
Reservations systems
M/G/1 queues with priority
Stability of queueing systems

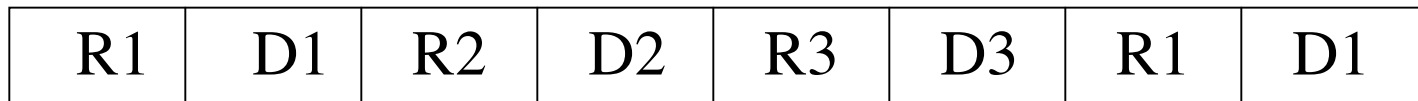
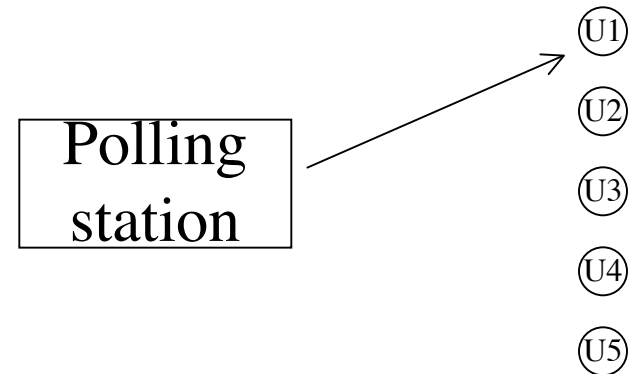
Eytan Modiano
MIT

RESERVATION SYSTEMS

- **Single channel shared by multiple users**
- **Only one user can use the channel at a time**
- **Need to coordinate transmissions between users**

- **Polling systems**

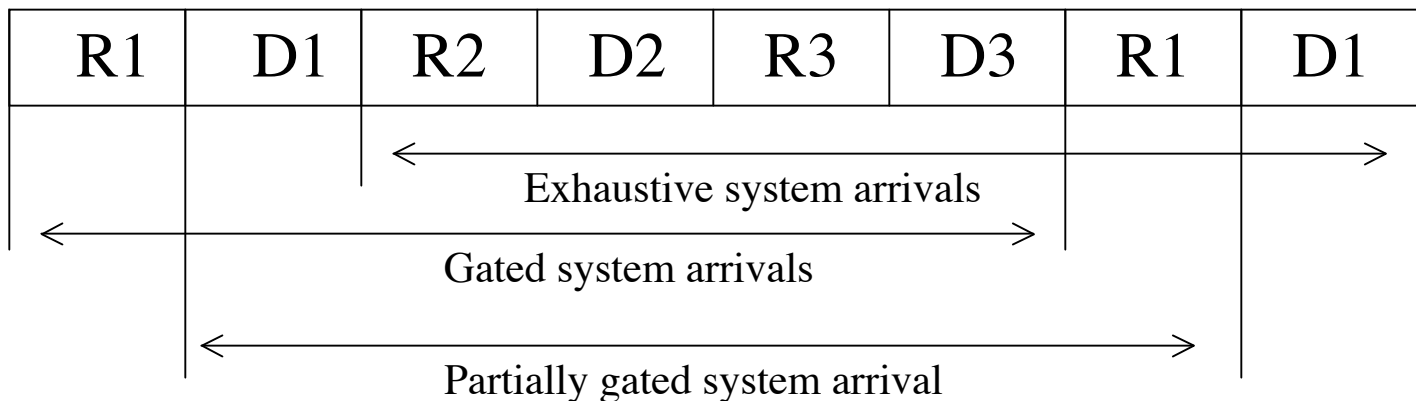
- **Polling station polls the users in order to see if they have something to send**
- **A scheduler can be used to receive and schedule transmission requests**



- **Reservation interval (R) used for polling or making reservations**
- **Data interval (D) used for the actual data transmission**

Reservations and polling systems

- **Gated system** - users can transmit only those packets that arrived prior to start of reservation interval
 - E.g., explicit reservations
- **Partially gated system** - Can transmit all packets that arrived before the start of the data interval
- **Exhaustive system** - Can transmit all packets that arrive prior to the end of the data interval
 - E.g., token ring networks
- **Limited service system** - only one (K) packets can be transmitted in a data interval



Single user exhaustive systems

- Let V_j be the duration of the j^{th} reservation interval
 - Assume reservation intervals are iid
- Consider the i^{th} data packet:

$$E[W_i] = R_i + E[N_i]/\mu$$

R_i = residual time for current packet or reservation interval

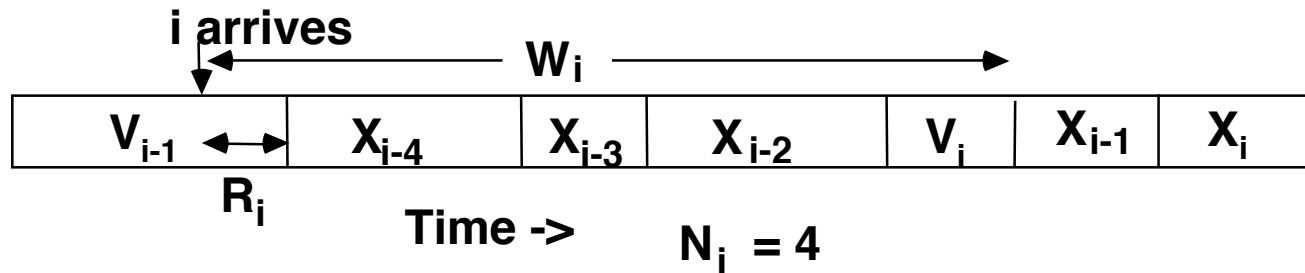
N_i = Number of packets in queue

- Identical to M/G/1 with vacations

$$W = \frac{\lambda E[X^2]}{2(1-\rho)} + \frac{E[V^2]}{2E[V]}$$

$$\text{When } V = A \text{ (constant)} \Rightarrow W = \frac{\lambda E[X^2]}{2(1-\rho)} + \frac{A}{2}$$

Single user gated system (e.g., reservations)



$$W_i = R_i + \sum_{j=i-N_i}^{i-1} X_j + V_i$$

$$E[W_i] = E[R_i] + E[N_i]E[X] + E[V]$$

$$W = R + N_Q E[X] + E[V] \quad (NQ = \lambda W)$$

$$W = (R + E[V]) / (1 - \rho)$$

SINGLE USER RESERVATION SYSTEM

- The residual service time is the same as in the vacation case,

$$R = \lambda \frac{E[X^2]}{2} + \frac{(1-\rho)E[V^2]}{2E[V]}$$

- Hence,

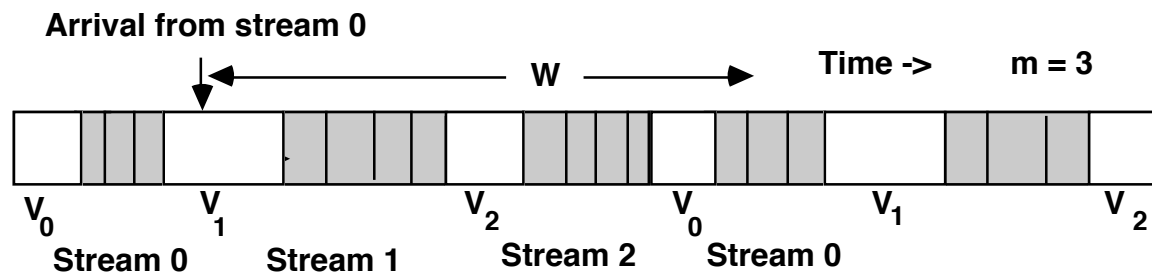
$$W = \lambda \frac{E[X^2]}{2(1-\rho)} + \frac{E[V^2]}{2E[V]} + \frac{E[V]}{1-\rho}$$

- If all reservation intervals are of constant duration A ,

$$W = \lambda \frac{E[X^2]}{2(1-\rho)} + \frac{A}{1-\rho} + \frac{A}{2}$$

Multi-user exhaustive system

- Consider m incoming streams of packets, each of rate λ/m
- Service times $\{X_n\}$ are IID and independent of arrivals with mean $1/\mu$, second moment $E[X^2]$.
- Server serves all packets from stream 0, then all from stream 1, ..., then all from $m-1$, then all from 0, etc.
- There is a reservation interval of fixed duration $V_i = V$ (for all i)



Multi-user exhaustive system

- Consider arbitrary packet i
- Let Y_i = the duration of whole reservation intervals during which packet i must wait ($E[Y_i] = Y$)

$$W = R + \rho W + Y$$

- Packet i may arrive during the reservation or data interval of any of the m streams with equal probability ($1/m$)
 - If it arrives during its own interval $Y_i = 0$, etc..., hence,

$$Y_i = \{iV \quad w.p. \quad 1/m \quad 0 \leq i < m$$

$$Y = E[Y_i] = \frac{V}{m} \sum_{i=0}^{m-1} i = \frac{V(m-1)}{2}$$

$$W = \frac{R + Y}{(1 - \rho)}, \quad R = \frac{(1 - \rho)V^2}{2V} + \frac{\lambda E[X^2]}{2}$$

Multi-user exhaustive system

$$W = \frac{(1-\rho)V + \lambda E[X^2] + V(m-1)}{2(1-\rho)},$$
$$= \frac{V}{2} + \frac{V(m-1)}{2(1-\rho)} + \frac{\lambda E[X^2]}{2(1-\rho)} = \frac{\lambda E[X^2]}{2(1-\rho)} + \frac{V(m-\rho)}{2(1-\rho)}$$

- In text, $V = A/m$ and hence,

$$W = \frac{A}{2m} + \frac{A(m-1)}{2m(1-\rho)} + \frac{\lambda E[X^2]}{2(1-\rho)} = \frac{\lambda E[X^2]}{2(1-\rho)} + \frac{A(1-\rho/m)}{2(1-\rho)}$$

Gated System

- When a packet arrives during its own reservation interval, it must wait m full reservation intervals

$$Y_i = \{iV \quad \text{w.p.} \quad 1/m \quad 1 \leq i \leq m$$

$$Y = E[Y_i] = \frac{V}{m} \sum_{i=1}^m i = \frac{V(m+1)}{2}$$

$$W = \frac{V}{2} + \frac{V(m+1)}{2(1-\rho)} + \frac{\lambda E[X^2]}{2(1-\rho)}$$

With $V = A/m$,

$$\frac{\lambda E[X^2]}{2(1-\rho)} + \frac{A}{2m} + \frac{A(1+1/m)}{2(1-\rho)} = \frac{\lambda E[X^2]}{2(1-\rho)} + \frac{A}{2} \left(\frac{1+(2-\rho)/m}{(1-\rho)} \right)$$

M/G/1 Priority Queueing

- **Priority classes 1, ..., n (class 1 highest and n lowest)**

$\lambda_k = \text{arrival rate for class } k$

$\mu_k = \text{service rate for class } k$

$E[X_k^2] = \text{second moment of service time (class } k)$

- **Non-preemptive system: Customer receiving service is allowed to complete service without interruption**

$$W_k = \frac{\sum_{i=1}^{i=n} \lambda_i E[X_i^2]}{2(1 - \rho_1 - \dots - \rho_{k-1})(1 - \rho_1 - \dots - \rho_k)}, \quad \rho_i = \frac{\lambda_i}{\mu_i}$$

- **Notice that the waiting time of high priority traffic is affected by lower priority traffic**

Preemptive-resume systems

- **When a higher priority customer arrives, lower priority customer is interrupted**
 - Service is resumed when no higher priority customers remain
 - Notice that the delay of high priority customers is no longer affected by that of lower priority customers
 - Preemption is not always practical and usually involves some overhead
- **Consider a class k arrival and let,**
 - W_k = waiting time for customers of class k or higher priority classes (1..K-1) already in the system
 - R_k = residual time for class k or higher customers
 - Notice that lower priority customers in service don't affect W_k because they are preempted
 - W_I = Waiting time for higher priority customers that arrive while priority k customer is already in the system
 - T_K = Average system time for priority K customer

$$T_k = W_k + W_I + 1/\mu_k$$

Preemptive-resume, continued...

$$W_k = \frac{R_k}{1 - \rho_1 - \dots - \rho_k}, \quad R_k = \frac{\sum_{i=1}^k \lambda_i E[X_i^2]}{2}$$

$$W_I = \sum_{i=1}^{k-1} (\lambda_i / \mu_i) T_k = \sum_{i=1}^{k-1} (\rho_i) T_k$$

$$T_k = \frac{1}{\mu_k} + \frac{R_k}{1 - \rho_1 - \dots - \rho_k} + T_k \sum_{i=1}^{k-1} \rho_i$$

$$T_k = \left(\frac{1}{\mu_k}\right) \frac{(1 - \rho_1 - \dots - \rho_k) + R_k}{(1 - \rho_1 - \dots - \rho_{k-1})(1 - \rho_1 - \dots - \rho_k)}$$

- **Notice independence of lower priority traffic**

Stability of Queueing Systems

- **Possible Definitions**

- **Average Delay is bounded**

$$E(\text{delay}) < \text{infinity}$$

- **Delay is finite with probability 1**

$$P(\text{delay} < \text{infinity}) = 1$$

- **Existence of a stationary occupancy distribution**

Occupancy does not drift to infinity

E(delay) < Infinity

- **Example: M/M/1 queue**

$$T = \frac{1}{\mu - \lambda} < \infty \quad \forall \lambda < \mu \Rightarrow \rho < 1$$

- **Example: M/G/1 queue**

$$T = \frac{1}{\mu} + \frac{\lambda E[X^2]}{2(1 - \rho)} < \infty \quad \text{if } (\rho < 1) \text{ and } (E[X^2] < \infty)$$

P(Delay < Infinity) = 1

- **Slightly weaker definition than E[delay] < infinity**
- **P(delay < infinity) = 1 even if E(delay) = infinity**

• **Example:**

$$f_D(d) = \frac{2}{\pi(1+d^2)}, d > 0$$

$$E[Delay] = \int_0^{\infty} \frac{2d}{\pi(1+d^2)} = \frac{\text{Log}[1+d^2]}{\pi} \Big|_0^{\infty} \Rightarrow \infty$$

$$P[Delay < x] = \int_0^x \frac{2}{\pi(1+d^2)} = \frac{2 \arctan(x)}{\pi} \xrightarrow{x \rightarrow \infty} 1$$

- **In general it can be shown that for any G/G/1 queue**
 - **Arrival and service time distributions may even be correlated!**

If $\lambda < \mu$, P(delay < Infinity) = 1 even if E(delay) not finite

Existence of a stationary occupancy distribution

- **Irreducible and Aperiodic Markov chain**

- $P_j > 0$ for all states $j \Rightarrow$ all states are visited infinitely often

- **Drift:**
$$D_i = E[X_{n+1} - X_n \mid X_n = i] = \sum_{k=-i}^{\infty} kP_{(i,i+k)}$$

- **When in state I ,**

- $D_i > 0 \Rightarrow$ state tends to increase

- $D_i < 0 \Rightarrow$ state tends to decrease

- **Intuitively, we don't want the state to drift to infinity, hence for large enough states the drift better get negative!**

- **Lemma: If $D_i < \text{infinity}$ for all i and for some $\delta > 0$ and $i' > 0$,**

- $D_i < -\delta$ for all $i > i'$, then the Markov chain has a stationary distribution

Irreducible: all states communicate (I.e., positive probability of getting from every state to every other state)

Periodic state : self transitions are possible only after a number of transitions (n)

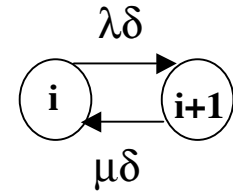
that is a multiple of some constant d (I.e., $n = 3, 6, 9, \dots$). Aperiodic \Rightarrow no state is periodic

Examples

- **M/M/1**

$$D_i = E[X_{n+1} - X_n | X_n = i] = 1(\lambda\delta) - 1(\mu\delta) = (\lambda - \mu)\delta$$

$$D_i < 0 \Rightarrow \lambda < \mu$$



- **M/M/m**

$$D_i = E[X_{n+1} - X_n | X_n = i] = 1(\lambda\delta) - 1(m\mu\delta) \quad \forall i \geq m$$

$$D_i < 0 \Rightarrow \lambda < m\mu \quad \forall i \geq m$$

- **M/M/Inf**

$$D_i = E[X_{n+1} - X_n | X_n = i] = 1(\lambda\delta) - 1(i\mu\delta)$$

$$D_i < 0 \Rightarrow \lambda < i\mu$$

For any $\lambda < \infty$ and $1/\mu < \infty \exists i'$ s.t., $D_i < 0 \forall i > i'$

