Noise-Free Multiple Access Networks over Finite Fields

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Abstract

We consider a time-slotted multiple access noise-free channel where, the received and transmit alphabets belong to a finite field and the transmitters interfere additively over that field. We derive the capacity region and the maximum *code rate* where, *code rate* is defined as the ratio of the information symbols recovered at the receiver to the symbols sent by the transmitters in a slot duration. We show that capacity grows logarithmically with the size of the field but the code rate is invariant with field size. We also show that codes achieve maximum code rate if and only if they achieve capacity and add no redundancy to the shorter of the two information codewords. The cases when both transmitters always transmit in a slot, as well as when each transmitter transmits according to a Bernoulli process, are considered. For the case when both transmitters always transmit, we propose a systematic code construction and show that it achieves the maximum code rate and capacity. We also show that a systematic random code achieves the maximum code rate and capacity with probability tending to 1 exponentially with codeword length and field size. For the case when transmitters transmit according to a Bernoulli process, we propose a coding scheme to maximize the expected code rate. We show that maximum code rate is achieved by adding redundancy only at the less bursty transmitter.

I. Introduction

Research on noise-free multiple access networks has generally assumed that the received alphabet size grows with the number of transmitters. The binary multiple access channel, where, interference is additive and the received alphabet grows with the number of transmitters, has been looked at in [1], [2] and [3]. In [4], transmission of information is considered for a modulo-2 multiple access channel. However, it considers only the case where a proper subset of the users transmit and the work is confined to $\mathbb{F}_2$ and not to finite field channels in general. We consider a time-slotted multiple access noise-free channel where two transmitters are transmitting to a single receiver without coordination in fixed time slots. All transmissions begin at the beginning of the slot. The multiple access interference is additive, but the transmit and receive alphabet sizes are the same. We call this a noise-free finite field adder channel over $\mathbb{F}_{2^k}$, where the transmitted and received elements belong to $\mathbb{F}_{2^k}$, for $1 \leq k$. We use two metrics: capacity and *code rate*. Code rate is defined as the ratio of the information symbols recovered at the receiver, to the symbols sent by the transmitters in a slot duration and represents the overhead required for reliable communication.

II. Single Slot Model

We consider a discrete time-slotted channel. Information codewords $\vec{a}$ and $\vec{b}$ are vectors containing elements from $\mathbb{F}_{2^k}$ of sizes n_a and n_b respectively, which represent independent information. From [5], we know that source-channel separation holds for this channel. Thus, the scheme of separate source and channel coding is optimal. For this model, all operations, matrices and vectors are in $\mathbb{F}_{2^k}$. We will refer to $\vec{a}$ and $\vec{b}$ as transmit vectors and assume without loss of generality that $n_a \geq n_b$. Let L_a and L_b be the generator matrices for the channel codes that act on $\vec{a}$ and $\vec{b}$, respectively. $\vec{X_a}$ and $\vec{X_b}$ are the codewords, with elements from $\mathbb{F}_{2^k}$, that are sent over the channel and the additive interference is over $\mathbb{F}_{2^k}$. At the decoder, matrix T acts upon the received vector $\vec{Y}$ to generate a subset of $\vec{a}$ and $\vec{b}$. $\vec{R}$ is the decoded output containing $n_a^{'}$ elements of $\vec{a}$ and $n_b^{'}$ elements of $\vec{b}$. Let m_a and m_b be the increase in the size of vectors $\vec{a}$ and $\vec{b}$, respectively, due to channel coding. We will denote l_a and l_b as the lengths of the vectors obtained by channel coding on $\vec{a}$ and $\vec{b}$, respectively. In general, $l_a \neq l_b$ and so both transmitters may not be transmitting for the entire slot duration. We have $l_a = n_a + m_a$ and $l_b = n_b + m_b$.

III. Capacity Region and Maximum Code Rate

The multiple access capacity region is the closure of the convex hull of all rate pairs (R_a, R_b) satisfying

$$R_a \leq k, \quad R_b \leq k, \quad R_{sum} = R_a + R_b \leq k. \tag{1}$$

The expression for the code rate is:

$$C_{rate} \leq \frac{n_a + n_b}{n_a + 2n_b}. \tag{2}$$

We will call codes that achieve capacity and maximum code rate optimal codes. We obtain the following theorems:

Theorem 1: The capacity region of a two-transmitter noise-free finite field adder channel grows logarithmically with the size of the field but the code rate remains the same for all field sizes.

Theorem 2: For a noise-free finite field adder channel, codes achieve the maximum code rate if and only if they are capacity achieving and no redundancy is added to the smaller transmit vector.

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IV. Code Construction

We describe a code construction that is shown to be optimal. By definition, $l_a \geq n_a$ and $l_b \geq n_b$. When $l_a = l_b = n_a + n_b$, all elements can be recovered after multiple access interference at the receiver. Thereby, we can safely reduce the region for finding optimal codes to the rectangle $n_a \leq l_a \leq n_a + n_b$ and $n_b \leq l_b \leq n_a + n_b$. All points outside this rectangle will have a lower code rate as the number of received elements remains same for increasing l_a and l_b. Hence, we confine our analysis for finding optimal codes to this rectangle. In this region we have :

$$\vec{X_a} = L_a\vec{a}, \quad \vec{X_b} = L_b\vec{b}, \quad \vec{Y} = L_a\vec{a} + L_b\vec{b}, \quad \vec{R} = T\vec{Y} = (TL_a)\vec{a} + (TL_b)\vec{b}. \tag{3}$$

Let $W_a = (TL_a)$ and $W_b = (TL_b)$. We define a 1row as a row vector having only one non-zero element. In order to recover $n_a^{'}$ elements of $\vec{a}$ and $n_b^{'}$ elements of $\vec{b}$, W_a should be a $(n_a^{'} + n_b^{'}) \times n_a$ size matrix with $n_a^{'}$ 1rows, W_b a $(n_a^{'} + n_b^{'}) \times n_b$ size matrix with $n_b^{'}$ 1rows and the 1row positions for these matrices should not overlap. Let $W = \left[\begin{array}{c|c} W_a & W_b \end{array} \right]$ and $L = \left[\begin{array}{c|c} L_a & L_b \end{array} \right]$. Thus, W should have $n_a^{'} + n_b^{'}$ unique 1rows. For given L, we need to find the maximum number of 1rows in W that can be generated by linear combinations of the rows of L. This maximizes $n_a^{'} + n_b^{'}$ for given $l_a + l_b$ which in turn maximizes the code rate and also specifies L_a, L_b and T. We have the following theorems:

Theorem 3: Optimal codes are not contained in the region $0 < m_b \leq n_a - n_b$ and on the line $m_a = m_b - [n_a - n_b]$.

Theorem 4: To achieve the maximum code rate, it suffices to add redundancy to only one vector.

Optimal codes exist over the following regions:

Region A where $n_a \leq l_a \leq n_a + n_b$ and $l_b = n_b$. Here $n_a^{'} = m_a + [n_a - n_b]$, $n_b^{'} = m_a$, and $C_{rate-R_A} = \frac{2m_a + [n_a - n_b]}{n_a + n_b + m_a}$.

Region B where $l_a = n_a$ and $n_a + 1 \leq l_b \leq n_a + n_b$. Here $n_a^{'} = m_b$, $n_b^{'} = m_b - [n_a - n_b]$ and $C_{rate-R_B} = \frac{2m_b - [n_a - n_b]}{n_a + n_b + m_b}$.

From Theorem 4, we see that it suffices to add redundancy at only one transmitter. Let the redundancy be m. In *Region A*, $C_{rate-R_A} = \frac{2m + [n_a - n_b]}{n_a + n_b + m}$ and in *Region B*, $C_{rate-R_B} = \frac{2m - [n_a - n_b]}{n_a + n_b + m}$. The highest code rate for the code is therefore

$$C_{rate} = \frac{n_a + n_b}{n_a + 2n_b}. \tag{4}$$

The transmission rates of the code are

$$R_a = \frac{kn_a}{n_a + n_b}, \quad R_b = \frac{kn_b}{n_a + n_b}, \quad R_{sum} = R_a + R_b = k. \tag{5}$$

We see from (4,5) that this code achieves the maximum code rate and capacity for this channel and is thus an optimal code. Moreover, this code obeys the property of any maximum code rate achieving code, i.e. no redundancy is added to the smaller transmit vector. We have the following theorem, which shows the asymptotic optimality of systematic random coding:

Theorem 5: For a two transmitter noise-free multiple access finite field adder channel over $\mathbb{F}_{2^k}$, as the codeword lengths or field size tends to infinity, a random systematic code becomes optimal with probability tending to 1 exponentially with codeword length and field size.

V. Multiple access when transmitters become bursty

We now look at the case when each transmitter transmits in a slot according to a Bernoulli process. Let the probability that transmitters *a* and *b* have a codeword to transmit in a slot be p_a and p_b respectively and the sizes of $\vec{a}$ and $\vec{b}$ be n_a and n_b respectively. We look at the average, or *expected code rate*. Let us denote $\alpha = \frac{p_a}{p_b}$ and $\beta = \frac{n_a}{n_b}$. We have $\alpha \in [0, \infty)$ and $\beta \geq 1$. The maximum expected code rate can be written as

$$E(CodeRate(\alpha, \beta)) = \frac{1 + \alpha\beta}{1 + \alpha\beta + \min(1, \alpha\beta)}. \tag{6}$$

For $\alpha \in [0, \frac{1}{\beta}]$, the mean size of $\vec{a}$ is less than or equal to the size of $\vec{b}$ and we add redundancy only at *b*. For $\alpha \in [\frac{1}{\beta}, \infty)$ the mean size of $\vec{a}$ is larger than or equal to the mean size of $\vec{b}$, and we add redundancy only at *a*. Note that, for $\alpha = \frac{1}{\beta}$, we can add redundancy at *a* or *b*. When $\beta = 1$, i.e. $n_a = n_b$, if $p_b \leq p_a$, we add redundancy only at *a* and when $p_b > p_a$ we add redundancy only at *b*. This gives rise to the following theorem:

Theorem 6: When the information codewords at the input to the channel encoders have the same size, maximum expected code rate is achieved by adding redundancy at the less bursty transmitter not adding any redundancy at the more bursty transmitter.

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