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Problem Set No.8 Due: Nov 4, 2008

Problem 1: Birkhoff-von Neumann Switch
An input-output matching in $N \times N$ input-buffered crossbar switch can be described as a permutation matrix $P = [p_{ij}, \forall i, j]$, i.e., $p_{ij} \in \{0, 1\}$, and $\sum_i p_{ij} = 1, \forall j$ and $\sum_j p_{ij} = 1, \forall i$ (See Figure 1). Let r_{ij} be the arrival rate of the traffic from input port i to output port j.

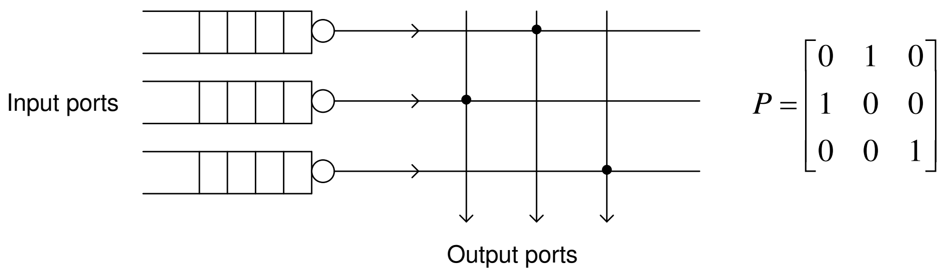

Figure 1: 3×3 input-queued switch

Assume the following:

$$\sum_{i=1}^{N} r_{ij} \le 1, \forall j \quad (1)$$

$$\sum_{j=1}^{N} r_{ij} \le 1, \forall i. \quad (2)$$

A matrix $R = [r_{ij}, \forall, i, j]$ satisfying the above inequality is said to be *doubly substochastic*, and if all the inequalities are met with equality, then it is called *doubly stochastic*. The following two results were respectively shown by von Neumann and Birkhoff:

- Von Neumann: If a matrix R is doubly substochastic, then there exists a doubly stochastic matrix $\tilde{R}$ such that $r_{ij} \le \tilde{r}_{ij}, \forall i, j$.
- Birkhoff: For a doubly stochastic matrix $\tilde{R}$, there exists a set of positive numbers ϕ_k and permutation matrices P_k such that

$$\tilde{R} = \sum_k \phi_k P_k \quad (3)$$

.

a) Prove in Birkhoff that $\sum_k \phi_k = 1$ (Hint: Multiply both side of (3) by a column vector with all its elements being 1).

Application to Switching: Such a set of positive numbers ϕ_k and permutation matrices $P_k, k = 1, ..., K$ can be found for some $K \leq N^2 - 2N + 2$, i.e.,

$$R \leq \tilde{R} = \sum_{k=1}^{K} \phi_k P_k \tag{4}$$

$$\sum_{k=1}^{K} \phi_k = 1. \tag{5}$$

This result implies that if the fraction of time that the matching P_k is used is ϕ_k, then the input traffic satisfying (1) and (2) with strict inequality can be stably supported. We will prove this argument more rigorously. First, consider the following switching algorithm using this concept:

1. Initialization: Generate a token for each class $k, k = 1, ..., K$. Assign the virtual finishing time of the first token of class k by

   $$F_k^1 = \frac{1}{\phi_k}.$$

   Sort these K tokens in an increasing order of the virtual finishing times.

2. The switch serves the tokens in an increasing order of the virtual finishing times. Only one token is served in a time slot. If it is decided that a token of class k should be served, then the switching connection pattern is set to the matching corresponding to P_k.

3. Once the $(l-1)$-th token of class k is served, the switch generates the l-th token of class k and assign the virtual finishing time by

   $$F_k^l = F_k^{l-1} + \frac{1}{\phi_k}.$$

   Insert the newly generated token in the sorted token list and repeat the algorithm from step 2.

b) Prove that the total number of tokens that have virtual finishing times not greater than F_k^l is given by

$$l + \sum_{j \neq k} \lfloor (l\phi_j)/\phi_k \rfloor,$$

where $\lfloor a \rfloor$ denotes the greatest integer less than or equal to a. (Hint: Use the fact $F_k^l = \frac{l}{\phi_k}$.)

c) Let τ_k^l be the index of the time slot that the l-th token of class k is served. Use the result in b) to prove:

$$\tau_k^l \leq F_k^l.$$

d) Let $D_k(t)$ be the cumulative number of class k tokens served by time t. Then, it follows $D_k(t) = \sup\{l : \tau_k^l \leq t\}$. Use the result in c) to prove:

$$D_k(t) \geq \lfloor \phi_k t \rfloor.$$

(Hint: Again, use the fact $F_k^l = \frac{l}{\phi_k}$.)

e) Let p_{kij} be the element at the (i, j) position of the permutation matrix P_k. Note that p_{kij} only takes values 0 and 1. Define $E_{ij} = \{k : p_{kij} = 1\}$ to be the subset of $\{1, 2, ..., K\}$ such that for all k in E_{ij}, the permutation matrix P_k has a nonzero element at the (i, j) position. Use the result in d) to prove:

$$\sum_{k \in E_{ij}} D_k(t) \leq \sum_{k \in E_{ij}} \phi_k t + (K - |E_{ij}|), \forall t.$$

(Hint: Use the fact $\sum_{k=1}^{K} D_k(t) = t, \forall t$, and divide it into two groups of E_{ij} and non-E_{ij}. Then, use the fact $\lfloor a \rfloor \geq a - 1$)

f) Let $C_{ij}(t)$ be the cumulative number of time slots that are assigned to the traffic from input i to output j by time t. Then, it follows $C_{ij}(t) = \sum_{k \in E_{ij}} D_k(t)$. Use the results in d) and e) to prove:

$$C_{ij}(t) - C_{ij}(s) \geq \sum_{k \in E_{ij}} \phi_k (t - s) - K.$$

g) Assume that the inequalities (1) and (2) strictly hold, i.e.,

$$\sum_{i=1}^{N} r_{ij} < 1, \forall j \tag{6}$$

$$\sum_{j=1}^{N} r_{ij} < 1, \forall i. \tag{7}$$

Explain that the result in f) implies the stability of the switching algorithm, given that the arrival rates satisfy (6) and (7). (Hint: Note that K is just a constant such that $K \leq N^2 - 2N + 2$)