Introduction

We study split cuts for the second-order conic sets of the form

\[ \text{LC} (B, c) := \{ (x, t) \in \mathbb{R}^n \times \mathbb{R} : |B x - c| \leq t \}, \]

where \( B \) is an invertible matrix.

In the context of the same set, Atamtürk et al. introduced an extended formulation in a higher dimensional space. Introducing auxiliary variables \( y \in \mathbb{R}^m \), one can reformulate \( \text{LC} (B, c) \) as

\[ \begin{align*}
|B y - c| - t y_i & = 0, \\
(B x - c) y_i & = t, \\
y_i & \geq 0,
\end{align*} \]

where \( y_i \) denotes the \( i \)-th row of matrix \( B \) and vectors \( y_i \) respectively.

Simple conic MIR

Proposition (Simple Conic MIR). Let \( S_k = (c, t) \in \mathbb{R}^m \times \mathbb{R}^n : |(c, t)| \leq t \), and let \( f = b - k \). Then

\[ \sum_{(c, t) \in S_k} \phi(c, t) \leq 2(1 - f) \left( |b| - \frac{1}{2} |b - t| \right), \]

where \( f = b - k \) is valid for \( S_k \).

Conic aggregation: Let \( P = \{ x \in \mathbb{R}^n | y \in \mathbb{R}^m : |A x - b| \leq t \} \), and let \( \lambda, \mu \in \mathbb{R}^n \). Then

\[ \lambda T y + \mu T (A x - b) \leq \frac{1}{2} \lambda T \mu + (A x - b) T (\lambda + \mu). \]

Nonlinear Split Cuts

equivlancy of Conic MIR and Split Cuts

Proposition. Every exponential conic MIR is a split cut for \( S^m \).

Moreover, we can show that split cuts and conic MIR cuts are equivalent even in the absence of non-negativity for the integer variables.

Proposition. Let \( x \in \mathbb{R}^n \) be such that \( x^2 = \pi \) and \( \lambda \notin \mathbb{Z} \). Then

\[ (ax - b)^2 = \ell, \]

where \( \ell = (ax - b)^2 - b^2 \). This proposition shows that split cuts can be obtained by applying the simple conic MIR to the simple aggregation in the next lemma.

Lemma. Let \( \lambda \in \mathbb{R}^n \). Then

\[ |x| |(\lambda x - b)| \leq |\lambda|^T t \]

is a valid inequality for \( P \).

Numerical Experiments

We report some preliminary numerical results for the closest vector problem of the form

\[ \min_x \| x - c \| \quad \text{subject to} \quad B x = 0. \]

Matrix \( B \) is generated uniformly at random in \([-1, 1]^n \). We report the averaged gap closed (in percent), which is the amount that the integrality gap is closed after adding the cuts. In the next table, \( d \) denotes the number of instances, \( NSC \) problem denotes the original problem after adding the nonlinear split cuts, conic MIR problem denotes the original problem after adding the conic MIR cuts, and \( NEI \) denotes the elementary cuts and NEI denotes non-elementary cuts.

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Future Work

Split cuts can be extended to more general conic sets such as \( p \)-order cones.

Proposition. Let \( \{ x \in \mathbb{R}^n | y \in \mathbb{R}^m : |A x - b| \leq t \} \), and let \( \lambda, \mu \in \mathbb{R}^n \). Then

\[ \lambda T y + \mu T (A x - b) \leq \frac{1}{2} \lambda T \mu + (A x - b) T (\lambda + \mu). \]

Proposition. Let \( \{ x : \| x \| \leq \pi \} \) be the second-order cone. Also let \( \| \cdot \| \) denotes the \( p \)-th unit vector. Then

\[ \{ x : \| x \| \leq \pi \} \]

is a valid inequality for \( P \).

References


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Split Cuts for Conic Programming

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