Differential flatness of a front-steered vehicle with tire force control

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Abstract: A trajectory tracking controller based on differential flatness is presented for a nonlinear bicycle model. This controller maps the bicycle dynamics into a point mass located at a center of oscillation with an additional degree of freedom of yaw dynamics. A state transformation is performed that reveals structure in the yaw dynamics resembling a Liénard system. A candidate Lyapunov function inspired by this structure is used to assess the stability of the yaw dynamics while tracking straight-line trajectories and steady turns. The basin of attraction of the controller is limited by actuator constraints and the presence of unstable equilibrium points during turns with high lateral acceleration. The controller properties and the stability of yaw dynamics are demonstrated in simulation.

I. INTRODUCTION

Autonomous navigation of robotic vehicles at high speeds has received substantial research attention recently, as highlighted by the DARPA Grand Challenge competitions [1, 2]. Applications of high speed robotic vehicles include military reconnaissance and material transport. In these scenarios, increasing vehicle speed and maneuverability may reduce danger to the vehicle and improve task efficiency. Research on safe autonomous operation robotic vehicles can also be applied to improve the safety of human-operated vehicles, for which accidents cause more than $200 billion in damage and 30,000 deaths annually in the United States [3].

High-speed robotic vehicle navigation systems require a number of capabilities for successful operation, including environment sensing, rapid trajectory planning, and robust control of the vehicle along the desired trajectory. This paper considers the problem of trajectory tracking control near the limits of tire-road friction for robotic vehicles with front wheel steering. A challenge of vehicle control near the limits of friction is the nonlinear behavior of the tire forces and potentially unstable dynamics for large slip angles. Unstable equilibrium points may be present, depending on the vehicle speed, steering angle, and tire properties [4].

A path tracking control algorithm commonly used for robotic vehicles with front wheel steering is pure pursuit [5]. This algorithm involves pointing the front wheels at a point on the desired path that lies a specified look-ahead distance in front of the vehicle. Controller stability requires a minimum look-ahead distance for the given vehicle speed.

Other approaches to trajectory tracking control include the use of receding horizon optimization with a predictive model [6, 7]. These approaches support a variety of models and performance objectives, though the computational demands of real-time numerical optimization may be prohibitive, particularly for low-cost microcontrollers.

Recent research in trajectory tracking control has focused on systems with a property known as differential flatness. A nonlinear system $\dot{x} = f(x,u)$ is differentially flat if an output $y$ can be found such that the states $x$ and inputs $u$ can be expressed in terms of $y$ and a finite number of its derivatives [8]. A benefit of flat systems is that flat outputs can follow arbitrary trajectories $y(t)$ provided that the trajectory is sufficiently smooth. For example, a front-steered bicycle driving in a plane without wheel slip is a flat system whose flat output is the position of the rear wheel [8], which has been exploited for vehicle tracking control [9], though the no-slip assumption restricts its applicability.

Another flat output is the position of the Huygens center of oscillation (C.O.) of certain types of rigid body systems, including a vertical take-off and landing aircraft [10, 11]. States related to this point have been identified as flat outputs for a bicycle model with friction forces acting at the front and rear tires, as described below.

Ackermann noted that there exists a point near the front wheels whose lateral body-fixed acceleration is decoupled from the rear lateral tire force. This decoupling was exploited to design a steering controller for the body-fixed acceleration at that point [12]. It was later noted by Fuchshumer that both the point identified by Ackermann at the front of the vehicle and a similar point at the rear of the vehicle are centers of oscillation. The body-fixed velocity components at the rear C.O. were identified as flat outputs and a corresponding flatness-based controller defined [13]. The position of the rear C.O. was chosen as the flat output in a trajectory tracking controller by Setlur [14]. Other flat outputs have been considered under the assumption of constant speed and linear tire force models [15, 16].

This paper presents a trajectory tracking controller for the flat output located at the front C.O. The position of this point is an advantageous choice of flat output as it can be controlled to track trajectories with finite acceleration, whereas the rear C.O. requires an additional degree of smoothness in reference trajectories.

As the position of the front C.O. represents two degrees of freedom, there is an additional degree of freedom...
corresponding to bicycle yaw dynamics. A linear analysis by Ackermann indicated that the bicycle yaw dynamics are a stable second-order system with damping inversely proportional to vehicle speed [12]. This linear analysis is limited to small slip angles for which the lateral tire forces are linear. In this paper, a nonlinear stability analysis is presented that is valid for a wider range of conditions.

Two important nonlinear effects in tire force modeling are the friction circle and load transfer effects [17]. The friction circle effect refers to the coupling between the longitudinal and lateral forces acting at each tire. The load transfer effect refers to the sensitivity of tire friction force magnitude to changes in the tire normal force caused by heavy braking or acceleration. For brevity, this paper considers a vehicle model in which these effects are not significant. The model employs front-wheel drive and braking so that there is no friction circle effect at the rear tire. Additionally, a low surface friction coefficient is assumed so that load transfer is not significant.

A nonlinear bicycle model and trajectory tracking controller for the front center of oscillation is presented in Section II. The structure and stability of the yaw dynamics are evaluated in Section III, followed by simulation results in Section IV demonstrating the tracking controller and verifying the stability of the yaw dynamics.

II. FLATNESS CONTROL OF BICYCLE MODEL

In II.A., a bicycle model with nonlinear kinematics and tire forces is presented. In II.B., the position of the front C.O. is identified as a flat output and a trajectory tracking controller based on tire force control is defined.

A. Dynamics of bicycle model

A nonlinear bicycle model is presented here and illustrated in Fig. 1 using similar notation to [10]. The vehicle is modeled as a rigid body in the plane formed by vectors \( \vec{i}, \vec{j}, \vec{k} \) from fixed frame \((\vec{i}, \vec{j}, \vec{k})\). A body-fixed frame is given as \((\vec{i}_b, \vec{j}_b, \vec{k}_b)\) with \(\vec{k}_b = \vec{k}\) and yaw angle \(\psi\). The vehicle center of gravity (c.g.) is located at point \(c\) and has position \(\vec{r}_c\), velocity \(\vec{v}_c\), acceleration \(\vec{a}_c\), and state vector \(\mathbf{x}\) as:

\[
\begin{align*}
\vec{r}_c &= X_c \vec{i} + Y_c \vec{j} \\
\vec{v}_c &= \dot{X}_c \vec{i} + \dot{Y}_c \vec{j} \\
\vec{a}_c &= \ddot{X}_c \vec{i} + \ddot{Y}_c \vec{j} \\
\mathbf{x} &= \begin{bmatrix} X_c & Y_c & \psi & \dot{X}_c & \dot{Y}_c & \dot{\psi} \end{bmatrix}^T
\end{align*}
\]

The forces acting on the vehicle are lateral tire friction forces \(F_{sf}\) and \(F_{yr}\) and longitudinal tire friction forces \(F_{sf}\). The vehicle dynamics at the c.g. are given as

\[
\begin{align*}
\mathbf{m} \ddot{\mathbf{x}}_c &= \begin{bmatrix} F_{sf} \cos \delta - F_{sf} \sin \delta \frac{\dot{\psi}}{\dot{\psi}} \\
F_{sf} \sin \delta + F_{sf} \cos \delta + F_{yr} \end{bmatrix} \vec{i}_b \[5]
\end{align*}
\]

\[
\begin{align*}
\dot{\psi} &= x_f \left( F_{sf} \sin \delta + F_{sf} \cos \delta \right) - x_y F_{yr} \\
\end{align*}
\]

where \(m\) is the vehicle mass, \(I_{zz}\) is the yaw inertia, \(\delta\) is the front steering angle, and the position of \(c\) relative to the wheels is given by \(x_f\) and \(x_r\).

Fig. 1. Nonlinear bicycle model states and inputs.

The lateral tire forces \(F_{sf}\) and \(F_{yr}\) are modeled as differentiable functions of the tire slip angles \(\alpha_f\) and \(\alpha_r\), which are computed as

\[
\begin{align*}
\alpha_f &= \tan^{-1} \left( \frac{\dot{\psi} + \dot{\psi}}{\dot{\psi}} \right) - \delta \\
\alpha_r &= \tan^{-1} \left( \frac{\dot{\psi} - \dot{\psi}}{\dot{\psi}} \right)
\end{align*}
\]

with an example tire with speed \(V\), slip angle \(\alpha\) and lateral force \(F_y\) illustrated in Fig. 2.

Fig. 2. Lateral tire force \(F_y\) and slip angle \(\alpha\) for an example tire.

A lateral tire force model based on Pacejka’s Magic Formula [4] is presented below with parameters \(B, C, D,\) and \(E\). The formula is modified to include rotational symmetry so that \(F_y(\alpha) = F_y(\beta) \Leftrightarrow \sin \alpha = \sin \beta\). The angle \(\bar{\alpha}(\alpha)\) is defined by mapping the angle \(\alpha\) into \([-\pi/2, \pi/2]\) as

\[
\bar{\alpha}(\alpha) = \sin^{-1}(\sin \alpha)\]

with the tire model given as

\[
F_y(\bar{\alpha}) = D \sin \left[ C \tan^{-1} \left( B(1 - E \bar{\alpha} + E \tan^{-1}(B \pi)) \right) \right]
\]

As mentioned in Section I, heavy braking and acceleration can cause variation in the normal forces at the front and rear, known as load transfer [17]. Since a low-friction tire model is used in this paper, the normal forces \(F_{sf}\) and \(F_{yr}\) are approximately constant and computed as

\[
\begin{align*}
F_{sf} &= mg \frac{x_f}{x_f + x_r} \\
F_{yr} &= mg \frac{x_y}{x_y + x_r}
\end{align*}
\]

(11) (12)
Parameters for the front and rear lateral forces $F_{xf}$ and $F_{xr}$ on a low-friction road taken from [4] are given in Table I. The normalized lateral forces $F_{xf}/F_{xf}$ and $F_{xr}/F_{xr}$ are plotted in Fig. 3 for two ranges of slip angles to show detail near the origin and the wrapping behavior at large slip angles.

Fig. 3. Normalized lateral tire forces $F_{xf}/F_{xf}$ and $F_{xr}/F_{xr}$, plotted with respect to slip angle. The top plot shows detail near the origin, and the bottom plot shows a wider range of slip angles.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
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<tr>
<td>$x_f$</td>
<td>Dist. From C.G. to front</td>
<td>1.2 m</td>
</tr>
<tr>
<td>$x_r$</td>
<td>Dist. from C.G. to rear</td>
<td>1.3 m</td>
</tr>
<tr>
<td>$m$</td>
<td>Mass</td>
<td>1500 kg</td>
</tr>
<tr>
<td>$I_z$</td>
<td>Yaw moment of inertia</td>
<td>3000 kg m$^2$</td>
</tr>
<tr>
<td>$B_l$</td>
<td>Front tire parameter</td>
<td>11.275 rad$^{-1}$</td>
</tr>
<tr>
<td>$C_l$</td>
<td>Front tire parameter</td>
<td>1.5600</td>
</tr>
<tr>
<td>$D_l$</td>
<td>Rear tire parameter</td>
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</tr>
<tr>
<td>$E_l$</td>
<td>Front tire parameter</td>
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</tr>
<tr>
<td>$\alpha_{pea1}$</td>
<td>Slip at peak $F_{xf}$</td>
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</tr>
<tr>
<td>$\mu_l$</td>
<td>Peak front friction</td>
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</tr>
<tr>
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</tr>
<tr>
<td>$C_r$</td>
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<td>$D_r$</td>
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<tr>
<td>$E_r$</td>
<td>Rear tire parameter</td>
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<td>$\alpha_{pea2}$</td>
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<tr>
<td>$\mu_r$</td>
<td>Peak rear friction</td>
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TABLE I

**MODEL PARAMETERS**

B. Flat outputs of bicycle model

The position of the bicycle's front center of oscillation (C.O.) will be shown to be a flat output, provided the steering and wheel torque actuators have sufficient bandwidth and control authority to control the tire forces $F_{xf}$ and $F_{xf}$.

The front C.O.'s is located a distance $x_p = I_{z}(mx_r)$ in front of $c$ and is labeled point $p$, as shown in Fig. 4. The position $\vec{r}_p$, velocity $\vec{v}_p$, and acceleration $\vec{a}_p$ of point $p$ are

$$\vec{r}_p = \vec{r}_c + x_p \vec{b} = X_p \vec{i} + Y_p \vec{j}$$  \hspace{1cm} (13)

$$\vec{v}_p = \vec{v}_c + x_p \vec{b} = \dot{X}_p \vec{i} + \dot{Y}_p \vec{j}$$  \hspace{1cm} (14)

$$\vec{a}_p = \vec{a}_c + x_p \vec{b} = \ddot{X}_p \vec{i} + \ddot{Y}_p \vec{j}$$  \hspace{1cm} (15)

and by combining this expression for $\vec{a}_p$ with the vehicle dynamics in (5)-(6), the dynamics of point $p$ are written as

$$m \ddot{X}_p = \left( F_{xf} \cos \delta - F_{xf} \sin \delta - m \dot{x}_p \dot{\psi}^2 \right) \dot{\beta}_p$$

$$+ \left( 1 + \frac{x_f}{x_r} \right) \left( F_{xf} \sin \delta + F_{xf} \cos \delta \right) \dot{\gamma}_p$$  \hspace{1cm} (16)

It can be seen that the rear lateral force $F_{xr}$ is eliminated from the expression for $m \ddot{X}_p$, and a term proportional to $\dot{\psi}^2$ is added.

It is assumed that the tire properties are known so that the forces $F_{xf}$ and $F_{xf}$ can be controlled with steering and traction/braking actuators. By mapping the front forces into inputs $u_{ib}$ and $u_{jb}$ as

$$u_{ib} = F_{xf} \cos \delta - F_{xf} \sin \delta$$  \hspace{1cm} (17)

$$u_{jb} = F_{xf} \sin \delta + F_{xf} \cos \delta$$  \hspace{1cm} (18)

and applying the feedback law

$$u_{ib} = m(x_p \dot{\psi}^2 + u_1(t) \cos \psi + u_2(t) \sin \psi)$$  \hspace{1cm} (19)

$$u_{jb} = - \frac{mx_r}{x_f + x_r} (-u_1(t) \sin \psi + u_2(t) \cos \psi)$$  \hspace{1cm} (20)

with reference inputs $u_1(t)$ and $u_2(t)$, the bicycle dynamics are partitioned into a flat subsystem based on the position of point $p$ and the yaw dynamics as

$$\ddot{\dot{X}}_p = u_1(t)$$  \hspace{1cm} (21)

$$\ddot{Y}_p = u_2(t)$$  \hspace{1cm} (22)

$$\dot{\psi} = \frac{x_f F_{xx}}{I_{zz}} \left( -u_1(t) - u_2(t) \frac{\sin \psi}{g} - \frac{u_2(t) \cos \psi}{F_{xf}} \right)$$  \hspace{1cm} (23)

As mentioned previously, an advantage of the front center of oscillation is that it can track arbitrary accelerations, while the rear center of oscillation requires an additional degree of smoothness in the reference trajectory [13, 14]. A limitation on the tire force controller is imposed by actuator constraints, in particular the maximum steering angle, which limit the conditions in which the controller can be applied. These conditions will be analyzed in Section IV.

The rear lateral tire force $F_{xr}$ was decoupled from the dynamics of point $p$, but it is still present in the yaw dynamics. The structure and stability of the yaw dynamics...
are evaluated in the following section.

III. ANALYSIS OF YAW DYNAMICS

In this section, the structure and stability of the yaw dynamics of the system controlled by the flatness-based controller are evaluated. While previous studies have provided a linear analysis of the stability of the yaw dynamics [12], this work presents a nonlinear analysis that is valid in a wider range of conditions.

A. Structure of yaw dynamics

Several state transformations are introduced below that reveal the structure of the yaw dynamics and enable a nonlinear stability analysis. It is assumed that the flatness-based controller tracks a desired reference trajectory with fixed frame position $\bar{r}_p(t)$, velocity $\bar{v}_p(t)$, and acceleration $\bar{a}_p(t)$. The speed $v_p(t)$ and angle $\theta_p(t)$ of the reference trajectory are shown in Fig. 4 and computed as:

$$v_p = \sqrt{\bar{v}_p(t)^2 + \bar{\dot{y}}_p(t)^2}$$
$$\theta_p(t) = \tan^{-1}\left(\frac{\bar{\dot{y}}_p(t)}{\bar{v}_p(t)}\right)$$

The fixed frame acceleration components $\bar{X}_p(t)$ and $\bar{\dot{Y}}_p(t)$ are expressed in terms of $v_p(t)$, $\theta_p(t)$, and derivatives as:

$$\bar{X}_p(t) = u_1(t) = \bar{v}_p(t) \cos \theta_p(t) - v_p(t) \bar{\dot{\theta}}_p(t) \sin \theta_p(t)$$
$$\bar{Y}_p(t) = u_2(t) = \bar{v}_p(t) \sin \theta_p(t) + v_p(t) \bar{\dot{\theta}}_p(t) \cos \theta_p(t)$$

so that the yaw dynamics are re-expressed as:

$$\dot{\psi} = \frac{x, F_{xz}}{I_{zz}} \left(\bar{v}_p(t) \sin \beta_p + \frac{v_p(t) \bar{\dot{\theta}}_p(t)}{g} \cos \beta_p - \frac{F_{zy}(\alpha_p)}{F_{zz}}\right)$$

where $\beta_p$ is the slip angle at point $p$ shown in Fig. 4 and defined as:

$$\beta_p(\psi, t) = \theta_p(t) - \psi$$

The angle $\gamma$ is defined as the difference between slip angles $\gamma = \beta_p - \alpha_p$. Some identities for $\gamma$ are provided below with $x_p = x + x_p$. Note that $\gamma$ is small when the scaled yaw rate $x_p \psi$ is small relative to the desired speed $v_p(t)$.

$$\gamma(\beta_p, \psi, t) = \tan^{-1}\left(\frac{x_p \psi \cos \beta_p}{v_p(t) - x_p \psi \sin \beta_p}\right)$$

$$\gamma(\alpha_p, \beta_p, t) = \sin^{-1}\left(\frac{x_p \bar{\dot{\theta}}_p(t) - \beta_p \cos \alpha_p}{v_p(t)}\right)$$

Using these definitions, the states of the yaw dynamics are transformed from $\psi, \psi$ to $\alpha_p, \beta_p$ using the definition of $\gamma$ and $\beta_p$. It can be shown that this state transformation is a valid diffeomorphism provided that the following conservative condition is met:

$$\cos \gamma \neq 0 \Rightarrow \left|\frac{x_p + x_p \psi}{v_p(t)}\right| < 1$$

The yaw dynamics are partitioned into autonomous and non-autonomous terms as:

$$\alpha_p, \beta_p, t) = \beta_p - \frac{f_3(\alpha_p) - f_4(\alpha_p, \beta_p, t)}{v_p(t)}$$

$$\beta_p = \frac{f_1(\alpha_p, \beta_p, t)}{v_p(t)} - f_2(\alpha_p, \beta_p, t) + \bar{\dot{\theta}}_p(t)$$

where $v_p$ is the speed at the rear wheel.

$$v_p(t) = v_p(t) \cos \gamma(\alpha_p, \beta_p, t) - x_p \psi(\beta_p, t) \sin \alpha_p$$

$$\psi(\beta_p, t) = \theta_p(t) - \beta_p$$

and

$$f_1(\alpha_p) = -\frac{x_p F_{xz}}{I_{zz}} \frac{F_{zy}(\alpha_p)}{F_{zz}}$$

$$f_2(\alpha_p, \beta_p, t) = \frac{x_p F_{xz}}{I_{zz}} \frac{v_p(t) \bar{\dot{\theta}}_p(t)}{g} \cos \alpha_p + \gamma(\alpha_p, \beta_p, t)$$

$$f_3(\alpha_p) = x_p \bar{\dot{\theta}}_p(t) \cos \beta_p$$

$$f_4(\alpha_p, \beta_p, t) = x_p \bar{\dot{\theta}}_p(t) \cos \alpha_p - \psi(\beta_p, t) \beta_p \sin \alpha_p$$

$$-v_p(t) \sin \gamma(\alpha_p, \beta_p, t)$$

The dynamics in (33)-(34) are similar to a Liénard system in the Liénard plane such as the system given below with states $x$ and $y$ [18, 19]. Many nonlinear oscillators can be expressed as Liénard systems, including the Van der Pol oscillator. Useful theorems and candidate Lyapunov functions are available to analyze the stability of Liénard systems, such as the candidate Lyapunov function $V$ below.

$$\dot{x} = \left|\begin{array}{c} y - g(x) \\ -p(x) \end{array}\right|$$

$$V(x, y) = \frac{1}{2} y^2 + \int_0^x p(s) ds$$

$$\dot{V}(x) = -p(x) q(x)$$

Although the yaw dynamics expressed in (33)-(34) don't precisely match the form of the Liénard system in (41), a Lyapunov function is adapted from $V(x, y)$ to characterize the stability of the yaw dynamics in the following sections.

B. Stability of unforced yaw dynamics

In this section, the stability of the yaw dynamics are assessed when tracking a straight-line trajectory at constant speed ($\bar{v}_p = \bar{\dot{\theta}}_p = 0$). In this case, $f_2$ and $f_1$ simplify to:

$$f_2 = 0$$

$$f_4(\alpha_p, \beta_p, t) = x_p \beta_p^2 \sin \alpha_p$$

and a candidate Lyapunov function $V(\alpha_p, \beta_p)$ is given as:

$$V(\alpha_p, \beta_p) = \frac{1}{2} \beta_p^2 + \frac{x_p F_{xz}}{I_{zz}} \Phi(\alpha_p)$$

$$\Phi(\alpha_p) = -\int_0^x \frac{F_{zy}(s)}{F_{zz}} ds$$

(47)
\[ V(\alpha, \beta) = -x_p \left( f_1(\alpha)^2 \cos \alpha + \beta^2 f_1(\alpha) \sin \alpha \right) \quad (48) \]

The function \( \Phi(\alpha) \) is plotted in Fig. 5 for the example tire model described in Section II. Both functions \( \Phi(\alpha) \) and \( V(\alpha, \beta) \) are locally positive definite at the origin.

\[ \sin \alpha = 0 \Rightarrow F_y(\alpha) = 0 \quad (49) \]
\[ \sin \alpha \neq 0 \Rightarrow F_y(\alpha) \sin \alpha < 0 \Rightarrow f_1(\alpha) \sin \alpha > 0 \quad (50) \]

which implies that for \( \cos \alpha > 0, \) \( V \) is non-increasing \( (V \leq 0) \) and \( V = 0 \) when \( \sin \alpha = 0. \) It can be proved using LaSalle's theorem [20] that the system will converge to a stable equilibrium point at the origin when

\[ V(\alpha, \beta) < V(\pi/2, 0) \quad (51) \]

This analysis is verified by simulating (33)-(34) with \( \theta_p = 0 \) for speeds of \( v_p = 5 \text{ m/s} \) and \( 35 \text{ m/s} \) and initial conditions of \( \alpha_r = -20 \text{ deg} \) and \( \beta_p = 0. \) A phase plane plot is given in Fig. 6 and a plot of the Lyapunov function in Fig. 7.

It can be seen that the unforced yaw dynamics converge to the origin of the phase space. Since \( \dot{V} \) is inversely proportional to the rear wheel speed \( v_r, \) the \( 35 \text{ m/s} \) trajectory requires more time to converge.

\[ \frac{\partial F_y}{\partial \alpha} \cos \alpha < 0 \quad (56) \]

By inspection of Fig. 3, it can be seen that (56) is satisfied for a small range of \( \alpha \) near the origin and between the peaks of the function \( F_y(\alpha) \).

A candidate Lyapunov function \( V_0(\alpha, \beta_p) \) is given as

\[ V_0(\alpha, \beta_p) = \frac{1}{2} \beta_p^2 + \Phi(\alpha) \quad (57) \]
\[ \Phi(\alpha_r) = \frac{x_r F_{cr}}{I_z} v_p \hat{\theta}_p \sin \alpha_r + \Phi(\alpha_r) \quad (58) \]

The function \( \Phi(\alpha_r) \) is plotted in Fig. 9 for several values of the lateral acceleration \( v_p \hat{\theta}_p \). It can be seen that \( \Phi(\alpha_r) \) can have zero, one, or more extrema away from the origin when \( v_p \hat{\theta}_p \) is nonzero. The minima near the origin for \( v_p \hat{\theta}_p = 0.1g, 0.2g \) correspond to stable equilibria, while the minima at large \( \alpha_r \) for \( v_p \hat{\theta}_p = 0.2g, 0.3g \) do not satisfy (56) and are not stable equilibria.

A sufficient condition for the existence of a stable equilibrium point near the origin is that the applied acceleration \( v_p \hat{\theta}_p \) not exceed the peak rear friction force as

\[ \left| \frac{v_p \hat{\theta}_p}{g} \right| < \mu_r \quad (59) \]

A sufficient condition for the stable equilibrium near the origin to be the only equilibrium point is given as

\[ \left| \frac{v_p \hat{\theta}_p}{g} - \frac{F_{yr}(\pi/2)}{F_{cr}} \right| < 0 \quad (60) \]

When the angle \( \gamma \) is small, it can be shown that the time derivative of \( V_\theta(\alpha_r, \beta_p) \) reduces to

\[ V_\theta = -x_p \cos \alpha_r x_r F_{cr} \left( \frac{v_p \hat{\theta}_p}{g} \cos \alpha_r - \frac{F_{yr}(\alpha_r)}{F_{cr}} \right) \left( \frac{v_p \hat{\theta}_p - F_{yr}(\alpha_r)}{g} \right) \]

which is negative semi-definite for \( \cos \alpha_r > 0 \) when (60) is satisfied except for a small area surrounding the equilibrium point.

This analysis is also verified by simulating (33)-(34) with \( v_p \hat{\theta}_p = 0.1g \) for speeds of \( v_p = 5 \text{ m/s} \) and \( 35 \text{ m/s} \) and initial conditions of \( \alpha_r = -20 \text{ deg} \) and \( \beta_p = 0 \). A phase plane plot is given in Fig. 10 and a plot of the Lyapunov function in Fig. 11.

At each speed, the yaw dynamics converge to the stable equilibrium point near the origin. It can be seen that the nonzero value of \( v_p \hat{\theta}_p \) causes the Lyapunov function contours to become distorted.

![Fig. 9. Plot of the function \( \Phi(\alpha_r) \) for several values of input \( v_p \hat{\theta}_p \) indicated as "ay" in the legend.](image)

![Fig. 10. Phase plane response of yaw dynamics with \( v_p \hat{\theta}_p = 0.1g \) with phase variables \( \alpha_r \) and \( \beta_p \). Contours of \( V_\theta \) are included.](image)

![Fig. 11. Lyapunov value for yaw dynamics with \( v_p \hat{\theta}_p = 0.1g \).](image)

The dynamics are also simulated for \( v_p \hat{\theta}_p = 0.2g \) at the same speeds. A phase plane plot is given in Fig. 12. For this input, there is a maxima of \( \Phi(\alpha_r) \) at about \( \alpha_r = -10 \text{ deg} \), which corresponds to an unstable equilibrium point. It can be seen that this unstable equilibrium point limits the basin of attraction of the stable equilibrium point near the origin.

![Fig. 12. Phase plane response of yaw dynamics with \( v_p \hat{\theta}_p = 0.2g \).](image)
D. Summary

A state transformation was performed to identify the structure in the yaw dynamics resembling a Liénard system. This structure motivated a choice of Lyapunov function, which was used to assess the stability of the yaw dynamics during straight and steady turning maneuvers. The analysis was verified with simulation results.

IV. Discussion

The trajectory tracking controller presented in Section II can track arbitrary bounded magnitude accelerations at the front center of oscillation provided a tire force controller with sufficient actuators. The acceleration magnitude limits are imposed by tire friction circle constraints. Additionally, to ensure that actuator constraints are satisfied, particularly the steering angle limit, the yaw dynamics must be bounded.

It was shown that during straight-line and steady turn maneuvers, the yaw dynamics converge to an equilibrium point near the origin, provided that the basin of attraction is not limited by the presence of an unstable equilibrium point. In future work, the Lyapunov analysis should be extended to consider the behavior of the yaw dynamics while tracking time-varying trajectories.

It was observed that the phase plane trajectories followed the Lyapunov function contours closely for high speeds. This implies that the Lyapunov function may be used to predict the peak slip angle expected during a high-slip maneuver under constant acceleration. Further study of the behavior of the yaw dynamics may be used to control the vehicle during aggressive maneuvers performed by expert drivers, such as drifting or trail-braking [21].

The flatness-based controller requires an intermediate controller that uses front wheel steering and traction/braking torques at the front and rear wheels to control the tire forces. While the details of this intermediate controller were not the focus of this work, it may be noted that the an accelerometer placed at the location of the flat output may provide useful feedback in implementing this controller even if direct measurement of the tire forces is not available [12].

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VI. Conclusion

The work presented in this paper allows the trajectory planning problem for a nonlinear bicycle to be reduced to that of planning for a point mass with yaw dynamics. The basin of attraction of the yaw dynamics during steady turns was shown to be limited by actuator constraints and the presence of unstable equilibria. An analysis of the yaw dynamics has also provided insight into the behavior of vehicle dynamics at high slip angles. Future work will address the stability of the yaw dynamics when tracking trajectories with time-varying acceleration and the use of energy shaping techniques for control of the yaw dynamics. The analysis will be extended to vehicles with all-wheel drive and rear-wheel drive as well.

References