Lecture 7:

Burke’s Theorem and Networks of Queues

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Burke’s Theorem

- An interesting property of an M/M/1 queue, which greatly simplifies combining these queues into a network, is the surprising fact that the output of an M/M/1 queue with arrival rate $\lambda$ is a Poisson process of rate $\lambda$
  - This is part of Burke's theorem, which follows from reversibility

- A Markov chain has the property that
  - $P[\text{future} \mid \text{present, past}] = P[\text{future} \mid \text{present}]$
    Conditional on the present state, future states and past states are independent

\[
P[\text{past} \mid \text{present, future}] = P[\text{past} \mid \text{present}]
\]

=> $P[X_n = j \mid X_{n+1} = i, X_{n+2} = i_2, \ldots] = P[X_n = j \mid X_{n+1} = i] = P^*_{ij}$
Burke’s Theorem (continued)

- The state sequence, run backward in time, in steady state, is a Markov chain again and it can be easily shown that
  \[ p_i P^*_{ij} = p_j P_{ji} \]
  e.g., M/M/1 \((p_n)\lambda=(p_{n+1})\mu\)

- A Markov chain is reversible if \(P^{*ij} = P_{ij}\)
  - Forward transition probabilities are the same as the backward probabilities
  - If reversible, a sequence of states run backwards in time is statistically indistinguishable from a sequence run forward

- A chain is reversible iff \(p_i P_{ij}=p_j P_{ji}\)

- All birth/death processes are reversible
  - Detailed balance equations must be satisfied
Implications of Burke’s Theorem

- Since the arrivals in forward time form a Poisson process, the departures in backward time form a Poisson process.

- Since the backward process is statistically the same as the forward process, the (forward) departure process is Poisson.

- By the same type of argument, the state (packets in system) left by a (forward) departure is independent of the past departures.
  - In backward process the state is independent of future arrivals.
NETWORKS OF QUEUES

- The output process from an $M/M/1$ queue is a Poisson process of the same rate $\lambda$ as the input.

- Is the second queue $M/M/1$?
Independence Approximation

(Kleinrock)

• Assume that service times are independent from queue to queue
  – Not a realistic assumption: the service time of a packet is determined by its length, which doesn't change from queue to queue

• $X_p$ = arrival rate of packets along path p

• Let $\lambda_{ij}$ = arrival rate of packets to link (i,j)

• $\mu_{ij}$ = service rate on link (i,j)

$$\lambda_{ij} = \sum_{P \text{ traverses link } (i,j)} X_p$$
Kleinrock approximation

- Assume all queues behave as independent M/M/1 queues

\[
N_{ij} = \frac{\lambda_{ij}}{\mu_{ij} - \lambda_{ij}}
\]

- \( N = \text{Ave. packets in network}, \ T = \text{Ave. packet delay in network} \)

\[
N = \sum_{i,j} N_{ij} = \frac{\lambda_{ij}}{\mu_{ij} - \lambda_{ij}}, \quad T = \frac{N}{\lambda}
\]

\[
\lambda = \sum_{\text{all paths}} X_p = \text{total external arrival rate}
\]

- Approximation is not always good, but is useful when accuracy of prediction is not critical
  - Relative performance but not actual performance matters
  - E.g., topology design
Slow truck effect

- Example of bunching from slow truck effect
  - long packets require long service at each node
  - Shorter packets catch up with the long packets

- Similar to phenomenon that we experience on the roads
  - Slow car is followed by many faster cars because they catch up with it
Jackson Networks

- **Independent external Poisson arrivals**
- **Independent Exponential service times**
  - Same job has independent service time at different queues
- **Independent routing of packets**
  - When a packet leaves node \( i \) it goes to node \( j \) with probability \( P_{ij} \)
  - Packet leaves system with probability \( 1 - \sum_j P_{ij} \)
  - Packets can loop inside network

- **Arrival rate at node \( i \),**
  
  \[
  \lambda_i = r_i + \sum_k \lambda_k P_{ki}
  \]
  
  - Set of equations can be solve to obtain unique \( \lambda_i \)'s
  - Service rate at node \( i = \mu_i \)
Jackson Network (continued)

- Customers are processed fast \((\mu >> \lambda)\)
- Customers exit with probability \((1-P)\)
  - Customers return to queue with probability \(P\)
  - \(\lambda = r + P\lambda \Rightarrow \lambda = r/(1-P)\)
- When \(P\) is large, each external arrival is followed by a burst of internal arrivals
  - Arrivals to queues are not Poisson
Jackson’s Theorem

- We define the state of the system to be \( \bar{n} = (n_1, n_2 \cdots n_k) \) where \( n_i \) is the number of customers at node \( i \).

- Jackson's theorem:

\[
P(\bar{n}) = \prod_{i=1}^{i=k} P_i(n_i) = \prod_{i=1}^{i=k} \rho_i^{n_i} (1 - \rho_i), \quad \text{where} \quad \rho_i = \frac{\lambda_i}{\mu_i}
\]

- That is, in steady state the state of node \( i \) (\( n_i \)) is independent of the states of all other nodes (at a given time):
  - Independent M/M/1 queues
  - Surprising result given that arrivals to each queue are neither Poisson nor independent
  - Similar to Kleinrock’s independence approximation
  - Reversibility
    Exogenous outputs are independent and Poisson
    The state of the entire system is independent of past exogenous departures

\[v_n = (n_1, n_2 \cdots n_k)\]
Example

\[ \lambda_1 = ? \]
\[ \lambda_2 = ? \]

\[ P(n_1, n_2) = ? \]