

# Some Generic Invariant Factor Assignment Results Using Dynamic Output Feedback

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## ABSTRACT

Working with input-output transfer functions in the frequency domain and exploiting a formulation involving generalized Sylvester resultants, we are able to derive necessary and sufficient conditions for generic invariant factor assignment, in several cases, using proper dynamic output feedback compensators.

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## I. INTRODUCTION

A very well-known fact in system theory is that feedback can be used in order to improve system performance. In the theory of finite dimensional linear time invariant systems one frequently deals with the question of how to improve the performance of systems described by the following differential equation:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (1.1)$$

where  $x(t)$  is an  $n$ -vector,  $u(t)$  an  $l$ -vector, and  $A$  ( $n \times n$ ),  $B$  ( $n \times l$ ) matrices

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over the reals  $R$ . If constant state feedback is used,

$$u(t) = v(t) - Kx(t),$$

where  $K$  is an  $l \times n$  real matrix,  $v(t)$  a reference input, then the closed-loop system is described by

$$\dot{x}(t) = (A - BK)x(t) + Bv(t). \quad (1.2)$$

A central result in the area of pole assignment [21] is that  $(A, B)$  is controllable iff for every symmetric set  $\Lambda$  of  $n$  complex numbers, there is a matrix  $K$  such that  $A - BK$  has  $\Lambda$  for its set of eigenvalues. This implies that, under the assumption of controllability, arbitrary pole assignment can be accomplished by constant state feedback.

Rosenbrock, in a subsequent publication [19], showed that more than pole assignment can be accomplished for the system in (1.1). This result can be stated in the following manner: Let  $(A, B)$  be a controllable pair with controllability indices  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l \geq 0$ . Let  $\phi_i$ ,  $1 \leq i \leq l$ , be given monic polynomials satisfying the divisibility conditions  $\phi_i \mid \phi_{i-1}$ , and with  $\sum_{i=1}^l \theta(\phi_i) = n$  [ $\theta(\cdot)$  denotes degree]. Then there exists a constant matrix  $K$  such that the given polynomials are the nonunity invariant factors of  $sI - A + BK$  if and only if

$$\sum_{i=1}^k \theta(\phi_i) \geq \sum_{i=1}^k \lambda_i, \quad k = 1, 2, \dots, l, \quad \text{with equality at } k = l. \quad (1.3)$$

This result implies that we can arbitrarily assign not only the eigenvalues of  $A - BK$ , but the size and entries of the cyclic blocks appearing on the main diagonal of the rational canonical form of  $A - BK$  [17]. It is important to note that since  $(A, B)$  is controllable, so is  $(A - BK, B)$  for any  $K$  (i.e.,  $sI - A + BK$  and  $B$  are left coprime). From a frequency domain input-output point of view this means that if

$$Q(s) = (sI - A)^{-1}B$$

is the input-output transfer function of the system in (1.1), where the output is actually the state, and state feedback (1.2) is used, the closed loop transfer function becomes

$$H(s) = (sI - A + BK)^{-1}B.$$

Making the invariant factors of  $sI - A + BK$  equal to a given set  $\{\phi_i\}$  is equivalent to saying [12] that  $M_H(s)$  the Smith-McMillan form of  $H(s)$  is given by

$$M_H(s) = \text{diag} \left( \frac{\varepsilon_i(s)}{\phi_i(s)} \right)$$

(with appropriate modification if  $n \neq l$ ).

In many practical applications, physical constraints frequently necessitate the use of output rather than state feedback,

$$u(t) = v(t) + Ky(t), \quad y(t) = Cx(t), \quad (1.4)$$

where  $y(t)$  is an  $m$ -vector and  $C$  an  $m \times n$  constant real matrix. In many situations static output feedback is insufficient and dynamic output feedback is introduced:

$$\begin{aligned} \dot{z}(t) &= Fz(t) + Gy(t), \\ u(t) &= Hz(t) + Ky(t), \end{aligned} \quad (1.5)$$

where  $z(t)$  is a  $q$ -vector and  $F, G, H, K$  appropriate matrices with real entries.

In light of the above, Rosenbrock and Hayton [20] attempted to generalize Rosenbrock's earlier result to the output feedback case. They proceeded by using the frequency domain input-output point of view, considering the strictly proper system

$$P(s) = C(sI - A)^{-1}B = D_{LP}^{-1}N_{LP}$$

and the proper compensator  $C(s) = A_{LC}^{-1}B_{LC}$  (no longer static) as given by the system matrices (Rosenbrock sense)

$$P_P(s) = \begin{bmatrix} I & 0 & 0 \\ 0 & D_{LP} & N_{LP} \\ 0 & -I & 0 \end{bmatrix}, \quad P_C(s) = \begin{bmatrix} I & 0 & 0 \\ 0 & A_{LC} & B_{LC} \\ 0 & -I & 0 \end{bmatrix}.$$

The input-output transfer functions  $P(s)$  ( $m \times l$ ) and  $C(s)$  ( $l \times m$ ) have elements in  $R(s)$ , the field of rational functions in  $s$  over  $R$ . The matrices  $N_{LP}, D_{LP}, A_{LC}, B_{LC}$  have elements in  $R[s]$ , the polynomials in  $s$ . If the two

systems are connected as in Figure 1 (output feedback configuration), a composite system for the resulting closed-loop system is obtained [20] and then brought by strict system equivalence to the form

$$P_C(s) = \begin{bmatrix} I & 0 & 0 \\ 0 & A_{LC}D_{RP} + B_{LC}N_{RP} & A_{LC} \\ 0 & -N_{RP} & 0 \end{bmatrix},$$

where  $D_{LP}^{-1}N_{LP} = N_{RP}D_{RP}^{-1}$  are left and right matrix fraction descriptions of the system, and  $A_{LC}^{-1}B_{LC}$  a left matrix fraction description of the compensator [5, 12]. The closed-loop transfer function is

$$G(s) = N_{RP}(A_{LC}D_{RP} + B_{LC}N_{RP})^{-1}A_{LC}.$$

The basic result in [20] is the following: Let  $P(s) = N_{RP}D_{RP}^{-1}$  be an  $m \times l$  strictly proper transfer function of order  $n$ , with  $\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix}$  column reduced with column degree  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l \geq 0$  ( $\lambda_i$  are controllability indices), and  $\mu_1$  the largest observability index. Let  $\phi_i$ ,  $1 \leq i \leq l$ , be given polynomials satisfying the divisibility conditions  $\phi_i \mid \phi_{i-1}$  and with  $\sum_{i=1}^l \theta(\phi_i) = n + l(\mu_1 - 1)$ . Then a sufficient condition for the existence of a proper  $l \times m$  compensator  $C(s) = A_{LC}^{-1}B_{LC}$  such that the invariant factors of  $A_{LC}D_{RP} + B_{LC}N_{RP}$  are the

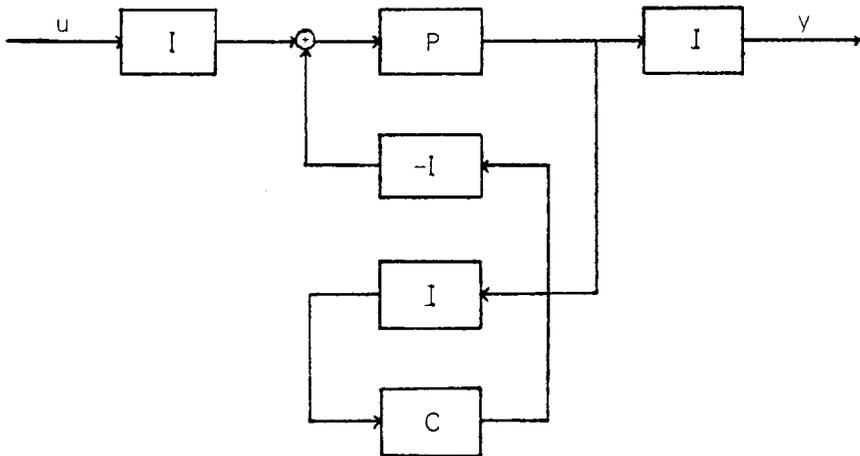


FIG. 1.

$\phi_i$  is

$$\sum_{i=1}^k \theta(\phi_i) \geq \sum_{i=1}^k \lambda_i + \mu_1 - 1, \quad k = 1, 2, \dots, l, \quad \text{with equality at } k = l. \quad (1.6)$$

One should immediately notice several basic differences of this result as compared with the earlier state feedback result.

(1) In addition to the controllability indices, other indices (namely observability) become important.

(2) This is only a sufficient condition.

(3) Dynamic feedback has been introduced.

(4) As no coprimeness conditions have been imposed, assigning the invariant factors of  $A_{LC}D_{RP} + B_{LC}N_{RP}$  does not imply that the invariant factors of some  $sI - A^*$  have been assigned, where  $A^*$  comes from a minimal realization, but rather that some realization (not minimal) can be found such that for the matrix  $\bar{A}$  corresponding to it,  $sI - \bar{A}$  has the given invariant factors.

(5) The order of the compensator used is  $l(\mu_1 - 1)$ . There exists the possibility that a better result can be stated employing a lower order compensator.

Several attempts have been made to "improve" the output feedback invariant factor result [6, 7, 9, 10, 15]. A partial list of some other recent publications on the general problem of pole assignment can be found in the references.

In this paper, working in the frequency domain with input-output transfer functions and using a formulation employing matrix fraction descriptions and generalized Sylvester resultants, we are able to give short new proofs of Rosenbrock's state and output feedback results. Furthermore we demonstrate that such a structure easily lends itself to a "generic" formulation of the invariant factor problem. This allows us to derive necessary and sufficient conditions for generic invariant factor assignment in several cases, and prove some other interesting results as well.

## 2. FORMULATION

Throughout the paper we assume the feedback configuration of Figure 2, where  $P(s)$  is the  $m \times l$  input-output transfer function of the given strictly

proper system, and  $C(s)$  the  $l \times m$  transfer function of a proper compensator which is to be computed. Both  $P(s)$  and  $C(s)$  have elements in  $R(s)$ . Without loss of generality we assume that  $m \geq l$ . In the case that  $l > m$  a "dual" formulation and results can be obtained. The closed loop transfer function is given by

$$G(s) = P(s)[I + C(s)P(s)]^{-1},$$

where we assume that  $[I + C(s)P(s)]^{-1}$  exists. Since  $P(s)$  and  $C(s)$  are rational matrices, they can be "factored" into polynomial matrices [5, 12]. We use the notation

$$\begin{aligned} P(s) &= B_{RP}A_{RP}^{-1}, & \text{a right matrix fraction description (MFD) of } P(s), \\ &= A_{LP}^{-1}B_{LP}, & \text{a left MFD of } P(s), \\ &= N_{RP}D_{RP}^{-1}, & \text{a right coprime (or irreducible) MFD of } P(s), \\ &= D_{LP}^{-1}N_{LP}, & \text{a left coprime (or irreducible) MFD of } P(s), \end{aligned}$$

where  $B_{RP}$ ,  $A_{RP}$ ,  $N_{RP}$ ,  $D_{RP}$ , etc. are polynomial matrices, and where the indeterminate  $s$  has been suppressed for simplicity. The closed loop transfer function can then be expressed in the following ways:

$$\begin{aligned} G(s) &= P(I + CP)^{-1} \\ &= B_{RP}(A_{LC}A_{RP} + B_{LC}B_{RP})^{-1}A_{LC} \end{aligned} \quad (2.1)$$

$$= N_{RP}(A_{LC}D_{RP} + B_{LC}N_{RP})^{-1}A_{LC} \quad (2.2)$$

$$= N_{RP} \underbrace{(D_{LC}D_{RP} + N_{LC}N_{RP})^{-1}}_{\Phi} D_{LC} \quad (2.3)$$

$$\begin{aligned} &= N_{RP}\Phi^{-1}D_{LC} \\ &= \tilde{N}_{RP}\tilde{\Phi}^{-1}\tilde{D}_{LC}, \quad \text{least order (or irreducible),} \end{aligned} \quad (2.4)$$

where  $\tilde{N}_{RP}$ ,  $\tilde{\Phi}$  are right coprime,  $\tilde{\Phi}$ ,  $\tilde{D}_{LC}$  left coprime. The description (2.4) is not unique, since if  $\tilde{N} = \tilde{N}_{RP}E$ ,  $\tilde{D} = H\tilde{D}_{LC}$ ,  $\tilde{\Phi} = H\tilde{\Phi}E$ ,  $E, H$  unimodular, then

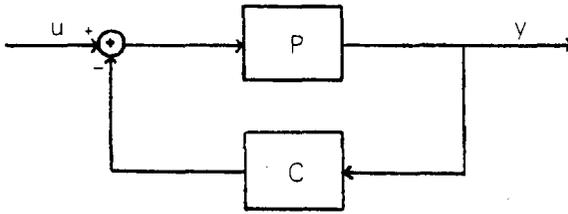


FIG. 2.

$\bar{N}\bar{\Phi}^{-1}\bar{D}$  is also a least order, irreducible polynomial matrix description (PMD) [12]. Clearly, since no coprimeness conditions have been imposed, the descriptions (2.1), (2.2), (2.3) are *not* least order.

If  $M_G(s)$  is the Smith-McMillan form of  $G(s)$ ,

$$M_G(s) = \begin{bmatrix} \frac{\epsilon_1}{\phi_1} & & & 0 \\ & \ddots & & \\ 0 & & \frac{\epsilon_l}{\phi_l} & \\ & & & 0 \end{bmatrix},$$

where  $\phi_i, 1 \leq i \leq l$ , are monic and satisfy the divisibility conditions  $\phi_1 \mid \phi_{i-1}, 2 \leq i \leq l$ , then using ideas of system equivalence one can show [7, 12] that

$$\tilde{\Phi} = E \begin{bmatrix} \phi_1 & & & 0 \\ & \ddots & & \\ 0 & & & \phi_l \end{bmatrix} H,$$

where  $E, H$  are unimodular matrices. Therefore we call the  $\{\phi_i\}$  the *invariant factors* (or polynomials) of the closed loop system. It is also clear that the  $\{\phi_i\}$  are the nonunity invariant factors of  $sI - A$ , where  $A$  comes from some minimal realization of  $G(s)$ .

If on the other hand we use a description of  $G(s)$  which is *not* least order—say  $G(s) = B\Psi^{-1}A$ , with the Smith form of  $\Psi$  being

$$\begin{bmatrix} \psi_l & & & 0 \\ & \ddots & & \\ 0 & & & \psi_1 \end{bmatrix},$$

where  $\psi_i \mid \psi_{i-1}$ ,  $2 \leq i \leq l$ , we shall have [7]  $\phi_i \mid \psi_i$ ,  $1 \leq i \leq l$ . The  $\{\psi_i\}$  will be the nonunity invariant factors of some  $sI - \bar{A}$ , where  $\bar{A}$  comes from a nonminimal realization of  $G(s)$ . The output feedback results of Rosenbrock and Hayton [20] deal with this problem.

The difference between the two approaches stems from the fact that one can work with either external or internal descriptions of systems [12]. From the results of Section 6 it is evident that in the "generic" case the difference disappears. It should also be mentioned that if no coprimeness conditions need to be satisfied as in Rosenbrock and Hayton [20], the proofs of these results are much easier to construct.

It is clear from the above that a very natural way to proceed with the invariant factor assignment problem is the following: Given a strictly proper system  $P(s) = N_{RP}D_{RP}^{-1}$ , find conditions for the existence of a polynomial solution  $X, Y$  to the polynomial equation

$$XD_{RP} + YN_{RP} = \Phi, \quad (2.5)$$

where:

- (1)  $\Phi$  is equivalent to  $\text{diag}(\phi_i)$ ,  $\phi_i$  a given set of desired closed loop invariant factors,
- (2)  $X^{-1}Y$  exists and is proper,
- (3)  $N_{RP}, \Phi$  are right coprime and  $X, \Phi$  left coprime.

If one considers invariant factor assignment as Rosenbrock and Hayton [20] do, then the condition (3) is dropped.

The way in which Rosenbrock and Hayton [20] proceed is to use the fact [18] that all polynomial solutions to (2.5) can be expressed as

$$\begin{aligned} X &= \Phi U - NN_{LP}, \\ Y &= \Phi V + ND_{LP}, \end{aligned}$$

where  $UD_{RP} + VN_{RP} = I$  (which is guaranteed, since  $D_{RP}, N_{RP}$  are right coprime [7, 12]) and  $N$  is an arbitrary polynomial matrix. By appropriately choosing  $N$  they show that  $X^{-1}Y$  exists and is proper [if the given  $\phi_i$  satisfy (1.6)].

We proceed in the following manner: Let  $N_{RP}, D_{RP}, X, Y$  be given as

$$\begin{aligned} D_{RP} &= D_t s^t + D_{t-1} s^{t-1} + \cdots + D_0, \\ N_{RP} &= N_t s^t + N_{t-1} s^{t-1} + \cdots + N_0, \\ X &= X_{k-1} s^{k-1} + X_{k-2} s^{k-2} + \cdots + X_0, \\ Y &= Y_{k-1} s^{k-1} + Y_{k-2} s^{k-2} + \cdots + Y_0, \end{aligned}$$

and let  $XD_{RP} + YN_{RP} = \Phi$ ,

$$\Phi = \Phi_{k+t-1}s^{k+t-1} + \dots + \Phi_0.$$

Then equating coefficients

$$[X_{k-1}, Y_{k-1}, \dots, X_0, Y_0]S_k(D_{RP}, N_{RP}) = [\Phi_{k+t-1}, \dots, \Phi_0],$$

where

$$S_k(D_{RP}, N_{RP}) = \begin{bmatrix} D_t & D_{t-1} & \dots & D_0 & 0 & \dots & 0 \\ N_t & N_{t-1} & \dots & N_0 & 0 & \dots & 0 \\ 0 & D_t & \dots & D_1 & D_0 & \dots & 0 \\ 0 & N_t & \dots & N_1 & N_0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & D_t & \dots & D_0 \\ 0 & 0 & \dots & 0 & N_t & \dots & N_0 \end{bmatrix}. \quad (2.6)$$

The real matrix  $S_k(D_{RP}, N_{RP})$  is the  $k$ th order Sylvester resultant of  $D_{RP}, N_{RP}$  [16] [a  $k(m+l) \times l(t+k)$  matrix]. Clearly if a  $\Phi$  exists which is equivalent to  $\text{diag}(\phi_i)$  in the range space of  $S_k(D_{RP}, N_{RP})$ , where  $X^{-1}Y$  exists and is proper with  $N_{RP}, \Phi$  right coprime,  $X, \Phi$  left coprime, then we have a sufficient condition for invariant factor assignment. It would therefore be very helpful if we knew the rank of the resultant operator. The following result, taken from [1, 16], is crucial for our investigation.

LEMMA 2.1. *Let  $ND^{-1}$  be proper,  $m \times l$ , with observability indices  $\mu_i$ . Then*

$$\text{rank } S_k(D, N) = (l+m)k - \sum_{i: \mu_i < k} (k - \mu_i).$$

It would also be very helpful to know under what conditions  $\text{diag}(\phi_i(s))$  is equivalent to some element in the range space of  $S_k(D_{RP}, N_{RP})$ . The following lemma, taken from [20], will be used for this purpose.

LEMMA 2.2. *Let  $\alpha_i, \beta_i$  be given integers satisfying  $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_l \geq 0$ ,  $\beta_1 \geq \beta_2 \geq \dots \geq \beta_l \geq 0$ . Let  $\{\phi_i\}$ ,  $1 \leq i \leq l$ , satisfy the divisibility conditions  $\phi_i \mid \phi_{i-1}$ ,  $2 \leq i \leq l$ . Then a necessary and sufficient condition for the existence of an  $l \times l$  polynomial matrix  $\Phi(s)$  equivalent to  $\text{diag}(\phi_i)$  and satisfying*

$\lim_{s \rightarrow \infty} [\text{diag}(s^{-\alpha_i})\Phi(s)\text{diag}(s^{-\beta_i})] = I$  is

$$\sum_{i=1}^k \theta(\phi_i) \geq \sum_{i=1}^k \alpha_i + \beta_i, \quad k = 1, 2, \dots, l, \quad \text{with equality at } k = l.$$

Throughout this paper we shall use the following definition of genericity:

**DEFINITION.** A set  $S \subseteq R^t$  is called *generic* if it contains a nonempty Zariski open set of  $R^t$  [24].

### 3. ROSENBRACK'S STATE FEEDBACK RESULT

In this section we shall use the formulation introduced in the previous section to give a new short proof of Rosenbrock's state feedback result [19].

**THEOREM 3.1.** Let  $P(s) = (sI - A)^{-1}B$  (where  $(A, B)$  is a controllable pair) with controllability indices  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l \geq 0$ . Let  $\phi_i, 1 \leq i \leq l$ , be given monic polynomials satisfying the divisibility conditions  $\phi_i \mid \phi_{i-1}, 2 \leq i \leq l$ , and with  $\sum_{i=1}^l \theta(\phi_i) = n$ .

Then there exists a constant  $C$  such that the given polynomials are the nonunity invariant factors of  $sI - A + BC$  if and only if

$$\sum_{i=1}^k \theta(\phi_i) \geq \sum_{i=1}^k \lambda_i, \quad k = 1, 2, \dots, l, \quad \text{with equality at } k = l. \quad (3.1)$$

*Proof. Sufficiency:*  $P(s)$  is  $n \times l$ , strictly proper, with observability indices all equal to  $\mu = 1$ . Let  $P(s) = N_{RP}D_{RP}^{-1}$ , where

$$\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix} = \begin{bmatrix} D_{hc} \\ 0 \end{bmatrix} \text{diag}(s^{\lambda_i}) + L(s),$$

in which  $D_{hc}$  is invertible and  $L(s)$  contains lower order terms [12]. This implies that

$$D_{RP} = D_{\lambda_1} s^{\lambda_1} + D_{\lambda_1-1} s^{\lambda_1-1} + \dots + D_0,$$

$$N_{RP} = N_{\lambda_1-1} s^{\lambda_1-1} + \dots + N_0,$$

$$S_1(D_{RP}, N_{RP}) = \begin{bmatrix} D_{\lambda_1} & D_{\lambda_1-1} & \dots & D_0 \\ 0 & N_{\lambda_1-1} & \dots & N_0 \end{bmatrix}.$$

From Lemma 2.1, rank  $S_1 = n + l$ . Now the number of nonzero columns of  $S_1$  is  $(\lambda_1 + 1) + (\lambda_2 + 1) + \dots + (\lambda_l + 1) = n + l$ . Let  $C = X^{-1}Y$ , where

$$X = D_{hc}^{-1},$$

$$Y = Y_0 \quad (\text{constant}).$$

Then by appropriately choosing  $Y_0$ , any polynomial  $\Phi(s)$  can be reached which satisfies  $\lim_{s \rightarrow \infty} [\Phi(s) \text{diag}(s^{-\lambda_i})] = I$ . But if the  $\{\phi_i\}$  satisfy (3.1), then from Lemma 2.2 with  $\alpha_i = \mu_i$ ,  $\beta_i = 0$  a polynomial  $\Phi^*(s)$  equivalent to  $\text{diag}(\phi_i(s))$  can be constructed which satisfies  $\lim_{s \rightarrow \infty} [\Phi^*(s) \text{diag}(s^{-\lambda_i})] = I$ . Let  $Y_0^*$  be the  $Y_0$  that corresponds to it. Then  $C = D_{hc} Y_0^*$  is the desired compensator, since  $G(s) = (sI - A + BC)^{-1}B = N_{RP} \Phi^*(s)^{-1}$ , both being irreducible representations [12, Lemma 6.5-9, p. 446].

*Necessity:* Suppose that  $C$  is a constant that makes the closed loop invariant factor equal to  $\{\phi_i\}$ . Let  $G(s) = N_{RP}(D_{RP} + CN_{RP})^{-1}$ . Clearly  $N_{RP}$ ,  $\Psi = D_{RP} + CN_{RP}$  are right coprime. This means that the invariant factors of  $\Psi$  are the  $\{\phi_i\}$ . Now

$$\lim_{s \rightarrow \infty} [\Psi \text{diag}(s^{-\lambda_i})] = \lim_{s \rightarrow \infty} \left( [I \quad C] \begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix} \text{diag}(s^{-\lambda_i}) \right)$$

$$= D_{hc},$$

where  $\det D_{hc} \neq 0$ . Clearly  $\Psi_1 = D_{hc}^{-1}\Psi$  satisfies  $\lim_{s \rightarrow \infty} [\Psi_1 \text{diag}(s^{-\lambda_i})] = I$  and has invariant factors  $\{\phi_i\}$ . From Lemma 2.2 the degrees of the  $\{\phi_i\}$  must satisfy (3.1). ■

#### 4. ROSENBRCK AND HAYTON'S OUTPUT FEEDBACK RESULT

Ideas developed in Section 2 can be used to give a new short proof of Rosenbrock and Hayton's output feedback result [20].

**THEOREM 4.1.** *Let  $P(s) = N_{RP} D_{RP}^{-1}$  be an  $m \times l$  strictly proper transfer function of order  $n$ , with  $\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix}$  column reduced, with column indices  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l \geq 0$  ( $\lambda_i$  controllability indices), and  $\mu_1$  the largest observability index. Let  $\phi_i$ ,  $1 \leq i \leq l$ , be given polynomials satisfying the divisibility conditions  $\phi_i \mid \phi_{i-1}$  and with  $\sum_{i=1}^l \theta(\phi_i) = n + l(\mu_1 - 1)$ . Then a sufficient condition for the existence of a proper  $l \times m$  compensator  $C(s) = A_{LC}^{-1} B_{LC}$*

such that the invariant factors of  $A_{LC}D_{RP} + B_{LC}N_{RP}$  are the  $\langle \phi_i \rangle$  is

$$\sum_{i=1}^k \theta(\phi_i) \geq \sum_{i=1}^k (\lambda_i + \mu_1 - 1), \quad k = 1, 2, \dots, l, \quad \text{with equality at } k = l. \tag{4.1}$$

*Proof.* We have

$$\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix} = \begin{bmatrix} D_{hc} \\ 0 \end{bmatrix} \text{diag}(s^{\lambda_i}) + L(s),$$

$D_{hc}$  invertible, with  $L(s)$  containing lower order terms. This implies that

$$\begin{aligned} D_{RP} &= D_{\lambda_1} s^{\lambda_1} + D_{\lambda_1-1} s^{\lambda_1-1} + \dots + D_0, \\ N_{RP} &= N_{\lambda_1-1} s^{\lambda_1-1} + \dots + N_0, \end{aligned}$$

and  $S_{\mu_1}(D_{RP}, N_{RP})$  is the  $\mu_1$ -order Sylvester resultant of  $D_{RP}, N_{RP}$  (2.6).

From Lemma 2.1,  $S_{\mu_1}(D_{RP}, N_{RP})$  has rank  $n + l\mu_1$ . Now the number of nonzero columns of  $S_{\mu_1}(D_{RP}, N_{RP})$  is  $(\lambda_1 + 1) + (\lambda_2 + 1) + \dots + (\lambda_l + 1) + (\mu_1 - 1)l = n + l\mu_1$ . Let  $C = X^{-1}Y$ , where

$$\begin{aligned} X &= X_{\mu_1-1} s^{\mu_1-1} + \dots + X_0 \\ Y &= Y_{\mu_1-1} s^{\mu_1-1} + \dots + Y_0. \end{aligned}$$

This means

$$\begin{bmatrix} X_{\mu_1-1}, Y_{\mu_1-1}, \dots, X_0, Y_0 \end{bmatrix} S_{\mu_1}(D_{RP}, N_{RP}) = \begin{bmatrix} \Phi_{\lambda_1 + \mu_1 - 1}, \dots, \Phi_0 \end{bmatrix}.$$

Since  $S_{\mu_1}(D_{RP}, N_{RP})$  has rank  $n + l\mu_1$ , any polynomial  $\Phi(s)$  which satisfies

$$\lim_{s \rightarrow \infty} \left[ \text{diag}(s^{-(\mu_1-1)}) \Phi(s) \text{diag}(s^{-\lambda_i}) \right] = H,$$

$H$  constant, can be reached—in particular, any polynomial  $\Phi(s)$  for which

the corresponding  $H$  is the identity. Since

$$[X, Y] \begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix} = X_{\mu_1-1} D_{hc} \text{diag}(s^{\lambda_i + \mu_1 - 1}) + \bar{L}(s)$$

[ $\bar{L}(s)$  contains lower order terms], the  $X$  corresponding to such a  $\Phi(s)$  must have  $X_{\mu_1-1} = D_{hc}^{-1}$ , i.e.,  $C = X^{-1}Y$  must exist and be proper [12].

But if the  $\{\phi_i\}$  satisfy (4.1), then from Lemma 2.2 with  $\alpha_i = \lambda_i$ ,  $\beta_i = \mu_1 - 1$ , a polynomial  $\Phi^*(s)$  equivalent to  $\text{diag}(\phi_i(s))$  can be constructed which satisfies

$$\lim_{s \rightarrow \infty} [\text{diag}(s^{-(\mu_1-1)}) \Phi^*(s) \text{diag}(s^{-\lambda_i})] = I.$$

Then  $C = X^{-1}Y$ , for any  $X, Y$  which map to  $\Phi^*(s)$ , is a compensator which satisfies the requirements of the theorem. ■

### 5. THE EQUATION $XD + YN = \Phi$

It has been mentioned in Section 2 that this polynomial matrix equation plays a key role in the analysis of the invariant factor problem. Using Sylvester resultants, we are able to prove the following three results.

**PROPOSITION 5.1.** *Let  $P = N_{RP} D_{RP}^{-1}$  be an  $m \times l$  strictly proper transfer function where  $\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix}$  is column reduced, with column degrees  $\lambda_i = \lambda$  and observability indices  $\mu_i = \mu$ , and where*

$$D_{RP} = Is^\lambda + D_{\lambda-1} s^{\lambda-1} + \dots + D_0, \quad N_{RP} = N_{\lambda-1} s^{\lambda-1} + \dots + N_0.$$

Let

$$Z = \left\{ \Phi \in R^{(\lambda+q)l^2} \mid \Phi = Is^{\lambda+q} + \Phi_{\lambda+q-1} s^{\lambda-1} + \dots + \Phi_0 \right\},$$

$$Q = \left\{ (X, Y) \mid X = Is^q + X_{q-1} s^{q-1} + \dots + X_0, Y = Y_q s^q + \dots + Y_0 \right\}$$

*A necessary and sufficient condition for the existence of a polynomial solution  $X, Y$  to  $XD_{RP} + YN_{RP} = \Phi$  in the class  $Q$ , for every  $\Phi$  in  $Z$ , is  $q \geq \mu - 1$ .*

Equation  $XD_{RP} + YN_{RP} = \Phi$  with the conditions imposed can be written as

$$[I, Y_q, \dots, X_0, Y_0] S_{q+1}(D_{RP}, N_{RP}) = [I, \Phi_{\lambda+q-1}, \dots, \Phi_0],$$

where  $S_{q+1}(D_{RP}, N_{RP})$  is thought of as an operator from  $R^{l(l+m)(q+1)}$  into  $R^{l(\lambda+q+1)l}$ . From Lemma 2.1,

$$\text{rank } S_k = \begin{cases} (l+m)k, & 1 \leq k \leq \mu, \\ (l+m)k - m(k-\mu), & \mu < k. \end{cases}$$

Therefore

- (a)  $S_1, S_2, \dots, S_{\mu-1}$  are not onto,
- (b)  $S_\mu$  is onto and one-one,
- (c)  $S_{\mu+1}, S_{\mu+2}, \dots$  are onto.

*Proof. Necessity:* Assume that  $q < \mu - 1$  and that  $XD_{RP} + YN_{RP} = \Phi$  has a solution in  $Q$  for every  $\Phi$  in  $Z$ . Thinking of  $(X, Y)$  as an element in  $R^{l(l+m)q+m}$  and  $\Phi$  as an element in  $R^{l(\lambda+q)l}$ , one can see that the  $\Phi$  which can be reached from elements in  $Q$  form a set of dimension less than  $l(q + \lambda)l$ . Therefore  $q \geq \mu - 1$ .

*Sufficiency:*  $S_{\mu+k}$  is onto for  $k \geq 0$ . This means that for any  $\Phi$  in  $Z$  there exist  $X, Y$  given by

$$\begin{aligned} X &= X_{\mu+k-1}S^{\mu+k-1} + \dots + X_0, \\ Y &= Y_{\mu+k-1}S^{\mu+k-1} + \dots + Y_0 \end{aligned}$$

such that

$$[X_{\mu+k-1}, Y_{\mu+k-1}, \dots, X_0, Y_0] S_{\mu+k} = [I, \Phi_{\lambda+\mu+k-2}, \dots, \Phi_0]$$

For this to happen,  $X_{\mu+k-1} = I$ , which implies that  $(X, Y) \in Q$ . ■

This result addresses the following question: Suppose we fix the order of the proper compensator (order is  $ql$ ), as well as the observability indices (all equal to  $q$ ). What are the possible  $\Phi$  (closed loop denominator matrices) that can be reached? Since  $\lambda_i = \lambda$  and  $\mu_i = \mu$ , the result concerns the “generic” case. In the next section these concepts will be used for obtaining necessary and sufficient conditions for generic invariant factor assignment.

A polynomial solution  $X, Y$  to  $XD + YN = \Phi$  is called an acceptable solution if  $X^{-1}Y$  exists (i.e.,  $\det X \neq 0$ ) and is proper.

**PROPOSITION 5.2.** *Let  $N$  ( $m \times l$ ),  $D$  ( $l \times l$ ),  $\Phi$  ( $l \times l$ ) be polynomial matrices,  $\lambda, q$  nonnegative integers,  $n = \lambda l$ ,  $\mu = n/m$ , and let  $W, Z, S$  be the sets*

$$W = \{ (N, D) \in R^{(m+l)n} \mid D = Is^\lambda + D_{\lambda-1}s^{\lambda-1} + \dots + D_0, \quad N = N_{\lambda-1}s^{\lambda-1} + \dots + N_0 \},$$

$$Z = \{ \Phi \in R^{(\lambda+q)l^2} \mid \Phi = Is^{\lambda+q} + \Phi_{\lambda+q-1}s^{\lambda+q-1} + \dots + \Phi_0 \},$$

$$S = \{ (N, D) \in R^{(m+l)n} \mid \text{there exists an acceptable solution } X, Y \text{ to } XD + YN = \Phi \text{ for every } \Phi \text{ in } Z \}.$$

A necessary and sufficient condition for  $S$  to be a generic subset of  $R^{(m+l)n}$  is  $q \geq \mu - 1$ .

*Proof. Necessity:* Suppose that  $S$  is generic and that  $q < \mu - 1$ . Show a contradiction.

The set  $F \subseteq R^{(m+l)n}$  for which  $\text{rank } S_i(D, N) = (m+l)i$  for  $1 \leq i \leq \mu$  and  $\text{rank } S_{\mu+1}(D, N) = (m+l)\mu + l$  is generic. It certainly contains a Zariski open set, and it is nonempty, since any  $(N, D)$  which gives rise to an  $ND^{-1}$  with equal observability indices must belong to it (Lemma 2.1). As a matter of fact any  $(N, D)$  in  $F$  must have equal observability indices, because otherwise one of the rank conditions would be violated (Lemma 2.1). It follows (Bitmead et al. [1, Corollary 1]) that for any  $(N, D)$  in  $F$ ,  $N, D$  are right coprime, and  $ND^{-1}$  has equal controllability indices as well.

Let  $N, D$  be an element in  $S \cap F$ . For any such element [20, Corollary 2, p. 848] it must be that  $\theta(Y) \leq q$  for some acceptable solution  $(X, Y)$ . But this implies that any acceptable solution must be of the form

$$X = Is^q + X_{q-1}s^{q-1} + \dots + X_0, \quad Y = Y_qs^q + \dots + Y_0,$$

which means  $(X, Y) \in Q$ , with  $Q$  as in Proposition 5.1, and  $q \geq \mu - 1$ . This is a contradiction.

*Sufficiency:* Let  $q \geq \mu - 1$ . From Proposition 5.1 we have that  $S_{q+1}(D, N)$  is onto for any  $(N, D)$  in  $F$ . This means that there exist  $X, Y$  such that

$$X = X_qs^q + \dots + X_0, \quad Y = Y_qs^q + \dots + Y_0,$$

and

$$[X_q, Y_q, \dots, X_0, Y_0] S_{q+1}(D, N) = [I, \Phi_{\lambda+q-1}, \dots, \Phi_0]$$

for every  $\Phi$  in  $Z$ . It follows that  $X_q = I$  and that  $(X, Y)$  is an acceptable solution. ■

This result is stronger than Proposition 5.1. The restriction of searching for a solution of  $XD + YN = \Phi$  that belongs to a certain class  $Q$  can be removed. This is because an acceptable solution  $(X, Y)$  to  $XD + YN = \Phi$  with  $N, D, \Phi$  as above must belong to the class  $Q$ : there are no acceptable solutions outside  $Q$ .

The previous two results dealt with the case when  $P = N_{RP} D_{RP}^{-1}$  is a strictly proper transfer function with equal controllability indices. Knowledge of the rank of the resultant operators  $S_i(D_{RP}, N_{RP})$  played a key role in their proof. Proposition 5.3 addresses the question of the generic rank of  $S_i(D, N)$  when  $P = ND^{-1}$  is an  $l \times l$  strictly proper transfer function with an arbitrary set of controllability indices  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l > 0$ .

**PROPOSITION 5.3.** *Let  $P = ND^{-1}$  be our  $l \times l$  strictly proper transfer function with controllability indices  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l > 0$ ,  $n = \lambda_1 + \dots + \lambda_l$ ,  $\mu = n/l$ , and*

$$\begin{bmatrix} D \\ N \end{bmatrix} = \begin{bmatrix} D_{nc} \\ 0 \end{bmatrix} \text{diag}(s^{\lambda_i}) + L(s).$$

*$L(s)$  contains lower order terms. The pair  $(N, D)$  can be thought of as an element of  $R^v$ . The set  $W \subset R^{2nl}$  of  $(N, D)$  for which  $S_i(D, N)$  has rank  $2li$  for  $1 \leq i \leq \mu$  is a generic subset of  $R^v$ , ( $v = 2nl + l^2$ ).*

*Proof.* It is clear that the set of  $(N, D)$  for which  $S_i(N, D)$  has rank  $2il$  for  $1 \leq i \leq \mu$  is a Zariski open set. The difficulty lies in proving that it is indeed nonempty.

We first construct the submatrix  $T_\mu(N)$  of  $S_\mu(D, N)$  and show that it is generically full rank:

$$\begin{aligned} T_\mu(N) &= \begin{bmatrix} N_{\lambda_1-\mu} & \cdots & N_1 & N_0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & \ddots & \vdots & \vdots \\ N_{\lambda_1-2} & N_{\lambda_1-3} & N_{\lambda_1-4} & \cdots & N_1 & N_0 & 0 \\ N_{\lambda_1-1} & N_{\lambda_1-2} & N_{\lambda_1-3} & N_{\lambda_1-4} & \cdots & N_1 & N_0 \end{bmatrix} \\ &= [B_{\lambda_1-1}, \dots, B_0], \end{aligned}$$

where  $B_i$  is the  $i$ th  $\mu l \times l$  block column of  $T_\mu(N)$ . Form  $T_\mu^*(N)$  from  $T_\mu(N)$  by deleting the columns indicated: From the  $l$ th columns of the  $B_j$ 's keep the  $l$ th columns of  $B_0, \dots, B_{\lambda_i-1}$  and remove the rest. From the  $(l-1)$ st columns of the  $B_j$ 's keep the  $(l-1)$ th columns of  $B_0, \dots, B_{\lambda_{l-1}-1}$  and remove the rest. And in general, from the  $(l-i)$ th columns of the  $B_j$ 's keep the  $(l-i)$ th columns of  $B_0, \dots, B_{\lambda_{l-i}-1}$  and remove the rest. What remains is the  $n \times n$  matrix  $T_\mu^*(N)$ .

Clearly the set  $V = \{N \in R^{nl} | T_\mu^*(N) \text{ has rank } n\}$  is a Zariski open set in  $R^{nl}$ . We claim that it is nonempty. This can be seen from the fact that an  $N^*$  exists that makes  $T_\mu^*(N^*)$ , after a proper rearrangement of its columns, equal to a lower triangular matrix with 1's on the main diagonal.

It is now easy to see that  $S_\mu(I \text{diag}(s^{\lambda_i}), N^*)$  has rank  $2\mu l$  [and that rank  $S_i(I \text{diag}(s^{\lambda_i}), N^*) = 2il$  for  $1 \leq i \leq \mu$ ].

Expressing it in a different way, Proposition 5.3 guarantees that "almost all"  $l \times l$  strictly proper transfer functions of McMillan degree  $n = \lambda_1 + \lambda_2 + \dots + \lambda_l$ ,  $\mu = n/l$ , with arbitrary controllability indices  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l$  have equal observability indices  $\mu$ .

## 6. INVARIANT FACTOR RESULTS

In Section 2 we made a distinction between what we call the invariant factors of the closed loop system and what has been used by Rosenbrock and Hayton [20]. The difference arises because we insist on working with irreducible polynomial matrix descriptions [PMDs (12)], whereas Rosenbrock and Hayton [20] choose not to. It can be said that since the difference is due to possible cancellations, in some "generic" sense the two definitions are the same. In this section we shall exploit the ideas presented thus far in order to obtain necessary and sufficient conditions for invariant factor assignment in several cases.

**THEOREM 6.1.** *Let  $P = ND^{-1}$  be an  $l \times l$  strictly proper transfer function where  $\lambda > 0$  and*

$$D = Is^\lambda + D_{\lambda-1}s^{\lambda-1} + \dots + D_0,$$

$$N = N_{\lambda-1}s^{\lambda-1} + \dots + N_0.$$

Let

$$q \geq 0 \quad \text{and} \quad \phi(s) = s^{\lambda+q} + \phi_{\lambda+q-1}s^{\lambda+q-1} + \dots + \phi_0.$$

Let

$W = \{ (N, D, \phi) \in R^{2\lambda l^2 + \lambda + q} \mid \text{there exists a proper compensator of order } lq \text{ making } \phi_1 = \phi_2 = \dots = \phi_l = \phi \text{ the invariant factors of the closed loop system} \}$ .

Then  $q \geq \lambda - 1$  is a necessary and sufficient condition for  $W$  to be a generic subset of  $R^{2\lambda l^2 + \lambda + q}$ .

*Proof. Sufficiency:* Let  $q \geq \lambda - 1$  and  $t = 2\lambda l^2 + \lambda + q$ . Show that  $W$  contains a nonempty Zariski open set. Let  $\Phi$  be the diagonal matrix

$$\Phi = \text{diag}(\phi_i), \quad \phi_i = \phi. \quad (6.1)$$

Since  $q \geq \lambda - 1$  (we are dealing with the square case  $l = m$ ,  $\mu = l\lambda/m = \lambda$ ), the set  $F \subseteq R^t$  of  $(N, D, \phi)$  for which  $S_{q+1}(D, N)$  is full rank is generic. Let  $\bar{S}_{q+1}$  be the matrix  $([(q+1)m + lq] \times (\lambda + q)l)$  obtained from  $S_{q+1}$  by removing the first  $l$  rows and  $l$  columns. Clearly  $\bar{S}_{q+1}$  is generically full rank. This implies that for any  $\Phi$  as in (6.1) there exists  $X = Is^q + \dots + X_0$ ,  $Y = Y_q s^q + \dots + Y_0$  such that

$$[Y_q, \dots, X_0, Y_0] \bar{S}_{q+1} = [\Phi_{\lambda+q-1}, \dots, \Phi_0] - [D_{\lambda-1}, \dots, D_0, 0, \dots, 0].$$

One such  $X, Y$  ( $C = X^{-1}Y$  proper of order  $lq$ ) is given by

$$\begin{aligned} [Y_q, X_{q-1}, \dots, X_0, Y_0] &= ([\Phi_{\lambda+q-1}, \dots, \Phi_0] - [D_{\lambda-1}, \dots, 0]) \\ &\quad \times (\bar{S}_{q+1}^T \bar{S}_{q+1})^{-1} S_{q+1}^T \end{aligned} \quad (6.2)$$

$\bar{S}_{q+1}^T \bar{S}_{q+1}$  is invertible, since  $\bar{S}_{q+1}$  is full rank and with fewer columns than rows.

This means that  $X_i, Y_i$  are rational expressions in the parameters of  $N, D, \phi$ . Now the set  $E \subseteq R^t$  for which  $\Phi, X$  are left coprime and  $N, \phi$  right coprime is a Zariski open set, since coprimeness is a condition satisfied when certain matrices are full rank [1, Corollary 1].

The key point to demonstrate is that  $E$  is nonempty. We claim that  $\alpha = (N_\alpha, D_\alpha, \phi_\alpha)$ , where

$$D_\alpha = Is^\lambda, \quad N_\alpha = I, \quad \Phi_\alpha = Is^{\lambda+q} + I,$$

is a point in  $E$ . Clearly  $N_\alpha, \Phi_\alpha$  are right coprime, since  $\begin{bmatrix} \Phi_\alpha \\ N_\alpha \end{bmatrix}$  is full rank for all  $s$  [19]. Now for this specific  $\Phi_\alpha$  the solution given in (6.2) can easily be computed to be

$$X_\alpha = Is^q, \quad Y_\alpha = I.$$

Clearly  $X_\alpha, \Phi_\alpha$  are left coprime.

Now  $G(s) = N(XD + YN)^{-1}X$  for any  $(N, D, \phi)$  in  $E$ , with  $X, Y$  given by (6.2), is a least order (irreducible) polynomial matrix fraction description of  $G(s)$ , and

$$XD + YN = \Phi.$$

Therefore the invariant factors of  $G(s)$  are the  $\phi_i$ . Clearly  $E \subseteq W$ , and thus  $W$  is a generic subset of  $R^l$ .

*Necessity:* Let  $W \subseteq R^l$  be the set of  $(N, D, \phi)$  for which there exists a proper compensator  $C = X^{-1}Y$  which makes  $\phi_1 = \phi_2 = \dots = \phi_l = \phi$  the closed loop invariant factors. Assume that it is generic.

For any  $\alpha \in W$  the following must be true. Let  $C = X^{-1}Y$  be the proper compensator of order  $lq$  that accomplishes the task. We choose  $X$  to be row reduced, and let  $q_1 \geq q_2 \geq \dots \geq q_l \geq 0$  be the row degrees  $q_1 + q_2 + \dots + q_l = lq$ , and  $X_{hc} = I$  the highest row degree coefficient matrix of  $X$ .

The following three statements must be true for such an  $\alpha$ :

(1) The matrices  $N, XD + YN$  must be right coprime and  $X, XD + YN$  must be left coprime. Since

$$\theta(\det(XD + YN)) = \lambda l + lq,$$

if there are cancellations, then the resulting denominator matrix will be  $\overline{XD + YN}$ , where

$$\theta(\det(\overline{XD + YN})) < \lambda l + lq.$$

But then the  $\{\phi_i = \phi\}$  could not be the invariant factors of the closed loop system, since  $\sum_{i=1}^l \theta(\phi_i) = \lambda l + \lambda q$ .

(2) In actuality the row degrees of a row reduced representation of  $C = X^{-1}Y$  must all be equal to  $q$ . Now

$$\lim_{s \rightarrow \infty} \text{diag}(s^{-q_i}) \begin{bmatrix} X & Y \end{bmatrix} \begin{bmatrix} D \\ N \end{bmatrix} \text{diag}(s^{-\lambda}) = I.$$

From (1),  $XD + YN$  has invariant factors  $\{\phi_i = \phi\}$ . From Lemma 2.2 this

implies that

$$\sum_{i=1}^k \theta(\phi_i) \geq \sum_{i=1}^k \lambda + q_i, \quad k = 1, 2, \dots, \quad \text{with equality at } k = l.$$

This implies  $\lambda + q \geq \lambda + q_1$ . If  $q_1 < q$  then  $q_1 + \dots + q_l \neq lq$ . Therefore  $q_1 = q$ . But  $2\lambda + 2q \geq 2\lambda + q_1 + q_2$ ,  $q_2 = q$ , and generally  $q_i = q$ .

(3) Since  $X, Y$  must be of the form

$$X = Is^q + X_{q-1}s^{q-1} + \dots + X_0, \quad Y = Y_qs^q + Y_{q-1}s^{q-1} + \dots + Y_0,$$

it must be that

$$XD + YN = \begin{bmatrix} \phi & & & 0 \\ & \phi & & \\ & & \ddots & \\ 0 & & & \phi \end{bmatrix}, \quad \phi \text{ the given polynomial.}$$

In general

$$XD + YN = Is^{\lambda+q} + \Phi_{\lambda+q-1}s^{\lambda+q-1} + \dots + \Phi_0 = \bar{\Phi}.$$

But since the invariant factors of  $\bar{\Phi}$  are  $\phi_1 = \phi_2 = \dots = \phi_l = \phi$ , this means that the gcd of  $1 \times 1$  minors in particular must be  $\phi$ . This is a polynomial of degree  $\lambda + q$ . All off diagonal entries in  $\bar{\Phi}$  are of lesser degree or zero. They cannot be of lesser degree; therefore they are zero.

With all this in mind, suppose now that  $q < \lambda - 1$ . Then for any  $\alpha$  in  $W$ , i.e. for generic  $\alpha$ , we must have

$$\begin{aligned} [I, Y_q, \dots, X_0, Y_0] S_{q+1}(D, N) &= [I, \underbrace{\Phi_{\lambda+q-1}, \dots, \Phi_0}_{= \Phi \text{ (diagonal)}}] \end{aligned}$$

Look at the first row of this matrix equation:

$$yS_{q+1}(D, N) = \left[ \underbrace{1, 0 \dots 0}_l, \underbrace{\phi_{\lambda+q-1}, 0 \dots 0}_l, \dots, \underbrace{\phi_0, 0 \dots 0}_l \right] = \phi$$

where  $y$  is a  $1 \times 2(q+1)l$  vector,  $S_{q+1}(D, N)$  a  $2(q+1)l \times (\lambda + q + 1)l$  matrix, and  $\phi$  a  $(\lambda + q + 1)l$ -vector.

Let

$$u = 2(q+1)l, \quad v = (\lambda + q + 1)l.$$

If we partition

$$S_{q+1}(D, N) = [A, B],$$

where  $A$  is  $u \times u$  and  $B$  is  $u \times v$ , then since  $q < \lambda - 1$ , it is clear that  $u < v$ . The matrix  $A$  is invertible for a generic subset of  $R^t$  ( $A$  has a resultant structure):

$$y = \phi_u A^{-1},$$

where  $\phi_u$  is the first  $u$  entries in the vector  $\phi$ . But since there are more equations than unknowns, it must also be that

$$yB = \phi_{v-u} \tag{6.3}$$

where  $\phi_{v-u}$  comprises the remaining  $v - u$  entries in  $\phi$ . But the relationship (6.3) is only satisfied on a Zariski closed set of  $R^t$ . This is a contradiction. This completes the necessity part of the theorem. ■

Theorem 6.1 in effect says that for almost all strictly proper transfer functions of McMillan degree  $n = \lambda l$  and equal controllability indices, a necessary and sufficient condition for the existence of a proper compensator of order  $lq$  which makes the closed loop invariant factors equal to  $\phi = \phi_1 = \dots = \phi_l$  for almost all  $\phi$  of degree  $\lambda + q$  is  $q \geq \lambda - 1$ . It is necessary therefore that the order of the compensator be greater than or equal to  $l(\lambda - 1)$ . It should be emphasized that here we are considering the square case where  $\lambda = \mu$ ,  $\mu$  the observability index of the transfer function  $P$ . Thus  $l(\mu - 1)$  is the more appropriate bound.

Theorem 6.1 addresses the case of strictly proper transfer function of McMillan degree  $n = \lambda l$  and equal controllability indices. Theorems 6.2, 6.3, and 6.4 deal with the more general case.

**THEOREM 6.2.** *Let  $P = ND^{-1}$  be an  $l \times l$  strictly proper transfer function where  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l \geq 0$ ,  $n = \lambda_1 + \lambda_2 + \dots + \lambda_l$ ,  $\mu = n/l$ , and*

$$\begin{bmatrix} D \\ N \end{bmatrix} = \begin{bmatrix} D_{hc} \\ 0 \end{bmatrix} \text{diag}(s^{\lambda_i}) + L(s)$$

with  $L(s)$  containing lower order terms. Let  $\{\phi_i\}$  be  $l$  monic polynomials,  $\phi_i \mid \phi_{i-1}$ ,  $2 \leq i \leq l$ , such that  $\theta(\phi_i) = \lambda_i + q$ . Let

$$W = \{(N, D, \phi_1, \dots, \phi_l) \in R^{2nl + \lambda_1 + q + l^2} \mid \text{there exists a proper compensator } C \text{ of order } lq \text{ making } \{\phi_i\} \text{ the invariant factors of the closed loop system, } C \text{ with equal observability indecies}\}.$$

Then  $q \geq \mu - 1$  is a necessary condition for  $W$  to be a generic subset of  $R^{2nl + \lambda_1 + q + l^2}$ .

*Proof.* Let  $W \subset R^t$  ( $t = 2nl + \lambda_1 + q + l^2$ ) be the set of  $(N, D, \phi, \dots, \phi_l)$  for which there exists a proper compensator  $C = X^{-1}Y$ , of order  $lq$ , which makes  $\{\phi_i\}$  the closed loop invariant factor. Assume that it is generic.

For any  $\alpha \in W$  the following must be true. Let  $C = X^{-1}Y$  be the proper compensator that accomplishes the task. We choose  $X$  to be row reduced and let  $q_1 = q_2 = \dots = q_l = 0q$  be the row degrees  $q_1 + q_2 + \dots + q_l = lq$  and  $X_{hc}$  the highest row degree coefficient matrix of  $X$ . The following three properties must hold for such an  $\alpha$ :

(1) The matrices  $N, XD + YN$  must be right coprime and  $X, XD + YN$  must be left coprime (follows proof of Theorem 6.1).

(2) The row degrees of a row reduced representation of  $C = X^{-1}Y$  are all equal to  $q$ . Now

$$\lim_{s \rightarrow \infty} \text{diag}(s^{-q}) [X \quad Y] \begin{bmatrix} D \\ N \end{bmatrix} \text{diag}(s^{-\lambda_i}) = D_{hc}.$$

From (1),  $XC + YN$  has invariant factors  $\{\phi_i\}$ . From Lemma 2.2 we have that (choose  $X_{hc} = D_{hc}^{-1}$ )

$$\sum_{i=1}^k \theta(\phi_i) \geq \sum_{i=1}^k \lambda_i + q, \quad i = 1, 2, \dots, k, \quad \text{with equality at } k = l.$$

(3) Since  $X, Y$  must be of the form

$$X = X_{hc}s^q + X_{q-1}s^{q-1} + \dots + X_0, \quad Y = Y_qs^q + \dots + Y_0,$$

it must be that

$$XD + YN = \begin{bmatrix} \phi_1 & & & & 0 \\ X & \phi_2 & & & \\ X & X & \phi_3 & & \\ \vdots & \vdots & \vdots & \ddots & \\ X & X & X & & \phi_l \end{bmatrix},$$

where  $\{\phi_i\}$  are the given polynomials and the X's indicate possible nonzero locations. In general

$$XD + YN = I \text{diag}(s^{\lambda_i + q}) + Q(s) = \Phi.$$

Since  $\{\phi_i\}$  are the invariant factors of  $\Phi$ , this means in particular that  $\phi_l$  [ $\theta(\phi_l) = \lambda_l + q$ ] is the gcd of  $1 \times 1$  minors of  $\Phi$ . Since off diagonal entries in the  $l$ th column are of degree less than  $\lambda_l + q$ , they must be zero. Furthermore the gcd of  $2 \times 2$  minors of  $\Phi$  is  $\phi_l \phi_{l-1}$  [ $\theta(\phi_l \phi_{l-1}) = \lambda_l + \lambda_{l-1} + 2q$ ]. Since off diagonal entries in column  $l - 1$  are of degree less than  $\lambda_{l-1} + q$ , above the diagonal entries must be zero. Continuing in this fashion, we see that claim (3) is true.

Suppose now that  $q < \mu - 1$ . Then for any  $\alpha$  in  $W$  (i.e. for generic  $\alpha$ ) we must have

$$[X_{hc}, Y_q, \dots, X_0, Y_0] \bar{S}_{q+1}(D, N) = [\Phi_{\lambda_1 + q}, \dots, \Phi_0],$$

where  $\bar{S}_{q+1}(D, N)$  is the  $(q + 1)2l \times [n + l(q + 1)]$  matrix obtained from  $S_{q+1}(D, N)$  by deleting its all zero columns. Looking at the first row of this matrix equation, we have

$$y \bar{S}_{q+1}(D, N) = [1, \dots, \phi_{\lambda_1 + q - 1}, 0, \dots, 0, \dots, \phi_0, 0, \dots, 0] = \phi.$$

A contradiction follows as in the proof of Theorem 6.1, where now use is made of Proposition 5.3. This completes the proof of Theorem 6.2. ■

We see that as in the case of equal controllability indices,  $q \geq \mu - 1$  is a necessary condition for generic invariant factor assignment, where  $\mu$  is the observability index of the transfer function  $P$ . We believe that  $q = \mu - 1$  is a sufficient condition as well, in view of the next two results, where it is shown

to be the case when the controllability indices satisfy either

- (a)  $2 \mid l$ ,  $k \geq 1$ ,  $\lambda_1 = \dots = \lambda_{l/2} = k + 2$ ,  $\lambda_{l/2+1} = \dots = \lambda_l = k$ , or  
 (b)  $3 \mid l$ ,  $k \geq 1$ ,  $\lambda_1 = \dots = \lambda_{l/3} = k + 2$ ,  $\lambda_{l/3+1} = \dots = \lambda_{2l/3} = k + 1$ ,  
 $\lambda_{2l/3+1} = \dots = \lambda_l = k$ .

**THEOREM 6.3.** *Let  $P = ND^{-1}$  be an  $l \times l$  strictly proper transfer function where  $l$  is even,  $k \geq 1$ ,*

$$\lambda_1 = \lambda_2 = \dots = \lambda_{l/2} = k + 2, \quad \lambda_{l/2+1} = \dots = \lambda_l = k, \quad \mu = k + 1,$$

$$\begin{bmatrix} D \\ N \end{bmatrix} = \begin{bmatrix} D_{hc} \\ 0 \end{bmatrix} \text{diag}(s^{\lambda_i}) + L(s).$$

Let  $\langle \phi_i \rangle$  be  $l$  monic polynomials,  $\phi_i \mid \phi_{i-1}$ ,  $2 \leq i \leq l$ , such that  $\theta(\phi_i) = \lambda_i + q$ . Let

$W = \{ (N, D, \phi_1, \dots, \phi_l) \in R^{2nl + \lambda_1 + q + l^2} \mid \text{there exists a proper compensator of order } lq \text{ making } \langle \phi_i \rangle \text{ the invariant factor of the closed loop system} \}$ .

Then  $q = \mu - 1$  is a sufficient condition for  $W$  to be a generic subset of  $R^{2n + \lambda_1 + q + l^2}$ .

*Proof.* Let  $q = \mu - 1$  and  $t = 2nl + \lambda_1 + q + l^2$ . Show that  $W$  contains a nonempty Zariski open set. Since  $q = \mu - 1$ , the set  $F \subseteq R^t$  of  $(N, D, \phi_1, \dots, \phi_l)$  for which  $S_\mu(D, N)$  is full rank is generic. This implies that for any  $\Phi$  of the form

$$\Phi = \text{diag}(\phi_i),$$

a unique solution  $X = Is^q + \dots + X_0$ ,  $Y = Y_q s^q + \dots + Y_0$  to  $XD + YN = \Phi$  exists and is given by

$$[I, Y_q, X_{q-1}, \dots, X_0, Y_0] = [\bar{\Phi}_{\lambda_1+q}, \dots, \bar{\Phi}_0] \bar{S}_\mu^{-1}$$

This means that  $X_i, Y_i$  are rational expressions in the parameters of

$N, D, \phi_1, \dots, \phi_l$ . Now the set  $E \subseteq R^l$  for which  $\Phi, X$  are left coprime and  $N, \Phi$  right coprime is a Zariski open set. The key point is to demonstrate that  $E$  is nonempty. It can be verified after some algebraic manipulations that if

$$D = \begin{bmatrix} I_{l/2} s^{k+2} & 0 \\ 0 & I_{l/2} s^k \end{bmatrix}, \quad N = \begin{bmatrix} I_{l/2} & 0 \\ I_{l/2} s & I_{l/2} \end{bmatrix},$$

$$\Phi = \begin{bmatrix} I_{l/2} \phi_1 & 0 \\ 0 & I_{l/2} \phi_2 \end{bmatrix},$$

where

$$\phi_2 = (s + 1)^{2k}, \quad \phi_1 = (s + 1)^{2k} (s^2 - 1),$$

then  $\bar{S}_\mu(D, N)$  is invertible, and  $X, \Phi$  are left coprime and  $\Phi, N$  right coprime. ■

**THEOREM 6.4.** *Let  $P = ND^{-1}$  be an  $l \times l$  strictly proper transfer function where  $3 \mid l, k \geq 1, \mu = k + 1,$*

$$\lambda_1 = \dots = \lambda_{l/3} = k + 2, \quad \lambda_{l/3+1} = \dots = \lambda_{2l/3} = k + 1,$$

$$\lambda_{2l/3+1} = \dots = \lambda_l = k,$$

$$\begin{bmatrix} D \\ N \end{bmatrix} = \begin{bmatrix} D_{hc} \\ 0 \end{bmatrix} \text{diag}(s^{\lambda_i}) + L(s).$$

*Let  $\{\phi_i\}$  be  $l$  monic polynomials,  $\phi_i \mid \phi_{i-1}, 2 \leq i \leq l,$  such that  $\theta(\phi_i) = \lambda_i + q.$  Let*

$$W = \{ (N, D, \phi_1, \dots, \phi_l) \in R^{2nl + \lambda_1 + q + l^2} \mid \text{there exists a proper compensator of order } lq \text{ making } \{\phi_i\} \text{ the invariant factors of the closed loop system} \}.$$

*Then  $q = \mu - 1$  is a sufficient condition for  $W$  to be a generic subset of  $R^{2nl + \lambda_1 + q + l^2}.$*

*Proof.* The proof proceeds in a similar fashion to that of Theorem 6.3. The point which shows the nonemptiness of  $W$  is the following:

$$D = \begin{bmatrix} I_{1/3}s^{k+2} & 0 & 0 \\ 0 & I_{1/3}s^{k+1} & 0 \\ 0 & 0 & I_{1/3}s^k \end{bmatrix}, \quad N = \begin{bmatrix} I_{1/3} & 0 & 0 \\ 0 & I_{1/3} & 0 \\ I_{1/3}s & 0 & I_{1/3} \end{bmatrix},$$

$$\Phi = \begin{bmatrix} I_{1/3}\phi_1 & 0 & 0 \\ 0 & I_{1/3}\phi_2 & 0 \\ 0 & 0 & I_{1/3}\phi_3 \end{bmatrix},$$

where  $\phi_3 = (s+1)^{2k}$ ,  $\phi_2 = (s+1)^{2k+1}$ ,  $\phi_1 = (s+1)^{2k+1}(s-1)$ . ■

**REMARK.** Sufficiency proofs for many more controllability index configurations have been constructed by using a *different* test point in each case. This makes it difficult to construct a general test point that can be used in every case.

Now it is evident that there are two basic issues concerning invariant factor assignment. One is the allowable degrees of the closed loop invariant factors [i.e. the sizes of the attainable cyclic blocks of  $A$ , where  $A$  comes from a minimal realization of  $G(s)$ ], and the other is the reachable invariant polynomials themselves. That is, assuming an allowable set of degrees, is it possible to reach all (or almost all) such polynomials? The necessary condition appearing in [20] addresses the allowable degrees issue. Here we have assumed a particular degree configuration (which incidentally is compatible with their conditions) and are investigating the order of the compensator needed for almost arbitrary invariant factor assignment.

It is important to mention that different degree assignments require different order compensators. This is evident from Theorem 6.5.

**THEOREM 6.5.** *Let  $P = ND^{-1}$  be an  $m \times l$  strictly proper transfer function where  $\lambda > 0$ ,  $n = \lambda l$ , and*

$$D = Is^\lambda + D_{\lambda-1}s^{\lambda-1} + \dots + D_0,$$

$$N = N_{\lambda-1}s^{\lambda-1} + \dots + N_0.$$

Let  $q \geq 0$  and  $\phi(s) = s^{n+q} + \phi_{n+q+1}s^{n+q+1} + \dots + \phi_0$ . Let

$W = \{ (N, D, \phi) \in R^{2\lambda l^2+n+q} \mid \text{there exists a proper compensator of order } q \text{ making } \phi_i = \phi, \phi_2 = \phi_3 = \dots = \phi_l = 1 \text{ the invariant factors of the closed loop system} \}$ .

Then a sufficient condition for  $W$  to be a generic subset of  $R^{2\lambda l^2+n+q}$  is  $q \geq \mu - 1$ .

*Proof.* Using the Sylvester resultant formulation, one can show [8] that if  $q \geq \mu - 1$ , then the set  $M \subseteq R^{2\lambda l^2+n+q}$  of  $(N, D, \phi)$  for which a proper compensator of order  $q$  exists and is of the form

$$\begin{aligned}
 X &= \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{bmatrix} S^q + \begin{bmatrix} {}_q x_1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{bmatrix} S^{q-1} + \dots \\
 &+ \begin{bmatrix} {}_0 x_1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & I & \\ 0 & & & \end{bmatrix}, \\
 Y &= \begin{bmatrix} {}_q y_1 & \dots & {}_q y_m \\ & & 0 \end{bmatrix} S^q + \dots + \begin{bmatrix} {}_0 y_1 & \dots & {}_0 y_l & \dots & {}_0 y_m \\ 0 & & & & \\ \vdots & I_{l-1} & & 0 & \\ 0 & & & & \end{bmatrix},
 \end{aligned}$$

which makes  $\phi$  the closed loop characteristic polynomial, is generic:

$$XD + YN = \Phi, \quad \text{where } \det \Phi = \phi.$$

But because of the structure of  $\Phi$  it can easily be shown that the set  $J \subseteq R^{2\lambda l^2+n+q}$  for which the gcd's of the  $i \times i$  minors of  $\Phi$  for  $1 \leq i \leq l - 1$  are all equal to 1 is generic. Therefore the set  $J \subseteq M$  for which  $\phi_1 = \phi, \phi_2 = \phi_3 = \dots = \phi_l = 1$  are the invariant factors of  $\Phi$  is a generic subset of  $R^{2\lambda l^2+n+q}$ . ■

Similar versions of this theorem have appeared in the past [2, 12, 20]. The approach taken here is different. As mentioned earlier, in the case when  $l > m$  a "dual" formulation and results can be obtained.

## 7. CONCLUSIONS

The problem of generalized pole assignment using output feedback has not been completely solved as yet. Great progress has been made, as evidenced by many important contributions (see references). In this paper, using a formulation involving generalized Sylvester resultants, we were able to give new short proofs of earlier results as well as suggest necessary and sufficient conditions for generic invariant factor assignment in several cases. We believe that the ideas presented here can be used to obtain many more results.

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## REFERENCES

- 1 R. R. Bitmead, S.-Y. Kung, B. D. O. Anderson, and T. Kailath, Greatest common divisors via generalized Sylvester and Bezout matrices, *IEEE Trans. Automat. Control* AC-23, No. 6 (Dec. 1978).
- 2 F. M. Brasch and J. B. Pearson, Pole placement using dynamic compensators, *IEEE Trans. Automat. Control* AC-15, No. 1 (Feb. 1970).
- 3 R. W. Brockett and C. I. Byrnes, Multivariable Nyquist criteria, root loci, and pole placement: A geometric viewpoint, *IEEE Trans. Automat. Control* AC-26, No. 1 (Feb. 1981).
- 4 E. J. Davison and S. H. Wang, On pole assignment in linear multivariable systems using output feedback, *IEEE Trans. Automat. Control* AC-20 (Aug. 1975).
- 5 C. A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties*, Academic, New York, 1975.
- 6 T. E. Djaferis and S. K. Mitter, The generalized pole assignment problem, in *Proceedings of 1979 IEEE CDC*, Fort Lauderdale, Florida, Dec. 1979.
- 7 T. E. Djaferis, General pole assignment by output feedback and solution of linear matrix equations from an algebraic viewpoint, Ph.D. Thesis, M.I.T., June 1979.
- 8 T. E. Djaferis, Generic pole assignment using dynamic output feedback, in *Proceedings of the 19th Allerton Conference on Communication, Control and Computing*, Monticello, Ill., Sept. 1981; *Internat. J. Control*, to appear.
- 9 E. Emre, The polynomial equation  $QQ_c + RP_c = \Phi$  with application to dynamic feedback, *SIAM J. Control Optim.* 18, No. 6 (Nov. 1980).
- 10 E. Emre, Pole assignment by dynamic output feedback, *Internat. J. Control* 33, No. 2 (1981).
- 11 R. Hermann and C. F. Martin, Applications of algebraic geometry to linear system theory, *IEEE Trans. Automat. Control* AC-22 (1977).
- 12 T. Kailath, *Linear Systems*, Prentice-Hall, Englewood Cliffs, N.J., 1980.

- 13 H. Kimura, Pole assignment by gain output feedback, *IEEE Trans. Automat. Control* 20 (1975).
- 14 H. Kimura, Further result in the problem of pole assignment by output feedback, *IEEE Trans. Automat. Control* 22 (1977).
- 15 T. Koussiouris, On the general problem of pole assignment, *Internat. J. Control* 30, No. 4 (1979).
- 16 S.-Y. Kung, T. Kailath, and M. Morf, A generalized resultant matrix for polynomial matrices, in *Proceedings of the IEEE Conference on Decision and Control*, Florida, 1976.
- 17 P. Lancaster, *Theory of Matrices*, Academic, New York, 1969.
- 18 L. Pernebo, Algebraic control theory for linear multivariable systems, Tekn. Dr. Thesis, Lund I.T., May 1978.
- 19 H. H. Rosenbrock, *State Space and Multivariable Theory*, Wiley, New York, 1970.
- 20 H. H. Rosenbrock and C. E. Hayton, The general problem of pole assignment, *Internat. J. Control* 27, No. 6 (1978).
- 21 W. M. Wonham, On pole assignment in multi-input controllable linear systems, *IEEE Trans. Automat. Control* AC-12, No. 6 (Dec. 1967).
- 22 J. C. Willems and W. H. Hesselink, Generic properties of the pole placement problem, Preprints 1978 IFAC Congress, Helsinki, Finland.
- 23 W. A. Wolovich, *Linear Multivariable Systems*, Applied Mathematical Sciences, Vol. II, Springer, New York, 1974.
- 24 O. Zariski and P. Samuel, *Commutative Algebra*, Vol. 1, Van Nostrand, Princeton, 1958.

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