Lecture 7:

Burke's Theorem and Networks of Queues

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Burke’s Theorem

- An interesting property of an M/M/1 queue, which greatly simplifies combining these queues into a network, is the surprising fact that the output of an M/M/1 queue with arrival rate λ is a Poisson process of rate λ
  - This is part of Burke's theorem, which follows from reversibility

- A Markov chain has the property that
  - P[future | present, past] = P[future | present]

    Conditional on the present state, future states and past states are independent

    P[past | present, future] = P[past | present]

    $\Rightarrow P[X_n=j \mid X_{n+1}=i, X_{n+2}=i_2,...] = P[X_n=j \mid X_{n+1}=i] = P^*_{ij}$

Burke’s Theorem (continued)

- The state sequence, run backward in time, in steady state, is a Markov chain again and it can be easily shown that

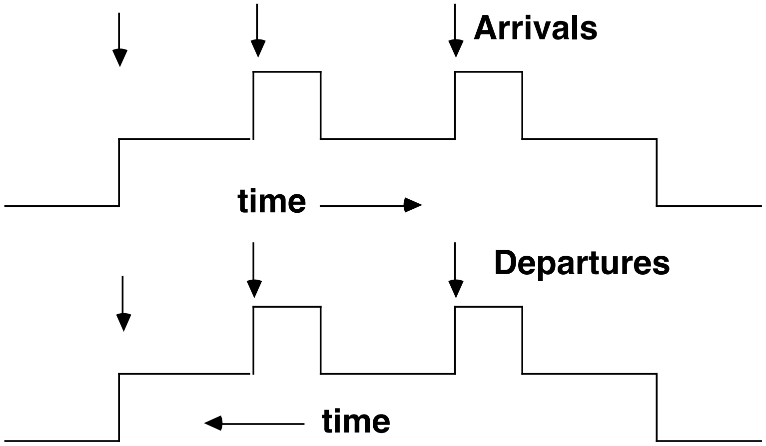
$p_iP^*_{ij} = p_jP_{ji}$ e.g., M/M/1 $(p_n)\lambda=(p_{n+1})\mu$

- A Markov chain is <u>reversible</u> if P*ij = Pij
  - Forward transition probabilities are the same as the backward probabilities
  - If reversible, a sequence of states run backwards in time is statistically indistinguishable from a sequence run forward

- A chain is reversible iff $p_iP_{ij}=p_jP_{ji}$

- All birth/death processes are reversible
  - Detailed balance equations must be satisfied

Implications of Burke’s Theorem

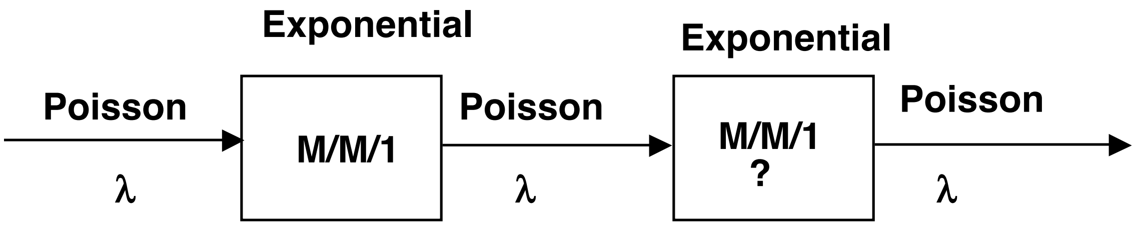


- Since the arrivals in forward time form a Poisson process, the departures in backward time form a Poisson process

- Since the backward process is statistically the same as the forward process, the (forward) departure process is Poisson

- By the same type of argument, the state (packets in system) left by a (forward) departure is independent of the past departures
  - In backward process the state is independent of future arrivals

NETWORKS OF QUEUES

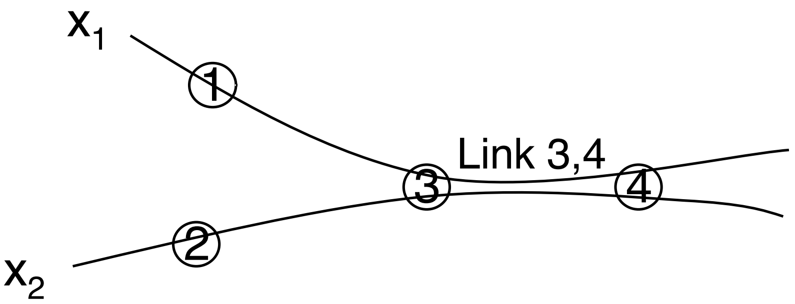


- The output process from an M/M/1 queue is a Poisson process of the same rate λ as the input

- Is the second queue M/M/1?

Independence Approximation (Kleinrock)

- Assume that service times are independent from queue to queue
  - Not a realistic assumption: the service time of a packet is determined by its length, which doesn't change from queue to queue



- X_p = arrival rate of packets along path p
- Let λ_{ij} = arrival rate of packets to link (i,j)
- μ_{ij} = service rate on link (i,j)

$$\lambda_{ij} = \sum_{\text{P traverses link (i,j)}} X_p$$

Kleinrock approximation

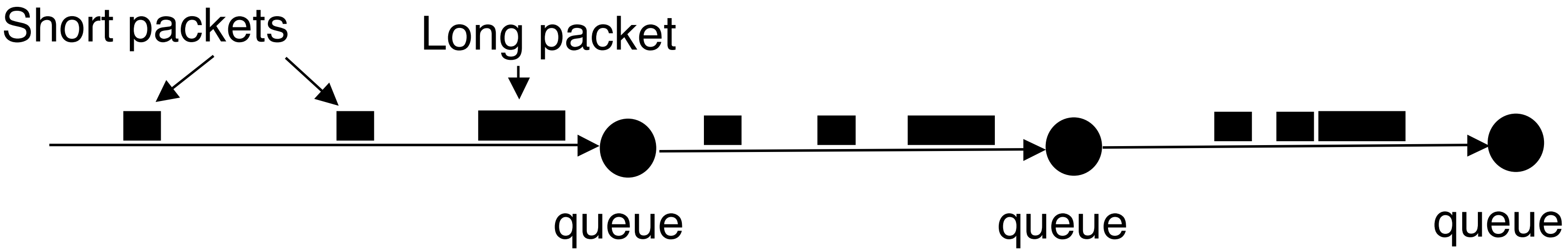
- Assume all queues behave as independent M/M/1 queues
- $$N_{ij} = \frac{\lambda_{ij}}{\mu_{ij} - \lambda_{ij}}$$
- N = Ave. packets in network, T = Ave. packet delay in network

$$N = \sum_{i,j} N_{ij} = \frac{\lambda_{ij}}{\mu_{ij} - \lambda_{ij}}, \qquad T = \frac{N}{\lambda}$$

$$\lambda = \sum_{all\ paths\ p} X_P = \text{ total external arrival rate}$$

- Approximation is not always good, but is useful when accuracy of prediction is not critical
  - Relative performance but not actual performance matters
  - E.g., topology design

Slow truck effect



- **Example of bunching from slow truck effect**
  - **long packets require long service at each node**
  - **Shorter packets catch up with the long packets**

- **Similar to phenomenon that we experience on the roads**
  - **Slow car is followed by many faster cars because they catch up with it**

Jackson Networks

- **Independent external Poisson arrivals**
- **Independent Exponential service times**
  - **Same job has independent service time at different queues**
- **Independent routing of packets**
  - **When a packet leaves node i it goes to node j with probability P_{ij}**
  - **Packet leaves system with probability** $1 - \sum_j P_{ij}$
  - **Packets can loop inside network**

- **Arrival rate at node i,**

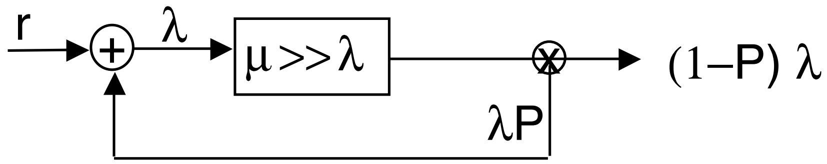
$$\lambda_i = r_i + \sum_k \lambda_k P_{ki}$$

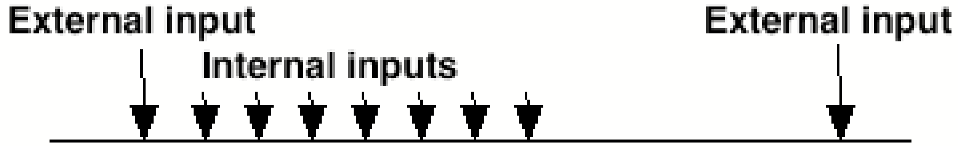
External arrivals

Internal arrivals from Other nodes

  - **Set of equations can be solve to obtain unique λ_i's**
  - **Service rate at node i = μ_i**

Jackson Network (continued)





- Customers are processed fast $(\mu >> \lambda)$
- Customers exit with probability (1-P)
  - Customers return to queue with probability P
  - $\lambda = r + P\lambda \Rightarrow \lambda = r/(1-P)$
- When P is large, each external arrival is followed by a burst of internal arrivals
  - Arrivals to queues are not Poisson

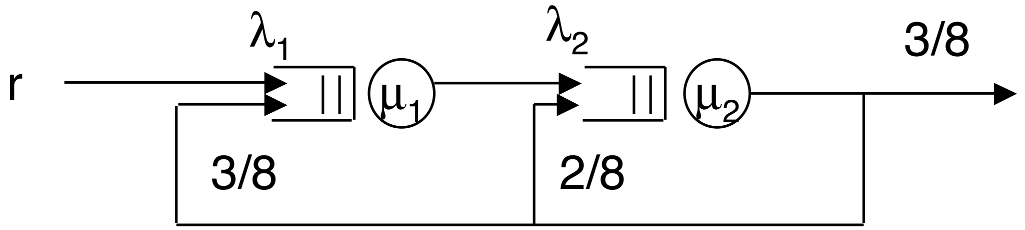
Jackson's Theorem

- We define the state of the system to be $\vec{n} = (n_1, n_2 \cdots n_k)$ where n_i is the number of customers at node i
- Jackson's theorem:

$$P(\vec{n}) = \prod_{i=1}^{i=k} P_i(n_i) = \prod_{i=1}^{i=k} \rho_i^{n_i} (1 - \rho_i), \qquad where\ \rho_i = \frac{\lambda_i}{\mu_i}$$

- That is, in steady state the state of node i (n_i) is independent of the states of all other nodes (at a given time)
  - Independent M/M/1 queues
  - Surprising result given that arrivals to each queue are neither Poisson nor independent
  - Similar to Kleinrock's independence approximation
  - Reversibility
    - Exogenous outputs are independent and Poisson
    - The state of the entire system is independent of past exogenous departures

Example



λ_1 = ?
λ_2 = ?

$P(n_1,n_2)$ = ?