material's crushing strength or, often sooner, by the stability of its neighboring elements, such as buttresses or walls [Ochsendorf et al 2004].

The different arch geometries and thrust lines in fig. 4 are generated with the same model. Since the geometries belong to the same type of structures, it suffices to change a few parameters. This enables the user, once a model is setup, to obtain the analyses for these different, but related, forms within minutes. This feature becomes very interesting when a vast number of structures have to be analyzed and compared, as applied to flying buttresses by Nikolinakou et al [2005]. Such an approach is possible since structures in masonry are scalable, and stability is more important than stress [Huerta 2004]. Rigid block model are therefore excellent models to understand unreinforced masonry structures.

The two arches of fig. 4 have the same thickness to radius ( $t/R$ ) ratio of 18%. Their minimum and maximum thrusts, expressed as a percentage of their self-weight, are respectively 16% and 25% for the semicircular arch (fig. 4a), and 14% and 23% for the pointed arch (fig. 4b). The self-weight of the two arches is nearly identical and the range of allowable thrusts is approximately the same. However, the pointed arch thrusts about 15% less than the circular arch. These results and the forces in the arches are obtained simply by measuring the rays of the force polygon and multiplying them with the appropriate scaling factor. Additionally, the size of the force polygon allows a quick visual comparison of the magnitude of the forces in the system. Romano and Ochsendorf [2006] provide a more detailed study on the performance of semi-circular versus pointed arches.

**Arch on spreading supports.** Masonry arches commonly collapse due to instability caused by large displacements (ground settlements, leaning buttresses etc.). Therefore a displacement analysis is crucial for unreinforced masonry structures. The introduction of displacement analyses to assess the safety of these structures and to understand collapse was proposed and investigated in detail by Ochsendorf [2002, 2005]. Collapse mechanisms and crack genesis with large displacements are not easy to simulate using traditional (such as analytical or finite element) analysis methods.

The approach demonstrated in this section shows a complex displacement analysis using a simple but powerful method. It is possible to illustrate the range of allowable displacements by superposing a static (thrust line) and a kinetic (rigid body) analysis. The example shows a simple arch on spreading supports (fig.5). The analysis demonstrates that very large displacements are possible before the structure becomes unstable, and that large cracks do not necessarily signify that the structure is in immediate danger of collapse.

To allow displacements, a rigid masonry structural element must develop cracks. There is a direct relationship between the thrust line and where the hinges occur: where the line touches the extremities of the structure, hinges are most likely to form (fig. 5a). From the moment the structure is cracked, the hinges define the location of the thrust line, and three hinges create a statically determinate structure with a unique equilibrium solution. Since compressive forces cannot travel through the air, the thrust line is forced to go through the hinging points (Fig. 5b).

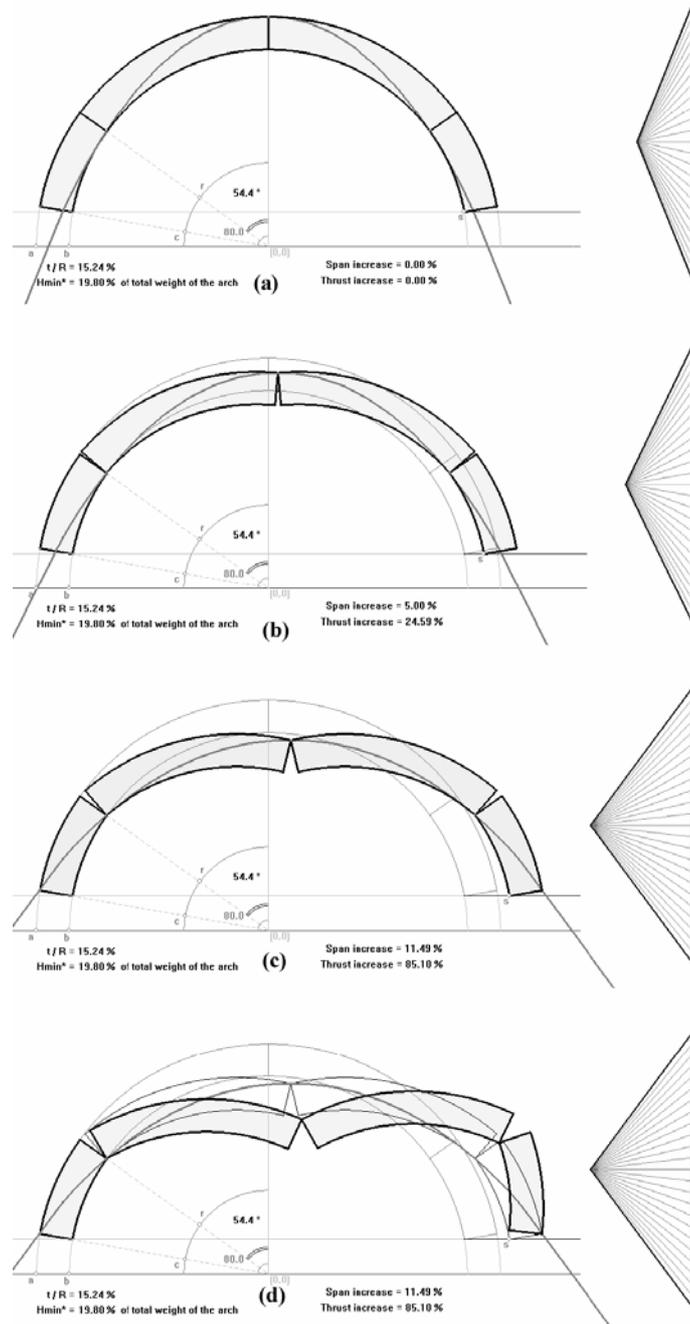


Fig. 5. This shows an arch on spreading supports with a  $t/R = 15\%$  and a total angle of embrace of  $160^\circ$  (a) initially in its minimum thrust state, (b) at an inter-mediate state, and (c) right at collapse. (d) shows a snapshot of the animation showing the collapse mechanism

The limit of displacement at which collapse occurs can be explored by applying displacements until it is no longer possible to fit a thrust line in the deformed structure. Failure occurs when there are more than three hinges, i.e. when the thrust line touches the boundaries of the structure in more than three places (fig. 5c). Fig. 5d shows a snapshot of an animation of a possible collapse mode due to spreading supports.

This model is interactive; the effects of spreading supports can be checked by the user by moving the corner or inputting values. While doing this, it can be noticed that the force polygon grows, indicating that the forces, specifically the horizontal thrust, in the system increase. This makes sense since the deforming arch becomes more and more shallow, resulting in an increase of horizontal thrust.

**Earthquake Analysis of Masonry Arches.** This approach can also be extended to assess the stability and safety of arched structures in areas prone to earthquakes. However, the assumptions adopted from Heyman [1995] must be reevaluated for dynamic loading. Clearly dynamic loading could increase local stresses causing crushing failure of the masonry, and could cause vibrations which would increase the likelihood of sliding. Despite the fact that such a purely geometrical analysis method cannot capture these more complicated effects of earthquake dynamics, the interactive graphical analysis can be useful.

It is common practice in structural engineering design to simulate earthquake loading by a constant horizontal force that is some fraction of the weight of the structure in magnitude. This is equivalent to applying a constant horizontal acceleration that is some fraction of the acceleration of gravity. Such an “equivalent static loading” does not capture all of the dynamics, but it does provide a measure of the lateral loading that the structure could withstand before collapse.

To implement equivalent static analysis for earthquake loading simulation, it is possible to modify the interactive thrust tool to include a tilting ground surface. Tilting the ground surface effectively applies a vertical acceleration (gravity,  $g$ ) and a horizontal acceleration ( $\lambda * g$ , where  $\lambda = \tan(\alpha)$  and  $\alpha$  is the ground surface inclination angle). The ground surface is tilted until the thrust line cannot be contained within the structure: the point where it passes through the exterior surface of the structure at four locations. At this point, four ‘hinges’ would form and the structure would collapse. Two examples of structures subjected to equivalent static analysis are shown at the point of collapse in Fig. 6.

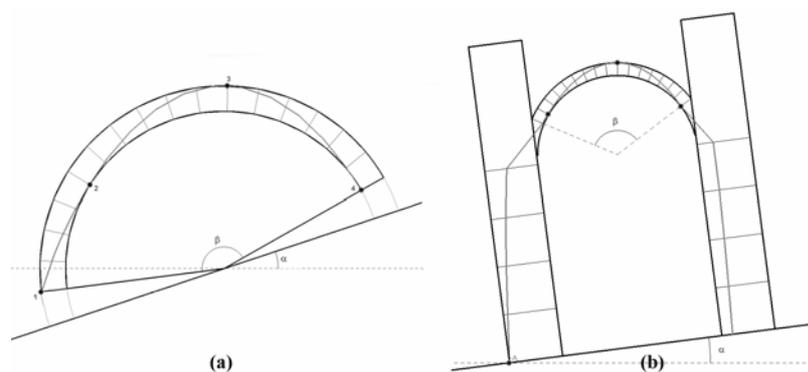


Fig. 6. Equivalent static analysis of (a) a masonry arch, and (b) a similar arch on buttresses. Points indicate hinge locations at collapse

The arch of sixteen voussoirs, inclusion angle ( $\beta$ ) of 157.5 degrees, and thickness/radius ratio of 0.15 would collapse at a constant horizontal acceleration of  $0.34^*g$  (fig. 6a). A similar arch ( $t/r = 0.15$ ,  $b = 120^\circ$ ) on buttresses with relative height and width dimensions of 1.5 and 0.5 times the vault span, respectively, would collapse at a constant horizontal acceleration of  $0.13^*g$  (fig. 6b). These equilibrium solutions can be verified through other common analysis techniques such as equilibrium equations or virtual work.

The resulting values of horizontal accelerations which cause collapse are conservative in that they assume an infinite duration of loading. Actual earthquake loadings are of much shorter duration which could allow the structure to “recover” due to inertial effects if higher horizontal accelerations are experienced. However, the analysis is also unconservative because the possibilities of local crushing and sliding have not been evaluated. Regardless, the method is valuable because it provides a rapid relative measure of the stability of vaulted masonry structures, and visually depicts the expected collapse mechanism. Therefore, while more rigorous dynamic analyses should be executed for structures which are known to be in danger of collapse, this equivalent static analysis method can provide a valuable tool for identifying those structures.

### ***Conclusion***

As has been shown through previous examples, the interactive analysis tools provide a useful method which uses graphic statics to achieve a rapid first order assessment of the stability of various masonry arch structures. Specifically, the real time graphic statics framework has been shown to allow the effects of geometrical changes such as arch thickness, buttress width, etc., to be quickly evaluated. Assessment of the stability of such arched structures with varying geometries would be considerably more difficult using traditional computational methods, but more importantly, the graphic statics framework inherently presents results visually as well as numerically, allowing results to be easily interpreted. Mistakes cannot hide in the equations and the validity of the results is guaranteed because of the visual nature of the technique.

The power of physical models and simulations can not be denied. For example, it was Hooke’s hanging chain that convinced his colleagues of the forces acting in tension under a given load as an inverted form of the forces in an arch. The methodology presented here, thanks to its integrated kinetic analysis, allows the same powerful simulations in a virtual environment with an added versatility to adapt the models easily in order to understand the effect of changes in geometry on the stability of arches.

The strong assumptions necessary to apply limit analysis to masonry provide complex analyses easily and quickly. After this first order analysis, the analyst must check if the assumptions have been violated. In some cases, the method will lead to unconservative results since second order effects such as local crushing, crack propagation and sliding have not been considered. This is most crucial in the kinetic analyses.

This paper has shown that there is great potential in using interactive thrust line analysis for masonry structures. It has also raised new research questions to be developed and has given possible paths to be considered for further exploration and development. Finally, the paper showed how graphical computation offers new possibilities in an old field of research, which is essential for conserving architectural heritage in the future. Finally, concept of the thrust line emphasizes the relationship between geometry and structural behavior of buildings as a fundamental principle for architectural designers in the future.

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