Value Iteration and Optimal Control of (Switched) Homogeneous Systems

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Abstract

In this note, we prove that dynamic programming value iteration converges uniformly for discrete-time homogeneous systems and continuous-time switched homogeneous systems. For discrete-time homogeneous systems, rather than discounting the cost function (which exponentially decreases the weights of the cost of future actions), we show that such systems satisfy approximate dynamic programming conditions recently developed by Rantzer, which provides a uniform bound on the convergence rate of value iteration over a compact set. For continuous-time switched homogeneous system, we present a transformation that generates an equivalent discrete-time homogeneous system with an additional "sampling" input for which discrete-time value iteration is compatible, and we further show that the inclusion of homogeneous switching costs results in a continuous value function. As an application, we rederive the results of Tuna for the optimal control of homogeneous systems specialized to switched homogeneous systems.

Index Terms

Switched systems, Optimal control, Dynamic programming, Homogeneous systems

I. INTRODUCTION

In this paper, we present new dynamic programming results for discrete-time homogeneous systems and continuous-time switched homogeneous systems. In particular, we provide conditions under which the value iteration algorithm [1] converges and the value function is continuous.

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Such convergence and continuity results may be used to compute approximately-optimal control laws for these systems. The problem formulation covers, as special cases, switched linear systems and nonlinear switched systems for which accurate homogeneous approximations can be developed.

Under fairly general conditions, value iteration is guaranteed to converge, but not necessarily to the value function. For infinite horizon formulations, a discounted cost function [1] in the Bellman equation may be used to guarantee the convergence of the value iteration algorithm, but at the price of changing the desired performance of the system. Specifically, the discount decreases the impact of future actions. Moreover, the closer the discount is to unity, the more the number of iterations that are required to reach a prescribed level of approximation.

In [2], a sufficient condition on the value function is presented that guarantees the convergence of value iteration. In particular, it is shown that if the value function is uniformly bounded by a fixed proportion of the incremental cost function, then value iteration converges uniformly, and, in this paper, we prove that discrete-time homogeneous systems and continuous-time switched homogeneous systems satisfy the conditions in [2] for the uniform convergence of value iteration. Furthermore, the CT value function is shown to be continuous. In the case of continuous-time systems, we present a method for transforming the system into a discrete-time system with an additional "sampling" input that makes it compatible with value iteration.

As an application of these results, we rederive the results of Tuna [3] for the optimal control of homogeneous systems specialized to switched homogeneous systems. Methods for computing optimal controllers for continuous-time homogeneous systems is reported in several works: [3], [4], [5], [6], [7], and [8] to mention a few, but we follow the methodology of [3], where a state-dependent sampling time is used to transform a continuous-time homogeneous system into a discrete-time, degree-1 homogeneous system, and a feedback control law is approximated using a quantization of the unit sphere. However, in specializing the results to continuous-time switched homogeneous systems, we are able to reduce the proofs of stability and performance to simple inductive proofs based in value iteration and continuity.

II. DISCRETE-TIME AND CONTINUOUS-TIME HOMOGENEOUS SYSTEMS

A. *Notation and Assumptions for Discrete-Time and Continuous-Time Homogeneous Systems*

We first begin with the definition of homogeneous systems used in this paper.

*Definition 1: A function $h : Y \times U \rightarrow H$ is **degree-d homogeneous-in-y** if there exists a matrix function $G(\alpha) = diag(\alpha^{r_1}, \ldots, \alpha^{r_n})$ for positive real constants α and r_i such that*

$$h(G(\alpha)y, u) = \alpha^{d-1}G(\alpha)h(y, u)$$

For ease, we will only consider the case $G(\alpha) = \alpha$, and the results of the paper may easily be rewritten for general G (in the switched-system case, the systems share the same G). A consequence of this assumption is that we can restrict our analysis of the system to the unit sphere in $\Re^n$, which we denote as S^{n-1}.

In this paper, we consider a discrete-time (DT) homogeneous system of the form

$$x(t+1) = f(x(t), u(t)) \tag{1}$$

where t is an integer, the state $x(t)$ is a vector in $\Re^n$, the input $u(t)$ is a vector in some compact set U, and f is degree-1 homogeneous-in-x.

Remark 1: As it is not generally desirable to apply unbounded u for bounded x, homogeneity in the parameter u in Definition 1 is not necessary [6]. If $\tilde{f}$ is degree-1 homogeneous in x and u, we can apply the transformation $f(x, u) = \tilde{f}(x, \|x\|u)$ and restrict u to some bounded set.

We also consider a continuous-time (CT) switched homogeneous system of the form

$$\dot{y}(\tau) = g_{i(\tau)}(y(\tau)) \tag{2}$$

where $\tau \in \Re$, the state $y(\tau)$ is a vector in $\Re^n$, the mode input $i(\tau)$ is a piecewise-constant function continuous from the right and taking values in a finite set Q (the set of *modes*), and the function g_i is a degree-d_i ($d_i \geq 1$) homogeneous-in-y function for each mode $i \in Q$.

We now define several important notations used throughout the paper for CT switched systems:

- let $\tau_0 = 0$ and successively define the k^{th} *switching instance* τ_k as the first time $i(\tau)$ changes value since time τ_{k-1}, i.e. $\tau_k = \min\{\tau > \tau_{k-1} \mid i(\tau^-) \neq i(\tau)\}$
- denote the *dwell period of the k^{th} switch* as $\Delta_k = [\tau_{k+1}, \tau_k)$,
- define $y_k = y(\tau_k)$ as the k^{th} *switching state*,
- and define $i_k = i(\tau_k)$ as the k^{th} *operating mode* and denote the *mode sequence* as the list $(i_0, i_1, \ldots)$.

If the mode becomes a constant after some switching time t_k, i.e. $i(\tau) = a$ is constant for $\tau \geq \tau_k$, then as there are no more switches, we define $\tau_j = \infty$ and $i_j = a$ for all integers $j > k$. Finally, let $i_{-1} = i_0$, which will help simplify notation.

At times, we equivalently express i by its mode and dwell period sequence $(i_k, \Delta_k)_k$, which will be useful for expressing the dynamics of (2) between switching instances.

Finally, it will be useful to explicitly express the trajectory of (2) as a function of time, the initial condition, and the input i. Denote the value at time τ of the trajectory originating from y_0 under a switching law i as $y(\tau, y_0, i)$ for those values of τ for which the trajectory is defined (since the trajectory may possess finite escape time).

Finally, we conclude this section with some assumptions about the CT and DT systems.

Assumption 1: The functions g_i are locally Lipschitz.

Assumption 2: f is bounded over $S^{n-1} \times U$.

We note that Assumption 2 automatically holds if f is continuous.

III. THE DT BELLMAN EQUATION AND VALUE ITERATION

For the DT system (1), let V be the *DT cost function* given by

$$V(x_0, u) = \sum_{t=0}^{\infty} L\left(x(t), u(t)\right) \tag{3}$$

where L is positive-definite and degree-d, $d > 1$, homogeneous in x. Define the *DT value function* as

$$V^*(x) = \inf_{u \in U} V(x, u)$$

By the homogeneity of L, it is clear that V^* is also degree-d homogeneous.

It is well known that the value function satisfies the Bellman equation

$$V^*(x) = \inf_{u \in U} \left\{V^*(f(x,u)) + L(x,u)\right\} \tag{4}$$

If the value function V^* is known, the optimal policy can be computed through an evaluation of the expression

$$u^*(x) \in \operatorname*{argmin}_{u \in U} \left\{V^*(f(x,u)) + L(x,u)\right\}$$

if the minimum exists.

A means for approximating the value function is by *value iteration*, where successively-improving approximations to the value function are computed iteratively in the following manner: pick some V^0 on $\Re^n$ and compute the sequence $(V^1, V^2, \ldots)$ iteratively by the relation

$$V^{k+1}(x) = \inf_{u \in U} \left\{V^k(f(x,u)) + L(x,u)\right\} \tag{5}$$

If the limit exists, denote $V^\infty = \lim_{k \to \infty} V^k$.

A. Convergence of DT Value Iteration and Continuity of the Value Function

While it is not generally true that value iteration will converge to the value function, certain assumptions may be imposed to guarantee convergence. In this paper, we make use of a convergence result given in [2], which we restate here in a form more amenable to our framework.

Proposition 1: If $V^*(f(x,u)) \leq \gamma L(x,u)$ holds uniformly for some constant $\gamma \geq 0$ and if V^* is bounded over a compact set E, then $(V^k)_k$ [1] converges uniformly to V^* over E.

Proof: According to [2], for $\alpha V^* \leq V^0 \leq \beta V^*$,

$$\left[1 + \frac{\alpha^{-1} - 1}{(1+\gamma^{-1})^k}\right]^{-1} V^*(x) \leq V^k(x) \leq \left[1 + \frac{\beta - 1}{(1+\gamma^{-1})^k}\right] V^*(x)$$

Uniform convergence is a consequence of the fact that V^* is bounded over E. ■

The results of this paper result in part by showing that homogeneous systems satisfy the conditions of Proposition 1. We now state an immediate corollary of this result.

Corollary 1: If $L(S^{n-1}, U)$ is lower bounded by a positive constant and $V^*(S^{n-1})$ is bounded, $(V^k)_k$ converges uniformly to V^* over S^{n-1}

Proof: By homogeneity, V^* is bounded over any compact set in $\Re^n$, in particular the compact set containing $f(S^{n-1}, U)$. Therefore, there exists a $\gamma > 0$ such that $V^* \circ f(x,u) < \gamma L(x,u)$ for all $(x,u) \in S^{n-1} \times U$. By homogeneity, the inequality extends over $\Re^n \times U$, and so uniform convergence results from Proposition 1 with $E = S^{n-1}$. ■

We now state a corollary concerning the continuity of the DT value function.

Corollary 2: If $L(S^{n-1}, U)$ is lower bounded by a positive constant, $V^*(S^{n-1})$ is bounded, and V^k is continuous for all k, then then $V^\infty = V^*$ and V^* is continuous.

Proof: By Corollary 1, value iteration is uniformly convergent. Since V^k is continuous for all k, V^* is continuous over S^{n-1} and, by homogeneity, continuous over $\Re^n$ as well. ■

Finally, in practice, it may be of interest to determine the boundedness of V^* from value iteration, and we state a useful result concerning this test.

Proposition 2: If $V^0 = 0$, equation (5) is minimized by some u_k^* for each k, and $L(S^{n-1}, U)$ is lower bounded by a positive constant, then if $V^\infty(S^{n-1})$ is bounded, $V^*(S^{n-1})$ is bounded as well.

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[1] We use the notation $(\cdot)_k$ to indicate a sequence over the index k, which will be useful in later sections when additional subscripts may be present in the sequences.

Proof: First, if $V^0 = 0$, then it can be shown that the sequence $(V^k(x))_k$ is monotonically increasing and bounded by $V^*(x)$.

Now, if the optimal input $u_k^*(x) \in \arg\min_u \{V^k(f(x,u)) + L(x,u)\}$ exists, then we let $V_i^K(x) = \sum_{t=0}^{K-1} L(x(t), u_{K-t}^*(x(t)))$. We term $\left(u_K^*, u_{K-1}^*, \ldots, u_1^*\right)$ the *K-step roll-out policy* [1].

Choose $\alpha < 1$. Let $\lambda > 0$ be such that $L(S^{n-1}, U) > \lambda$. By homogeneity and by our assumptions, $L(x,u) > \|x\|^d \lambda$ for all x and u (note that d is the degree-of-homogeneity of L).

By the boundedness of $V^\infty(S^{n-1})$, there exists an integer K such that $V^\infty(S^{n-1}) < K\lambda\alpha^d$, which, since $V_i^K(x) \le V^\infty(x)$, yields $V_i^K(x_0) < K\lambda\alpha^d \|x_0\|^d$.

Therefore, letting $x(t)$ result from an application of u_k^*, we have

$$\lambda \sum_{l=0}^{K-1} \|x(t)\|^d \le \sum_{t=0}^{K-1} L(x(t), u_{K-t}^*(x(t))) = V_i^K(x_0) < K\lambda\alpha^d \|x(0)\|^d$$

$$\text{Thus, } \sum_{l=0}^{K-1} \|x(t)\|^d < K\alpha^d \|x(0)\|^d$$

Therefore, for some time $t(x_0) < K$, $\|x(t(x_0))\|^d < \alpha^d \|x(0)\|^d$. By repeated application of the K-step roll-out policy, it can be shown that the resulting cost can be bounded over $S^n - 1$ (the cost can be bounded by a geometric series since $d > 1$). Therefore, the optimal cost is bounded over S^{n-1} as well. ■

IV. THE CT VALUE FUNCTION

For the CT system (2), consider the *CT cost function* $J(y_0, i)$ for an input trajectory i as

$$\begin{aligned} J(y_0, i) &= \int_0^\infty \|y(\tau, y_0, i(\tau))\|^{1+d_{i(\tau)}} d\tau + \sum_{k=0}^\infty \|y(\tau_k, y_0, i)\|^2 K_{i(\tau_k^-) i(\tau_k)} \\ &= \sum_{k=0}^\infty \left[\int_{\Delta_k} \|y(\tau, y_k, i_k)\|^{1+d_{i_k}} d\tau + \|y_k\|^2 K_{i_{k-1} i_k} \right] \end{aligned} \tag{6}$$

where the switching-cost constants K_{mn} are nonnegative for $m \neq n$ and zero otherwise.

Remark 2: Although any cost function with a continuous, degree-$(2 + d_i)$ homogeneous integrand and a degree-2 homogeneous switching cost could be considered, we restrict our analysis to (6) for its simplicity.

Notable in (6) is the power of the integrand: $1 + d_{i(\tau)}$. To gain some insight into this choice, consider the systems $\dot{y} = -y^3$ and $\dot{\tilde{y}} = -\tilde{y}$. Although their (phase space) trajectories are identical for the same initial condition $y_0 = \tilde{y}_0$, y is faster outside the unit sphere and slower inside the unit sphere. If we applied a quadratic integrand $\|y(\tau)\|^2$, the costs of the two systems would be different. However, by raising the integrand to the power $1 + d_i$, the costs are the same because the cubic system is penalized more heavily outside the unit sphere, where it travels quickly, and penalized less heavily inside the unit sphere, where is travels slowly.

Optimizing over all switching laws i with initial mode i_0, we obtain the *CT value function*,

$$J^*_{i_0}(y_0) = \inf_{\{i|i(0)=i_0\}} J(y_0, i) \tag{7}$$

A. Degree-1 Transformation of the CT System

To simplify the proofs of this section, we apply a useful transformation that will generate a degree-1 system having the same trajectories as the CT system (2). As in [4], let

$$\dot{z}(\tau) = \tilde{g}_{i(\tau)}(z(\tau)) = \|z(\tau)\|^{-d_{i(\tau)}+1} g_{i(\tau)}(z(\tau)) \tag{8}$$

Under suitable choices for each switching law [2], both (2) and (8) generate the same trajectories, but (8) is degree-1 homogeneous by this time scaling of (2).

Define a new cost function $\tilde{J}$ for system (8) as

$$\tilde{J}(z_0, i) = \int_0^\infty \|z(\tau, z_0, i)\|^2 d\tau + \sum_{k=0}^{\infty} \|z_k\|^2 K_{i_{k-1}i_k} \tag{9}$$

$\tilde{J}^*_{i_0}$ is defined similarly. It is clear that $\tilde{J}^*_i$ is degree-2 homogeneous. We now state the useful consequence of this transformation.

Proposition 3: $J^*_i = \tilde{J}^*_i$.

Proof: In a fixed mode i_0, let θ_{i_0,y_0} be the time-scaling between the z and y trajectories which satisfies

$$z(\theta_{i_0,y_0}(\tau), y_0, i_0) = y(\tau, y_0, i_0)$$

for all $\tau \geq 0$. It can be deduced by taking the derivative of both sides with respect to τ that $\dot{\theta}(\tau) = \|z(\theta(\tau))\|^{d_{i_0}-1}$.

Let i be any switching law given by $(i_k, \Delta_k)_k$ and assume without loss of generality that $y_k \neq 0$. Define the switching law $\tilde{i}$ given by $(\tilde{i}_k, \tilde{\Delta}_k)_k$ where $\tilde{i}_k = i_k$ and $\tilde{\Delta}_k = \theta_{i_k,y_k}(\Delta_k)$. It

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[2] The switching laws need to be scaled in time in order for the switchings to occur at the same location in the state space (i.e., so that $z_k = y_k$).

is clear that this is a bijective mapping. By induction, it can be shown that the switching states $(y_k)_k$ and $(z_k)_k$ are identical.

Starting with the cost function (6), we use the change of variable $v_k = \theta_{i_k,z_k}(\tau)$, yielding

$$\dot{\theta}_{i,z_k}(\tau) = \|z(\theta_{i,z_k}(\tau), z_k, i)\|^{d_{i_k}-1} = \|z(v_k, z_k, i_k)\|^{d_{i_k}-1}$$

and obtain

$$\begin{aligned} J_i(z_0, i) =& \sum_{k=0}^{\infty} \left[\int_{\theta_{i_k,z_k}(\Delta_k)} \|z(v_k, z_k, i_k)\|^{1+d_{i_k}} \frac{1}{\|z(v_k, z_k, i_k)\|^{d_{i_k}-1}} dv_k + \|z_k\|^2 K_{i_{k-1}i_k} \right] \\ =& \tilde{J}_i(z_0, \tilde{i}) \end{aligned}$$

It is straight-forward to show that this implies $J_i^* = \tilde{J}_i^*$. ■

Clearly, Proposition 3 implies that J_i^* is degree-2 homogeneous as well.

B. *Continuity of the CT Value Function*

In the case of the CT system being asymptotically controllable, it is of interest to prove that the value function is continuous. To this end, we impose the following assumption on the system

Assumption 3: The CT system (2) is asymptotically controllable [9], and there exists such a stabilizing control law that has a finite number of switches in any finite time interval.

To prove that J_i^* is continuous, we seek to leverage Corollary 2, but this result only applies to DT systems [3]. We now present a transformation of the CT to a DT system that will allow us to apply the DT value iteration results. First, we define a new function h_i representing the sampled dynamics of the normalized CT system (8) for a "sampling period" τ

$$h_i(x, \tau) = z(\tau, x, i)$$

We also define a new incremental cost function l as a sampling of the normalized cost (9)

$$l(x, \tau, i, j) = \int_0^{\tau} \|z(\gamma, x, i)\|^2 d\gamma + \|z(\tau, x, i)\|^2 K_{ij}$$

If we treat i and τ as control inputs, we have a DT system

$$x(t+1) = h_{i(t)}(x(t), \tau(t)) \tag{10}$$

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[3] Clearly, the results apply to DT switched systems by extending the input set U to $U \times Q$ in order to include the mode input.

where the time t is a nonegative integer. By substitution and by optimality, we can express $\tilde{J}_i^*$ by [4]

$$\tilde{J}_i^*(x) = \inf_{\{j, 0 \leq \tau \leq T_0\}} \{\tilde{J}_j^*(h_i(x,\tau)) + l(x,\tau,i,j)\} \tag{11}$$

for any $T_0 > 0$. In essence, all we have done is split-up the expression of the value function by the switching times, which is possible by optimality. Also, by allowing "switches" to the current mode, we are able to restrict τ to a compact set.

We can now use value iteration to prove that J_i^* is continuous. Define the sequence $(\tilde{J}_i^k)_k$ by

$$\tilde{J}_i^{k+1}(x) = \inf_{\{j, 0 \leq \tau \leq T_0\}} \{\tilde{J}_j^k(h_i(x,\tau)) + l(x,\tau,i,j)\} \tag{12}$$

We first prove that value iteration converges for the CT system.

Proposition 4: If $K_{ij} > 0$ for $i \neq j$, then $(\tilde{J}_i^k)_k$ converges uniformly over S^{n-1} to $\tilde{J}_i^*$.

Proof: Define a new incremental cost $\hat{l}$ as

$$\hat{l}(x,\tau,i,j) = \begin{cases} \infty, & i = j \text{ and } t < T_0 \\ l(x,\tau,i,j), & \text{otherwise} \end{cases}$$

The Bellman equation (11) may be equivalently written using $\hat{l}$ instead of l.

Let $I = [0, T_0]$. By Proposition A.1 (see Appendix), $\tilde{J}_i^*$ is bounded over any compact set, and therefore $\tilde{J}_i^* \circ h_i(S^{n-1})$ is bounded for all i.

For $j \neq i$, $\hat{l}(S^{n-1},\tau,i,j) = l(S^{n-1},\tau,i,j) > K_{ij}$. Therefore, there exists a positive constant γ_{ij} such that $\tilde{J}_j^* \circ h_i \leq \gamma_{ij}\hat{l}(\cdot,\cdot,i,j)$ over $S^{n-1} \times I$.

For $j = i$, $\hat{l}(x,\tau,i,i) = \infty$ if $\tau < T_0$, so any constant will suffice. For $\tau = T_0$, $\hat{l}(x,\tau,i,i) = l(x,T_0,i,i)$, which is strictly positive and continuous over S^{n-1}. Therefore, there exists a positive constant γ_i such that $\tilde{J}_i^* \circ h_i \leq \gamma_i\hat{l}(\cdot,\cdot,i,i)$ over $S^{n-1} \times I$.

The boundedness condition of Proposition 1 (and hence uniform convergence over S^{n-1}) follows. ■

We now prove that the value function is continuous.

Theorem 1: If $K_{ij} > 0$ for $i \neq j$, then J_i^* is continuous.

Proof: We will construct a value iteration sequence to prove the claim. If we use Corollary 2, we need only to show that each $\tilde{J}_i^k$ of such a sequence is continuous. We proceed by induction.

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[4] We note that i is actually a state of the DT system, but, for clarity, we write the value function using the index i_0 as in $J_{i_0}^*(x_0)$ instead of writing $J^*(x_0, i_0)$.

Let $I = [0, T_0]$. First, define sets T_m satisfying 1) T_m is finite, 2) $T_m \subset T_{m+1}$,and 3) for all $\tau \in I$, there exists a $\hat{\tau} \in T_m$ such that $|\tau - \hat{\tau}| < \frac{1}{m}$. Basically, we are quantizing the values for τ.

Assume $\tilde{J}_j^k$ is continuous for all j. By continuity over the compact controller set $Q \times I$, the minimizers τ^* and j^* of (11) exist. Define $\tilde{J}_i^{k+1,m}$ by

$$\tilde{J}_i^{k+1,m}(x) = \min_{\{j, \tau \in T_m\}} \{\tilde{J}_j^k(h_i(x,\tau)) + l(x,\tau,i,j)\}$$

Clearly, $\tilde{J}_i^{k+1,m}(x) \geq \tilde{J}_i^{k+1}(x)$. Since $\tilde{J}_i^{k+1,m}$ is the minimum over a finite set of continuous functions, it is continuous.

Choose any $\epsilon > 0$. By the uniform continuities of $\tilde{J}_j^k \circ h_i$ and l over $S^{n-1} \times I$, there exists a δ such that

$$|(\tilde{J}_j^k(h_i(x,\tau)) + l(x,\tau,i,j)) - (\tilde{J}_j^k(h_i(x,\hat{\tau})) + l(x,\hat{\tau},i,j))| < \epsilon$$

for $|\tau - \hat{\tau}| < \delta$ and for all $x \in S^{n-1}, i, j$.

Therefore, for all $x \in S^{n-1}$, there exists an M such that for all $m > M$,

$$\begin{aligned}
|\tilde{J}_i^{k+1,m}(x) - \tilde{J}_i^{k+1}(x)| = &\min_{\{j, \tau \in T_m\}} \{\tilde{J}_j^k(h_i(x,\tau)) + l(x,\tau,i,j)\} - (\tilde{J}_{j^*}^k(h_i(x,\tau^*)) + l(x,\tau^*,i,j^*)) \\
&\leq \tilde{J}_{j^*}^k(h_i(x,\tau_m^*)) + l(x,\tau_m^*,i,j^*) - (\tilde{J}_{j^*}^k(h_i(x,\tau^*)) + l(x,\tau^*,i,j^*)) \\
&\leq \epsilon
\end{aligned}$$

where $\tau_m^* = \mathrm{argmin}_{\tau \in T_m} |\tau - \tau^*|$. Consequently, $(\tilde{J}_i^{k+1,m})_m$ converges uniformly to $\tilde{J}_i^{k+1}$ over S^{n-1} and, hence, $\tilde{J}_i^{k+1}$ is continuous over S^{n-1}.

If we let $\tilde{J}_i^0 = 0$ (which is continuous), then by induction, $\tilde{J}_i^k$ is continuous for all k. Hence, $\tilde{J}_i^*$ is continuous. ■

V. APPLICATION TO THE CONTROL OF SWITCHED HOMOGENEOUS SYSTEMS

In this section, we briefly apply the previous results to the control of CT switched homogeneous systems. In [3], it is shown that a state-dependent sampling time can be used to transform a CT homogeneous system into a degree-1 DT homogeneous system, and a feedback control law can be approximated using a quantization of the unit sphere. In specializing the results to continuous-time switched homogeneous systems, we are able to reduce the proofs of stability and performance to simple inductive proofs based on the value iteration and continuity.

After we establish the existence of a sampling time that will allow us to stabilize the CT system with a DT controller, we present a method for constructing a stabilizing DT controller by approximating the DT approximation of the CT value function over a quantization of the unit sphere. The algorithm itself is a specialization of the algorithm presented in [3] to switched systems, but we use the value iteration results developed earlier to provide simple recursive proofs of stability and optimality. The controller is ultimately computed using a linear program.

A. Discrete-Time (DT) Switched Systems: Definitions and Notations

Because a controller for (2) will require switching logic to compute the control signal $i(\tau)$, it is practical to consider implementing it in discrete time. As in [3], we apply a special state-dependent sampling period that will yield a degree-1 homogeneous DT system from (2). The new system is defined as follows: let t be an integer and let $i(t)$ be a DT input (i.e., $i = (i(0), i(1), i(2), \ldots)$). The DT system is

$$x(t+1) = y(T(x(t), i(t)), x(t), i(t))$$

It can be seen that the sampling period of the DT system at time t is given by $T(x(t), i(t))$. For ease, from here on we denote these dynamics as $f_{i(t)}(x(t))$.

Now, we define the *DT cost function* as

$$V(x_0, i) = \sum_{t=0}^{\infty} L\left(x(t), i(t), i(t+1)\right)$$

where $L(x, i, j) = T_0\|x\|^2 + \|x\|^2 K_{ij}$ and T_0 is the base sampling period. It can be seen that this DT cost function resembles a time-discretization of the CT cost function.

The actual form for T is adopted from [3], and we formally present it along with a special consequence of its application below.

Proposition 5: For a positive constant T_0, the state-dependent sampling period given by $T(x, i) = T_0\|x\|^{-d_i+1}$ yields a continuous, degree-1 homogeneous function f_i for all x and i.

We call T_0 the *base sampling period.*

In the remainder, τ is the CT time variable, t is the DT time variable, y is the CT state, x is the DT state, and i is the mode input in both settings.

B. Objectives

The remainder of this section will be focused on determining conditions under which (2) can be sampled quickly enough to yield a DT system that is asymptotically controllable. Ultimately, we apply concepts from dynamic programming to construct a DT optimal feedback policy.

Formally, we want to find a base sampling period T_0 and a DT feedback control law u so that

$$\begin{aligned} x(t+1) &= f_{i(t)}(x(t)) \\ i(t+1) &= u(x(t), i(t)) \end{aligned} \tag{13}$$

is stable.

C. Approximating the CT Value Function in DT

First, we begin with a simple convergence result.

Proposition 6: If $V_i^*(S^{n-1})$ is bounded, then $V_i^* = V_i^\infty$ and V_i^* is continuous.

Proof: By Corollary 1, $V_i^* = V_i^\infty$. We construct a new value iteration sequence to prove the continuity claim. Let $(\tilde{V}_i^k)_k$ be a value iteration sequence. If $\tilde{V}_i^k$ is continuous for all i, then it is easy to see that $\tilde{V}_i^{k+1}$ (as given by (5)) is continuous for all i. If we let $\tilde{V}_i^0$ be any continuous function, then induction holds. Therefore, by Corollary 2, V^* is continuous. ■

We present some background results for the main approximation result, the proofs of which may be found in the appendix.

Proposition 7: For any $\epsilon > 0$, there exists a positive $\overline{T}_0$ such that $|l(x,\tau,i,j) - L(x,i,j)| < \epsilon$ for all $x \in S^{n-1}$, for all $0 \le \tau \le \overline{T}_0$, and for all i, j.

Proposition 8: For any $\epsilon > 0$, there exists a positive $\overline{T}_0$ such that $|\tilde{J}_j^*(h_i(x,\tau_1)) - \tilde{J}_j^*(h_i(x,\tau_2))| < \epsilon$ for all $x \in S^{n-1}$, for all $0 \le \tau_1, \tau_2 \le \overline{T}_0$, and for all i, j.

We now state the main result of this section.

Theorem 2 (Approximation of the CT value function): For any $\epsilon > 0$, there exists a positive time $\overline{T}_0$ such that for all base sampling periods $T_0 \le \overline{T}_0$, $|J_i^* - V_i^*| < \epsilon$ over S^{n-1}.

Proof: Let $\lambda > 0$ be such that $\|f_i(x)\|^2 < \lambda L(x,i,j)$ for all i, j and $x \in S^{n-1}$, and if $\epsilon \ge \frac{1}{2\lambda}$, make it smaller, and choose $T_0 < \overline{T}_0$ for $\overline{T}_0$ given by Propositions 7 and 8 for the choice of ϵ. We now construct a value iteration sequence to prove the claim. Let $V_i^0 = 0$ and assume that for all $x \in S^{n-1}$ and i,

$$V_i^k(x) \le \frac{\tilde{J}_i^*(x) + 2\epsilon\|x\|^2}{1 - 2\lambda\epsilon}$$

An upper bound for V_i^{k+1} over S^{n-1} is

$$\begin{aligned}V_i^{k+1}(x) =& \min_j\{\|f_i(x)\|^2 V_j^k\left(\frac{f_i(x)}{\|f_i(x)\|}\right)+L(x,i,j)\}\\ \leq& \frac{1}{1-2\lambda\epsilon}\min_j\{\tilde{J}_j^*(f_i(x))+2\lambda\epsilon L(x,i,j)+(1-2\lambda\epsilon)L(x,i,j)\}\\ =& \frac{1}{1-2\lambda\epsilon}\min_j\{\tilde{J}_j^*(f_i(x))+L(x,i,j)\}\end{aligned}$$

Since

$$\tilde{J}_j^*(f_i(x)) = \tilde{J}_j^*(h_i(x,T_0)) < \tilde{J}_j^*(h_i(x,\tau))+\epsilon$$

$$L(x,i,j) < l(x,\tau,i,j)+\epsilon$$

for all $0<\tau<T_0$, we have

$$\begin{aligned}V_i^{k+1}(x) \leq& \frac{1}{1-2\lambda\epsilon}\min_{\{j,0<\tau\leq T_0\}}\{\tilde{J}_j^*(h_i(x,\tau))+2\epsilon+l(x,\tau,i,j)+\epsilon\}\\ =& \frac{\tilde{J}_i^*(x)+2\epsilon}{1-2\lambda\epsilon}\end{aligned}$$

A lower bound is similarly determined. Since $V_i^0=0$, induction holds, and by Proposition 2 and Corollary 1, value iteration converges. Because $\tilde{J}_i^*$ is upper and lower bounded over S^{n-1}, then, for sufficiently small ϵ, the approximation claim holds. ■

We now formally propose the existence of a stabilizing DT control law for the CT system.

Corollary 3 (Stability of the CT system via DT control): There exists a positive base sampling period T_0 such that the CT system (2) is asymptotically stable using the DT control law

$$u^*(x,i) \in \operatorname*{argmin}_j\{V_j^*(f_i(x))+L(x,i,j)\}$$

Proof: The proof that the boundedness of V_i^* over S^{n-1} implies that the DT system (13) is asymptotically stable follows from a standard Lyapunov argument that leverages the homogeneities of V_i^* and (13), and so we omit that proof.

We now prove that the asymptotic stability of (13) implies the asymptotic stability of (2). Assume that T_0 is chosen sufficiently small so that (13) is asymptotically stable.

For all $x_0 \in S^{n-1}$ and $i_0 \in Q$, there exists an K such that $\|z(\tau,x_0,i_0)\| < K$ for all $0 \leq \tau \leq T_0$. For a given x_0, let $x(t)$ be the optimal state trajectory $i(t)$ be the optimal mode sequence.

First, by homogeneity, $\|z(\tau,x(t),i(t))\| \leq \|x(t)\|K$ for all $tT_0 \leq \tau \leq (t+1)T_0$. Since $\|x(t)\| \to 0$ as $t\to\infty$, then $\|z(\tau,x_0,i)\| \to 0$ as $\tau\to\infty$. Therefore, (2) is convergent.

Finally, choose any ϵ and choose x_0 so that $\|x(t)\| < \epsilon$ for all $t \geq 0$. Then $\|z(\tau, x_0, i)\| < K\epsilon$ for all $\tau \geq 0$. Therefore, (2) is Lyapunov stable. ■

Remark 3: It is important to note that, in practice, the DT controller can only be semiglobally stabilizing since it is not possible to sample a CT system using arbitrarily short sampling periods as the state grows unbounded.

From here on, we assume that the sampling period is small enough to yield a finite V_i^*.

D. Approximating the DT Value Function over a Finite Set

While knowledge of V_i^* automatically yields an optimal control law for the system, computing the value function is generally intractable, and so we are left to approximate the function instead. To this end, we approximate V_i^* by reducing our state space to a finite set over the unit sphere and compute the approximation over the finite set.

For a scalar $\delta > 0$, let $\hat{S}_\delta^{n-1}$ be a finite subset of S^{n-1} such that for each $x \in S^{n-1}$, there is an approximating state $\hat{x}$ in $\hat{S}_\delta^{n-1}$ that is within a distance δ to x. Let the *quantization function* Θ_δ be given by

$$\Theta_\delta(x) \in \|x\| \operatorname*{argmin}_{\hat{x} \in \hat{S}_\delta^{n-1}} \{\|\hat{x} - \frac{x}{\|x\|}\|\}$$

From here on, we drop the δ subscript notation.

Value iteration over the set $\hat{S}^{n-1}$ is given by

$$\hat{V}_i^{k+1}(x) = \min_j \{\hat{V}_j^k(\Theta(f_i(x))) + L(x, i, j)\} \tag{14}$$

Let $\hat{V}_i^*$ be the corresponding value function for the quantized system.

Proposition 9: If $\hat{V}_i^\infty(\hat{S}^{n-1})$ is finite, $(\hat{V}_i^k)_k$ converges to $\hat{V}_i^*$ over $\hat{S}^{n-1}$ for $\hat{V}_i^0 = 0$.

Proof: Because $\Theta \circ f_i$ is degree-1 homogeneous and bounded over S^{n-1}, $\hat{V}_i^\infty(\hat{S}^{n-1})$ is finite, $\hat{V}_i^0 = 0$, and since the minimizer j_k^* of (14) exists for each k, by Proposition 2, $\hat{V}_i^*$ is bounded. Therefore, by Corollary 1, convergence holds in general. ■

Of interest in the use of a finite state space is that the value function can be computed using a simple linear program. The program for computing this approximate value function is given below.

$$\begin{aligned} &\max \sum_{i \in Q, x \in \hat{S}^{n-1}} \hat{V}_i^*(x) \text{ subject to} \\ &\hat{V}_i^*(x) \leq \|f_i(x)\|^2 \hat{V}_j^* \left(\Theta \left(\frac{f_i(x)}{\|f_i(x)\|} \right) \right) + L(x,i,j) \\ &\text{for all } i,j \in Q \text{ and } x \in \hat{S}^{n-1} \end{aligned} \tag{15}$$

The continuity results of come into play once we begin to consider the application of approximation states. If V_i^* is δ-ϵ uniformly continuous over S^{n-1}, then for all $x \in S^{n-1}$,

$$V_i^*(f_i(x)) = \|f_i(x)\|^2 V_i^* \left(\frac{f_i(x)}{\|f_i(x)\|} \right) \begin{smallmatrix} < \\ > \end{smallmatrix} \|f_i(x)\|^2 \left[V_i^* \left(\Theta \left(\frac{f_i(x)}{\|f_i(x)\|} \right) \right) \pm \epsilon \right]$$

This relationship allows us to approximate the value function. From here on, however, we will use value iteration to prove results concerning the approximation $\hat{V}_i^*$, and we begin with the following proposition concerning the approximation quality of $\hat{V}_i^*$ to V_i^*.

Proposition 10: For $\epsilon > 0$, there exists a δ such that $|\hat{V}_i^* - V_i^*| < \epsilon$ over $\hat{S}^{n-1}$.

Proof: Let $\lambda > 0$ be such that $\|f_i(x)\|^2 < \lambda L(x,i,j)$ for all i,j and $x \in S^{n-1}$. If $\epsilon \geq \frac{1}{\lambda}$, make it smaller. Let δ be such that V_i^* is δ-ϵ uniformly continuous over S^{n-1}. Now, for all x and i, assume a value iteration sequence satisfying

$$\hat{V}_i^k(x) \leq \frac{V_i^*(x)}{1 - \lambda\epsilon}$$

An upper bound for $\hat{V}_i^{k+1}(x)$ is

$$\begin{aligned} \hat{V}_i^{k+1}(x) \leq& \min_j \{ \|f_i(x)\|^2 \left[V_j^* \left(\Theta \left(\frac{f_i(x)}{\|f_i(x)\|} \right) \right) \frac{1}{1-\lambda\epsilon} \right] + L(x,i,j) \} \\ =& \frac{1}{1-\lambda\epsilon} \min_j \{ V_j^*(f_i(x)) + L(x,i,j) \} \\ =& \frac{1}{1-\lambda\epsilon} V_i^*(x) \end{aligned}$$

A lower bound is similarly determined. Letting $\hat{V}_i^0 = 0$, induction holds. Because V_i^* is upper bounded over S^{n-1}, $\hat{V}^\infty$ is equally bounded, and so by Proposition 9, $\hat{V}_i^k \to \hat{V}_i^*$. The approximation claim follows. ∎

E. Stability and Performance

Given $\hat{V}_i^*(x)$, define the DT control law u as

$$u(x,i) \in \operatorname*{argmin}_j \{ \hat{V}_j^*(\Theta(f_i(x))) + L(x,i,j) \}$$

Now, to prove stability of the closed-loop system, we show the system yields a finite cost

$$\begin{aligned}\tilde{V}_i(x) =& \sum_{t=0}^{\infty} L\left(x(t), i(t), u(x(t), i(t))\right)\\ =& \tilde{V}_{u(x,i)}(f_i(x)) + L(x, i, u(x, i))\end{aligned}$$

To derive conditions under which $\tilde{V}_i$ is bounded, we use value iteration. To this end, we define the sequence of functions $(\tilde{V}_i^k(x))_k$ by

$$\tilde{V}_i^{k+1}(x) = \tilde{V}_{u(x,i)}^k(f_i(x)) + L(x, i, u(x, i))$$

with $\tilde{V}_i^0 = 0$. Clearly, $\lim_{k\to\infty} \tilde{V}_i^k(x)$ exists and is equal to $\tilde{V}_i(x)$ if the sequence is bounded.

We now state the main stability result of this section. First, define the finite family of functions $\Gamma = \{V_i^*\}_i \bigcup \{V_i^* \circ f_j(\cdot)\}_{i,j}$. Clearly, by finiteness, Γ is equicontinuous over S^{n-1}.

Theorem 3: For $\epsilon > 0$, there exists a δ such that $|V_i^* - \tilde{V}_i| < \epsilon$ over S^{n-1} (and, hence, the closed-loop system has near-optimal performance).

Proof: Let λ be such that $\|f_i(x)\|^2 < \lambda L(x, i, j)$ for all $x \in S^{n-1}$ and all i, j. Let δ be such that Γ is δ-ϵ equicontinuous over S^{n-1} and such that $|\hat{V}_i^* - V_i^*| < \epsilon$ on $\hat{S}^{n-1}$.

Assume

$$\tilde{V}_i^k(x) \leq V_i^*(x) + 3\lambda\epsilon\tilde{V}_i^k(x) + \epsilon\|x\|^2$$

Then, using Proposition A.2 in the appendix,

$$\begin{aligned}\tilde{V}_i^{k+1}(x) \leq& \left[V_u^*(f_i(x)) + 3\lambda\epsilon\tilde{V}_u^k(f_i(x)) + \epsilon\|f_i(x)\|^2\right] + L(x, i, u)\\ \leq& \left[(2\|f_i(x)\|^2 + \|x\|^2)\epsilon - L(x, i, u) + V_i^*(x)\right] + 3\lambda\epsilon\tilde{V}_u^k(f_i(x)) + \epsilon\|f_i(x)\|^2\\ &+ L(x, i, u)\\ \leq& V_i^*(x) + \|x\|^2\epsilon + \left[3\lambda\epsilon\tilde{V}_u^k(f_i(x)) + 3\lambda\epsilon L(x, i, u)\right]\\ =& V_i^*(x) + 3\lambda\epsilon\tilde{V}_i^{k+1}(x) + \epsilon\|x\|^2\end{aligned}$$

Therefore, for all k

$$\tilde{V}_i^k(x) \leq \frac{V_i^*(x) + \epsilon\|x\|^2}{1 - 3\lambda\epsilon}$$

Since $\tilde{V}_i^0(x) = 0$, induction follows. Setting $k \to \infty$ gives an upper approximation bound for $\tilde{V}_i$. By optimality, however, $\tilde{V}_i \geq V_i^*$. Therefore, by the boundedness of V_i^* over S^{n-1}, there exists a δ such that $|\tilde{V}_i - V_i^*| < \epsilon$. Asymptotic stability further follows from homogeneity. ■

F. Lipschitz Special Case

Of course, in general, we do not know the δ-ϵ relationship for V_i^*, and so the results above only assert the existence of a level of approximation that provide these benefits. If we strengthen our assumptions about L and f_i, though, we compute an upper bound for δ to offer a prescribed ϵ.

Proposition 11: If $\{L(\cdot,\cdot,i,j)\}_{i,j}$ and $\{f_i\}_i$ are Lipschitz functions over S^{n-1} with respective Lipschitz constants ζ and $\eta < 1$, then V_i^* is Lipschitz over S^{n-1} with a Lipschitz constant $\frac{\zeta}{1-\eta}$.

Proof: Assume $|V_i^k(x+\delta) - V^k(x)| < \beta_k\|\delta\|$ for some β_k and all $x, (x+\delta) \in S^{n-1}$. Then, for the case $V_i^k(x+\delta) \geq V^k(x)$, we get

$$\begin{aligned}
|V_i^{k+1}(x+\delta) - V_i^{k+1}(x)| &= \min_j \{V_j^k(f_i(x+\delta,w)) + L(x+\delta,i,j)\} - V_i^{k+1}(x)\\
&\leq \min_j\{|V_j^k(f_i(x+\delta,w)) - V_j^k(f_i(x))| + V_j^k(f_i(x))\\
&\quad + |L(x+\delta,i,j) - L(x,i,j)| + L(x,i,j)\} - V_i^{k+1}(x)\\
&\leq \min_j\{\beta_k\|f_i(x+\delta,w) - f_i(x)\| + V_j^k(f_i(x)) + \zeta\|\delta\| + L(x,i,j)\}\\
&\quad - V_i^{k+1}(x)\\
&\leq \min_j \{\beta_k\eta\|\delta\| + V_j^k(f_i(x)) + \zeta\|\delta\| + L(x,i,j)\} - V_i^{k+1}(x)\\
&= V_i^{k+1}(x) + (\beta_k\eta + \zeta)\|\delta\| - V_i^{k+1}(x)\\
&= (\beta_k\eta + \zeta)\|\delta\|
\end{aligned}$$

The other case provides the same result. Therefore, $\beta_{k+1} \leq \beta_k\eta + \zeta$. Starting with $\beta_0 = 0$ since $V_i^0(x) = 0$, we get for general k

$$\beta_k \leq \zeta \sum_{l=0}^{k} \eta^l$$

which converges for $\eta < 1$ as $k \to \infty$. ■

It is noteworthy that the constraint on η translates into the requirement each f_i is a contraction.

VI. SIMULATIONS

The example comes from a slight modification of the example switched system from [10]. The dual-mode switched system is given by

$$g_1(y) = \begin{bmatrix} 0.1y_1^3 - y_2^3 \\ 10y_1^3 + 0.1y_2^3 \end{bmatrix}, \; g_2(y) = \begin{bmatrix} 0.1y_1 - 10y_2 \\ y_1 + 0.1y_2 \end{bmatrix}$$

where y_1 and y_2 are the components of the vector y. Both g_1 and g_2 are unstable systems that "spiral" away from the origin. Note that g_1 is degree-3 homogeneous while g_2 is degree-1 homogeneous.

To construct a DT stabilizing control, we use a base sampling time of 0.1s and a quantization spacing of 0.1 radians along a semi-sphere[5], yielding 32 approximation states. Finally, we use the incremental cost function $L(x, i, j) = \|x\|^2 + \|x\|^2 K_{ij}$ where $K_{ij} = 1$ for $i \neq j$.

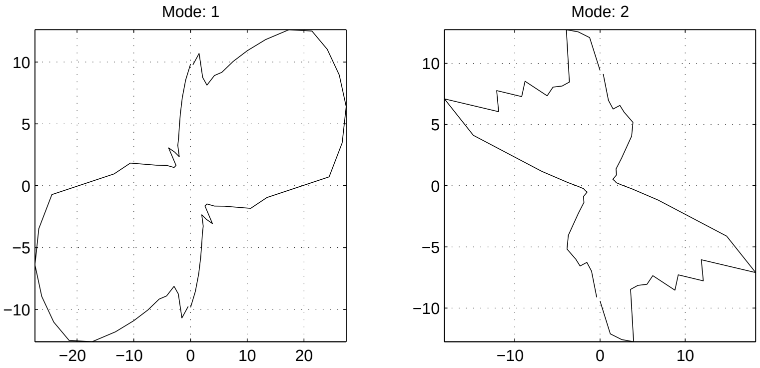

Fig. 1. Plot of the linear interpolations of $\hat{V}_1^*$ and $\hat{V}_2^*$ over S^{n-1}. For $x \in S^{n-1}$, the value of $\hat{V}_i^*(x)$ is plotted as its distance from the origin with the same angle as x.

Figure 1 shows the plots of the the linear interpolation of $\hat{V}_1^*$ and $\hat{V}_2^*$ over S^{n-1} for each optimization criterion. The magnitude of $\hat{V}_i^*(x)$ is represented as its distance from the origin.

The optimal control laws u are plotted in Figure 2 as a function of angle because they are independent of the magnitude of y.

Finally, Figure 3 shows a plot of CT closed-loop trajectories resulting from an initial state $y_0 = (1, 0)$ and $i_0 = 1$. Notice that although the system is executed for 30 time samples, the

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[5] By homogeneity, we need not consider the entire sphere.

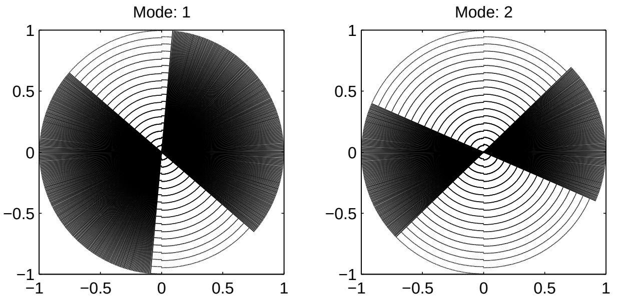

Fig. 2. Plot of u for each initial mode. As u is a degree-0 homogeneous function, it is just a function of the angle in the phase space of x. Striped regions represent where u takes a value of 1, and solid regions represent where u takes a value of 2.

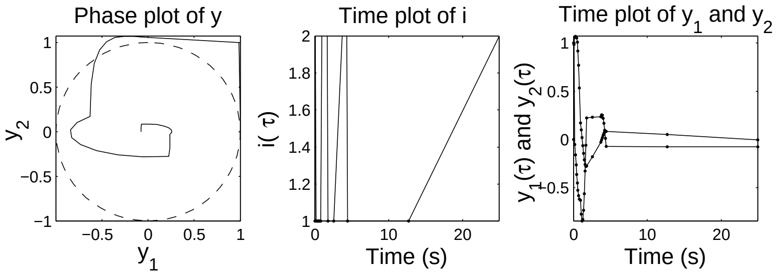

Fig. 3. Simulation of the closed-loop systems from the initial pair $y_0 = (1, 0)$ and $i_0 = 1$.

simulations take nearly 30 seconds to complete, despite a 0.1 base sampling period. This is because as the trajectory approaches the origin, g_1 reacts far more slowly, a consequence of using $T(x, 1) = \|x\|^{-2}T_0$, which approaches ∞ as $\|x\| \to 0$.

VII. CONCLUSIONS AND FUTURE WORK

In this paper, we presented conditions under which value iteration converges for discrete-time homogeneous and continuous-time switched homogeneous systems as well as conditions under which the value functions are continuous. Homogeneity was leveraged to show that the uniform convergence of value iteration results from the fact that such systems have value functions satisfying a boundedness condition presented in [2]. For continuous-time systems, a transformation of the system to a discrete-time homogeneous system was presented, and it was shown that the application of homogeneous switching costs guarantees the continuity of the value function. We applied these results and techniques to deriving simple proofs regarding the control of CT switched homogeneous systems.

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BOUNDEDNESS OF THE CT VALUE FUNCTION

Proposition A.1: $J_i^*(S^{n-1})$ is bounded.

Proof: Choose any $\epsilon < 1$. For each $z_0 \in S^{n-1}$, there exists a control law $i(z_0)$ and time $T(z_0)$ such that $\|z(\tau, z_0, i(z_0))\| < \epsilon$ for all $\tau \geq T(z_0)$. By continuity, there exists a distance $\delta(z_0)$ such that $\|z(T(z_0), \hat{z}_0, i(z_0))\| < \epsilon$ for all initial states $\|\hat{z}_0 - z_0\| < \delta(z_0)$. Choose M points $Z = \{z^1, z^2, \ldots, z^M\}$ on S^{n-1} such that the $\delta(z^k)$-neighborhoods about these points cover S^{n-1}. Let the function $\Gamma(z_0)$ map z_0 to its closest point in $\|z_0\|Z$ (basically, scale the quantization set Z).

Letting $K(i,t) = \mathrm{argmax}_k \{\tau_k \leq t\}$, define the truncated cost at time t as

$$\tilde{J}^t(z_0, i) = \int_0^t \|z(\tau, z_0, i)\|^2 d\tau + \sum_{k=0}^{K(i,t)} \|z_k\|^2 K_{i_{k-1} i_k}$$

which is continuous over z_0 so long as the trajectory does not suffer finite-escape time. Define the quantized control law $\hat{i}(z_0) = i(\Gamma(z_0))$ and quantized time $\hat{T}(z_0) = T(\Gamma(z_0))$. We will bound the cost of control using

$$\tilde{J}_{\max} = \max_{z_0 \in S^{n-1}} \tilde{J}^{\hat{T}(z_0)}(z_0, \hat{i}(z_0))$$

By homogeneity, $\|z(\hat{T}(z_0), z_0, \hat{i}(z_0))\| < \|z_0\|\epsilon$. Now since

$$\tilde{J}^{T(\Gamma(z_0))}(z_0, \hat{i}(z_0)) \leq \|z_0\|^2 \tilde{J}_{\max}$$

we construct a stabilizing quasi-feedback control law as follows: execute $\hat{i}(z_0)$ until $\|z\| < \epsilon$, then execute $\hat{i}(z)$ until $\|z\| < \epsilon^2$, and so on. The cost of this non-optimal control law is bounded (by a geometric series). ■

APPROXIMATION RESULTS

Proof: [Proposition 7] Let $L(x, T_0, i, j)$ be the DT incremental cost L with the additional explicit notation of the base sampling period parameter T_0. We have $L(x, 0, i, j) = \|x\|^2 K_{ij} = l(x, 0, i, j)$. Take a decreasing sequence of sampling periods $(T_k)_k$ that converges to 0. $(L(\cdot, T_k, i, j))_k$ is a monotonically decreasing sequence of continuous functions that is pointwise convergent to the continuous function $l(x, 0, i, j)$ over S^{n-1}. Hence, by uniform convergence, there is a positive sampling period $\overline{T}_L$ such that $|L(x, \overline{T}_L, i, j) - l(x, 0, i, j)| < \epsilon$ for $x \in S^{n-1}$.

Now, let

$$z_{max}(T) = \underset{\{i_0 \in Q, x_0 \in S^{n-1}, \tau \in [0,T]\}}{\text{argmax}} \|z(\tau, x_0, i_0)\|^2$$

It is clear that $(z_{max}(T_k))_k$ monotonically decreases to 1. By uniform convergence, there exists a positive sampling period $\overline{T}_z$ such that for all $T_0 < \overline{T}_z$, $|z_{max}(T_0) - 1| < \epsilon$.

Now, let

$$\hat{l}(x, \tau, i, j) = l(x, \tau, i, j) - \|z(\tau, x, i)\|^2 K_{ij}$$

Basically, we are removing the switching cost. It is clear that $(\hat{l}(x, T_k, i, j))_k$ monotonically decreases to $\hat{l}(x, 0, i, j)$. By uniform convergence, there exists a positive sampling period $\overline{T}_l$ such that for all $T_0 < \overline{T}_l$, $|\hat{l}(x, T_0, i, j) - \hat{l}(x, 0, i, j)| < \epsilon$ for $x \in S^{n-1}$. Therefore, for all $T_0 < \min\{\overline{T}_z, \overline{T}_l\}$, $|l(x, 0, i, j) - l(x, T_0, i, j)| < \epsilon(1 + K_{max})$ for $x \in S^{n-1}$ and where $K_{max} = \max_{i,j} K_{ij}$.

The claim of the proposition follows. ■

Proof: [Proposition 8] Let I be any compact set $[0, T]$ for some positive T. $\tilde{J}_j^* \circ h_i$ is uniformly continuous over $S^{n-1} \times I$, and therefore there exists a $\overline{T}_0$ such that for any $|\tau_1 - \tau_2| < \overline{T}_0$ and $\tau_1, \tau_2 \in I$, $|\tilde{J}_j^* \circ h_i(x, \tau_1) - \tilde{J}_j^* \circ h_i(x, \tau_2)| < \epsilon$ for $x \in S^{n-1}$. ■

Proposition A.2: For a given $\epsilon > 0$, there exists a δ such that

$$V_u^*(f_i(x)) \leq -L(x, i, u) + (2\|f_i(x)\|^2 + \|x\|^2)\epsilon + V_i^*(x)$$

Proof: Choose δ such that $|\hat{V}_i^* - V_i^*| < \epsilon$ on $\hat{S}^{n-1}$. By the equicontinuity of the set Γ, for $x \in S^{n-1}$

$$\begin{aligned} V_u^*(f_i(x)) =& \|f_i(x)\|^2 V_u^*\left(\frac{f_i(x)}{\|f_i(x)\|}\right) \\ \leq& \|f_i(x)\|^2 V_u^*\left(\Theta\left(\frac{f_i(x)}{\|f_i(x)\|}\right)\right) + \|f_i(x)\|^2\epsilon \end{aligned}$$

$$
\begin{aligned}
&= \left[\|f_i(x)\|^2 \hat{V}_u^* \left(\Theta\left(\frac{f_i(x)}{\|f_i(x)\|}\right)\right) + L(x,i,u)\right] \\
&\quad - L(x,i,u) + 2\|f_i(x)\|^2\epsilon - V_i^*(x) + V_i^*(x) \\
&\leq \hat{V}_i^*(x) - L(x,i,u) + 2\|f_i(x)\|^2\epsilon - \left[\hat{V}_i^*(x) - \epsilon\right] + V_i^*(x) \\
&= (2\|f_i(x)\|^2 + 1)\epsilon - L(x,i,u) + V_i^*(x)
\end{aligned}
$$

■

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