Computational Effectiveness of Split Cuts for Second-order Conic Programming

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(* Joint work with Mustafa Kilinc, University of Pittsburgh, and Juan Pablo Vielma, Massachusetts Institute of Technology)

INFORMS Annual Meeting, Phoenix, 2012
Outline

• Introduction
• Conic Mixed-integer Rounding (MIR) cuts
• Conic split cuts
• Summary
Second-order Conic Programming

• Second-order (Quadratic) cone

\[ L^m = \left\{ (x_1, \ldots, x_{m-1}, x_m) \in \mathbb{R}^m : \sqrt{\sum_{i=1}^{m-1} x_i^2} \leq x_m \right\} \]
Second-order Conic Programming

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• Second-order conic mixed-integer programing (SOCMIP)

\[
\begin{align*}
\min & \quad c^T x + g^T y \\
\text{s.t.} & \quad \|A^i x + G^i y - b^i\| \leq f_i^T x + r_i^T y - d_i, \quad i = 1, \ldots, q \\
& \quad x \in \mathbb{Z}^n, \quad y \in \mathbb{R}^p
\end{align*}
\]
Split Cuts

- A convex set $C \subseteq \mathbb{R}^n$ and $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$
Split Cuts

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$C^{\pi, \pi_0}_L := \{x \in C : \langle \pi, x \rangle \leq \pi_0 \}$

$C^{\pi, \pi_0}_G := \{x \in C : \langle \pi, x \rangle \geq \pi_0 + 1 \}$
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$C_{\pi, \pi_0} := \text{conv} (C_{\pi, \pi_0}^L \cup C_{\pi, \pi_0}^G)$
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• A convex set $C \subseteq \mathbb{R}^n$ and $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$

\[ C^\pi_{\pi_0} := \{ x \in C : \langle \pi, x \rangle \leq \pi_0 \} \]

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\[ C^\pi_{\pi_0} := \text{conv} \left( C^\pi_{\pi_0} \cup C^\pi_{\pi_0} \right) \]

• Split cuts of $C$
  
  ▪ Any valid (linear or nonlinear) inequality for $C^\pi_{\pi_0}$ for some $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$
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- A convex set $C \subseteq \mathbb{R}^n$ and $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$
Previous Work on SOCMIP

• Second-order conic sets of the form

\[ LC(B, c) = \{(x, t_0) \in \mathbb{R}^n \times \mathbb{R}_+ : \|B(x - c)\| \leq t_0\} \]

\[ B \text{ is invertible} \]
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• Atamturk and Narayanan (2008)
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\[ |b_i (x - c)| \leq t_i, \quad i = 1, \ldots, n \]

\[ \|t\| \leq t_0, \]

where \( b_i \) denotes the \( i\)-th row of matrix \( B \).
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Linear Part \( |b_i(x - c)| \leq t_i, \quad i = 1, \ldots, n \)

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- Atamturk and Narayananan (2008)
  
  **Linear Part**  \[ |b_i (x - c)| \leq t_i, \quad i = 1, \ldots, n \]
  
  **Nonlinear Part**  \[ \|t\| \leq t_0, \]

  where \( b_i \) denotes the \( i \)-th row of matrix \( B \).
Previous Work on SOCMIP

• Second-order conic sets of the form

\[ LC(B, c) = \{(x, t_0) \in \mathbb{R}^n \times \mathbb{R}_+ : \|B(x - c)\| \leq t_0 \}\]

• Other quadratic sets
  - (See Belotti, Goez, Polik, Ralphs, Terlaky 2012)

• Split cuts for ellipsoids
  - (See Dadush, Dey, Vielma 2011)
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• MD01-West 101-CC, Monday, 16:50 – 17:10
Proposition (Simple Conic MIR). Let 

\[ S_0 = \{(x, t) \in \mathbb{Z} \times \mathbb{R}_+ : |x - b| \leq t\}, \]

and \( f = b - \lfloor b \rfloor \). Then

\[ (1 - 2f)(x - \lfloor b \rfloor) + f \leq t \]

is valid for \( S_0 \) and \( \text{conv}(S_0) = \{(x, t) \in \mathbb{R} \times \mathbb{R}_+ : |x - b| \leq t, (1)\}. \)
Conic MIR Cuts

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• More general cuts using a superadditive function
Conic MIR Cuts and Linear Split Cuts

\[ LC(B, c)^H := \{(x, t, t_0) \in \mathbb{Z}^n \times \mathbb{R}_+^n \times \mathbb{R}_+ : |B(x - c)| \leq t_0\} \]
Conic MIR Cuts and Linear Split Cuts

\[ LC(B, c)^H := \{(x, t, t_0) \in \mathbb{Z}^n \times \mathbb{R}_+^n \times \mathbb{R}_+ : |B(x - c)| \leq t_0\} \]

\[ \pi \in \mathbb{Z}^n, \mu \in \mathbb{R}^n \text{ such that } B^T \mu = \pi, \pi^T c \notin \mathbb{Z} \]

\[ (1 - 2f)(\pi^T x - \lfloor \pi^T c \rfloor) + f \leq |\mu|^T t \]
Conic MIR Cuts and Linear Split Cuts

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- Equivalency between all versions of conic MIR cuts (simple and superadditive) and linear split cuts
Conic MIR Cuts and Linear Split Cuts

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\[ (1 - 2f)(\pi^T x - [\pi^T c]) + f \leq |\mu|^T t \]

- Equivalency between all versions of conic MIR cuts (simple and superadditive) and linear split cuts
- Also true for conic MIR with non-negativity
- A single conic MIR per disjunction
Projected Conic MIR

• Conic MIR cut might project to nonlinear cuts

\[ S = \left\{ (x, y, t_0) \in \mathbb{Z} \times \mathbb{R} \times \mathbb{R}_+ : \sqrt{(x - b)^2 + y^2} \leq t_0 \right\} \]

• Simple nonlinear conic MIR inequality

\[ \sqrt{((1 - 2f)(x - \lfloor b \rfloor) + f)^2 + y^2} \leq t_0 \]
Projected Conic MIR

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• Simple nonlinear conic MIR inequality

\[ \sqrt{((1 - 2f)(x - \lfloor b \rfloor) + f)^2 + y^2} \leq t_0 \]

• Not true in general!
Nonlinear Split Cuts

\[ LC(B, c)^{\pi, \pi_0} := \{(x, t_0) \in \mathbb{R}^n \times \mathbb{R}_+: \|B(x - c)\| \leq t_0, \]
Nonlinear Split Cuts

\[ LC (B, c)^{\pi, \pi_0} := \{(x, t_0) \in \mathbb{R}^n \times \mathbb{R}_+ : \|B(x - c)\| \leq t_0, \|A_{B,\pi,c}x - b_{B,\pi,c}\| \leq t_0\} \]
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Original Set
Nonlinear Split Cuts

\[ LC(B, c)^{\pi_0} := \{(x, t_0) \in \mathbb{R}^n \times \mathbb{R}_+ : \| B(x - c) \| \leq t_0, \| A_{B,\pi,c}x - b_{B,\pi,c} \| \leq t_0\} \]
Nonlinear Split Cuts vs. Conic MIR Cuts

• A single split disjunction
Nonlinear Split Cuts vs. Conic MIR Cuts

• A single split disjunction
  ▪ Nonlinear split cut (strictly) dominates conic MIR cut
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• A group of split disjunctions
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• A group of split disjunctions
  ▪ More complicated!
Nonlinear Split Cuts vs. Conic MIR Cuts

• A single split disjunction
  ▪ Nonlinear split cut (strictly) dominates conic MIR cut

• A group of split disjunctions
  ▪ More complicated!
  ▪ Combination of conic MIR cuts in the extended formulation $\Rightarrow$ More strength
A Group of Split Disjunctions (Example)

\[
LC(B, c) = \{(x, t_0) \in \mathbb{R}^n \times \mathbb{R}_+ : \|x - c\| \leq t_0\}
\]

\[
c_i = 1/2, \; i \in \{1, \ldots, n\}
\]
A Group of Split Disjunctions (Example)

\[ LC(B, c) = \{(x, t_0) \in \mathbb{R}^n \times \mathbb{R}_+ : \|x - c\| \leq t_0\} \]
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- Elementary conic MIR cuts give the conic MIR closure
A Group of Split Disjunctions (Example)

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\[ z_0^* = \min_{x} \{ t_0 : \|x - c\| \leq t_0, \; x \in \mathbb{Z}^n \} \]
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- Elementary conic MIR cuts give the conic MIR closure

\[ z_0^* = \min_{x} \{ t_0 : \|x - c\| \leq t_0, \ x \in \mathbb{Z}^n \} \]

Minimizing \( t_0 \) over conic MIR closure

\[ z_{cmir}^* = \sqrt{n/2} = z_0^* \]
A Group of Split Disjunctions (Example)

\[(\bar{x}, \bar{t}_0), \quad \bar{x}_i = 1/2, \quad i \in \{1, \ldots, n\}, \quad \bar{t}_0 = 1/2\]
A Group of Split Disjunctions (Example)

\((\bar{x}, \bar{t}_0), \bar{x}_i = 1/2, i \in \{1, \ldots, n\}, \bar{t}_0 = 1/2\)

\((\bar{x}, \bar{t}_0) \notin \text{conic MIR closure}\)
A Group of Split Disjunctions (Example)

\[(\bar{x}, \bar{t}_0), \quad \bar{x}_i = 1/2, \quad i \in \{1, \ldots, n\}, \quad \bar{t}_0 = 1/2\]

\[(\bar{x}, \bar{t}_0) \notin \text{conic MIR closure}\]

\[(\bar{x}, \bar{t}_0) \in \text{split closure}\]

Conic MIR closure
A Group of Split Disjunctions (Example)

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\[(\bar{x}, \bar{t}_0) \in \text{split closure}\]

\[\tilde{z}^*_{\text{split}} \leq 1/2\]
A Group of Split Disjunctions (Example)

\((\bar{x}, \bar{t}_0), \bar{x}_i = 1/2, \ i \in \{1, \ldots, n\}, \ \bar{t}_0 = 1/2\)

\((\bar{x}, \bar{t}_0) \notin \text{conic MIR closure}\)

\((\bar{x}, \bar{t}_0) \in \text{split closure}\)

\(\bar{z}_{\text{split}}^* \leq 1/2 < \bar{z}_{\text{cmir}}^* = \sqrt{n}/2\)

Conic MIR closure
A Group of Split Disjunctions (Example)

- Split cuts can still cut off points from the sides of the conic MIR closure

$$\pi = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Nonlinear split cut

Conic MIR closure
Computational Experiments

• Closest Vector Problem (CVP)

\[
\min_x \left\{ \|B(x - c)\| : x \in \mathbb{Z}^n \right\}
\]
Computational Experiments

• Closest Vector Problem (CVP)

\[
\min_{x} \{ \| B(x - c) \| : x \in \mathbb{Z}^n \}
\]

\(B:\ \text{Uniformly at random in } \{-3, \ldots, 3\}\)

\(c:\ \text{Uniformly at random in } [-1, 1]\)
Computational Experiments

<table>
<thead>
<tr>
<th>n</th>
<th>#</th>
<th>Split Cuts (% gap closed)</th>
<th>Conic MIR Cuts (% gap closed)</th>
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<td></td>
<td></td>
<td>El</td>
<td>El + NEl</td>
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<td>10</td>
<td>45.73</td>
<td>77.05</td>
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Table 1: Performance of the nonlinear split cuts and conic MIR cuts when $n$ elementary and $n$ non-elementary disjunctions are added.
## Computational Experiments

### Table 1: Performance of the nonlinear split cuts and conic MIR cuts when \( n \) elementary and \( n \) non-elementary disjunctions are added

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Summary

• Equivalency between conic MIR cuts and linear split cuts
• Dominance of a single conic split cut over conic MIR
• Groups of conic MIR could be “strong”!