Today
- NP-completeness, poly-time reductions

11/4/05

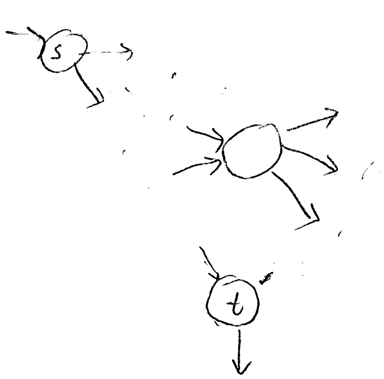
Q| Show UHAMPATH = $\{ \langle G, s, t \rangle \mid G$ is an undirected graph and $\exists$ path from $s \to t$ going though each vertex exactly once$\}$ is NP-complete

Proof| UHAMPATH is easily seen to be in NP (certificate is the path)

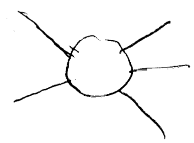
We show HAMPATH $\leq_p$ UHAMPATH

Must give poly-time computable function $f(\langle G, s, t \rangle) = \langle G', s', t' \rangle$ such that $\langle G, s, t \rangle \in$ HAM iff $\langle G', s', t' \rangle \in$ UHAM

G



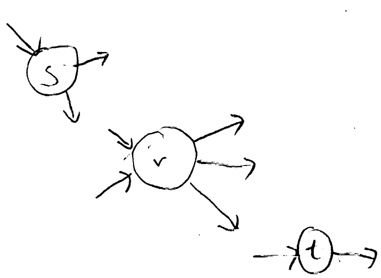
Bad G': replace all directed edges with undirected



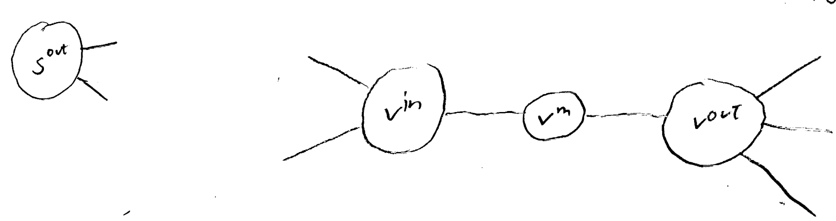
Now G' may have HAM path when G does not

Example:

G



Good G': replace every node in G besides s and t with 3 nodes as follows:



Also, drop incoming edges from s, outgoing edges from t

Now, if G has a HAM path $\Rightarrow$ G' will have a HAM path

And, if G' has a HAM path $\Rightarrow$ G will have a HAM path

A graph cut is a partition of vertices into two sets

Size of cut is # edges spanning the cut (one endpoint in each set)

MAX-CUT = { ⟨G,K⟩ | G has a cut of size at least K }

Q) Show MAX-CUT is NP-complete

Proof) MAX-CUT is easily seen to be in NP (certificate is the partition)

We show ≠SAT $\le_p$ MAX-CUT

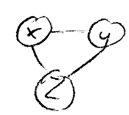
Give $f(\langle \phi \rangle) = \langle G, K \rangle$ st $\langle G,K\rangle \in$ MAX-CUT iff $\langle \phi \rangle \in$ ≠SAT

First Attempt

ϕ (e.g. $(x \vee y \vee z) \wedge (x \vee \bar{y} \vee z)$) G

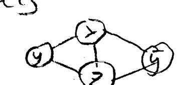
n vars, each variable x — two nodes x, $\bar{x}$

c clauses, each clause $(x \vee y \vee z)$ — edges connecting



Suppose x = true, y = false, z = true, ... is an ≠ satisfying assignment ⟹ Then cut defined by $x, \bar{y}, z \ldots \mid \bar{x}, y, \bar{z}$ has size $\ge 2c$? Not quite. Problem if clauses use same literals

For example, could get

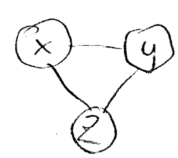


First Fix

ϕ G

each variable x — 3c nodes labeled x, 3c nodes labeled $\bar{x}$

each clause $(x \vee y \vee z)$



but never use x or y or z in another clause gadget

Suppose ϕ has ≠SAT assignment ⟹ G has a cut of size $\ge 2c$

But if G has a cut of size $\ge 2c$ ⟹ Does ϕ have an ≠sat assignment?

Maybe not. In fact, G always has a cut of size 2c

Final Fix: connect all nodes labelled x to all nodes labelled $\bar{x}$

Now,

ϕ has $\exists$ SAT assign $\Longrightarrow$ G has a cut of size $\geq n(3c)^2 + 2c$

G has a cut of size $\geq n(3c)^2 + 2c$ $\Longrightarrow$ ϕ has $\exists$ SAT assignment

Q] Show SUBSET-SUM $= \{\langle S, t\rangle \mid S$ is a set of #s, some subset of which sums to $t\}$

Proof] Clearly SUBSET-SUM $\in$ NP (since the subset is the certificate)

We show $3SAT \leq_p$ SUBSET-SUM

$$f(\langle\phi\rangle) = \langle S, t\rangle$$

e.g. $\phi = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_3 \vee \bar{x}_3)$ ..

First Attempt

				c_1	c_2
x_1	1	0	0	1	0
$\bar{x}_1$	1	0	0	0	1
x_2		1	0	0	0
$\bar{x}_2$		1	0	1	0
x_3			1	1	1
$\bar{x}_3$			1	0	1
t	1	1	1	1	1

Must Fix this

See book for full construction