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Problem Set No.3 Solutions Posted: Oct 2, 2008

Problem 1: text problem 3.5

The expected time in the question equals

$$\begin{aligned} E\{\text{Time}\} &= (5 + E\{\text{stay of 2nd student}\})P(\text{1st stays less or equal to 5 minutes}) \\ &+ (E\{\text{stay of 1st}|\text{stay of 1st} \geq 5\} \\ &+ E\{\text{stay of 2nd student}\}P(\text{1st stays more than5minutes}). \end{aligned}$$

We have $E\{\text{stay of 1st}|\text{stay of 1st } \geq 5\} = 30$, and, using the memoryless property of the exponential distribution,

$$E\{\text{stay of 1st}|\text{stay of 1st} \geq 5\} = 5 + E\{\text{stay of 1st}\} = 35$$

Also,

$$P(\text{1st student stays less or equal to 5 minutes}) = 1 - e^{-5/30}$$
$$P(\text{1st student stays more than 5 minutes}) = e^{-5/30}$$

By substitution, we obtain

$$P(\text{Time}) = (5 + 30)(1 - e^{-5/30}) + (35 + 30)e^{-5/30} = 60.394$$

Problem 2: text problem 3.6

a) The probability that the person will be the last to leave is 1/4 because the exponential distribution is memoryless, and all customers have identical service time distribution. In particular, at the instant the customer enters service, the remaining service time of each of the other three customers served has the same distribution as the service time of the customer.
b) The average time in the bank is 1 (the average customer service time) plus the expected time for the first customer to finish service. The latter time is 1/4 since the departure process is statistically identical to that of a single server facility with with 4 times large

service rate. More precisely we have

$$\begin{aligned} P(\text{no customer departs in the next t mins}) &= P(\text{1st does not departs in the next t mins}) \\ &* P(\text{1st does not departs in the next t mins}) \\ &* P(\text{1st does not departs in the next t mins}) \\ &* P(\text{1st does not departs in the next t mins}) \\ &= (e^{-t})^4 = e^{-4t} \end{aligned}$$

Therefore,

$$P(\text{the first departure occurs within the next t mins}) = 1 - e^{-4t}$$

and the expected time to the next departure is 1/4. So the answer is 5/4 mins.
c) The answer will not change because the situation at the instant when the customer begins service will be the same under the conditions for (a) and the conditions for (c).

Problem 3: text problem 3.9

a) For each session the arrival rate is $\lambda = 150/60 = 2.5$ packets/sec. When the line is divided into 10 lines of capacity 5 Kbits/sec, the average packet transmission time is $1/\mu = 0.2$ sec. The corresponding utilization factor is $\rho = \lambda/\mu = .5$. We have for each session $N_Q = \rho^2/(1-\rho) = .5$, $N = \rho/(1-\rho) = 1$, and $T = N/\lambda = .4$ sec. For all sessions collectively N_Q and N must be multiplied by 10 to give $N_Q = 5$ and $N = 10$.

When statistical multiplexing is used, all sessions are merged into a single session with 10 times larger λ and μ; $\lambda = 25$, $\mu = .02$. We obtain $\rho = .5$, $N_Q = .5$, $N = 1$, and $T = .04$ sec. Therefore N_Q, N, and T have been reduced by a factor of 10 over the TDM case.
b) For the sessions transmitting at 250 packets/min we have $\rho = (250/60) * .2 = .833$ and we have $N_Q = .833^2/(1 - .833) = 4.158$, $N = 5$, $T = N/\lambda = 5/(250/60) = 1.197$ sec. For the sessions transmitting at 50 packets/min we have $\rho = (50/60) * .2 = .166$, $N_Q = 0.033$, $N = .199$, $T = .199/(50/60) = .239$.

The corresponding averages over all sessions are $N_Q = 5 * (4.158 + .033) = 21$, $N = 5 * (5 + .199) = 26$, $T = N/\lambda = N/(5\lambda_1 + 5\lambda_2) = 26/5/(250/60 + 50/60) = 1.038$ sec.

When statistical multiplexing is used the arrival rate of the combined session is $5*(250+50) = 1500$ packets/sec and the same value for N_Q, N and T as in (a) are obtained.

Problem 4: text problem 3.22

a) When all the courts are busy, the expected time between two departures is $40/5 = 8$ minutes. If a pair sees k pairs waiting in the queue, there must be exactly $k+1$ departures from the system before they get a court. Since all the courts would be busy during this whole time, the average waiting time required before $k+1$ departures is $8(k+1)$ minutes.

b) Let X be the expected waiting time given that courts are found busy. We have $\lambda = 1/10$, $\mu = 1/40$, and $\rho = \lambda/(5\mu) = .8$. By the M/M/m results, we have

$$W = \frac{\rho P_Q}{\lambda(1-\rho)}$$

Since $W = XP_Q$, we obtain $X = \rho/[\lambda(1-\rho)] = 40$ min.

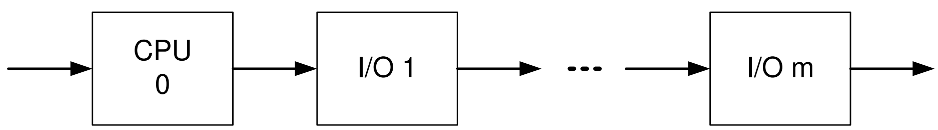
Problem 5: text problem 3.61

We have $\sum_{i=0}^{\infty} p_i = 1$. The arrival rate at the CPU is λ/p_0 and the arrival rate at the i^{th} I/O port is $\lambda p_i/p_0$. By Jackson's Theorem, we have

$$P(n_0, n_1, ..., n_m) = \prod_{i=0}^{m} \rho_i^{n_i}(1-\rho_i),$$

where $\rho = \frac{\lambda}{\mu_0 p_0}$ and $\rho_i = \frac{\lambda p_i}{\mu_i p_0}$ for $i > 0$.

The equivalent tandem system is as follows: The arrival rate is λ. The service rate for



queue 0 is $\mu_0 p_0$ and for queue $i (i > 0)$ is $\mu_i p_0/p_i$.

Problem 6: Derivation of Poisson Process from Random Variables

a) For $n = 1$, we can easily see that the formula for g_n holds because $S_1 = X_1$ whose density is $\alpha e^{-\alpha x}$. Assume $g_n(x) = \alpha\frac{(\alpha x)^{n-1}}{(n-1)!}e^{-\alpha x}$, then obtain the formula for g_{n+1}. Because $g_{n+1}(t) = \int_0^t g_n(t-x)g_1(x)dx$, it can be computed as:

$$\begin{aligned} g_{n+1}(t) &= \int_0^t \alpha\frac{(\alpha(t-x))^{n-1}}{(n-1)!}e^{-\alpha(t-x)} \cdot \alpha e^{\alpha x}dx \\ &= \alpha\frac{\alpha^n e^{-\alpha t}}{(n-1)!}\int_0^t (t-x)^{n-1}dx \\ &= \alpha\frac{(\alpha t)^n e^{-\alpha t}}{n!}, \end{aligned}$$

which verifies the formulation for g_n. Taking derivative of G_n yields g_n, and this completes the proof.

b) Let $N(t)$ be the number of the packets that have arrived up to time t. Then, the probability that $N(t) = n$ can be expressed as:

$$\Pr[N(t) = n] = \Pr[T_n \le t, T_{n+1} > t],$$

because T_n is the arrival time of the n-th packet. Let A and B denote the sample spaces of $\{T_n \leq t\}$ and $\{T_n > t\}$, respectively. Then, $\Pr[A \bigcap B]$ is the probability we want to find. It follows that

$$\begin{aligned}
\Pr[N(t) = n] &= \Pr[A \bigcap B] \\
&= \Pr[A - A \bigcap B^C] \\
&= \Pr[A] - \Pr[A \bigcap B^C] \\
&= \Pr[T_n \leq t] - \Pr[T_n \leq t, T_{n+1} \leq t] \\
&= \Pr[T_n \leq t] - \Pr[T_{n+1} \leq t] \\
&= G_n(t) - G_{n+1}(t) \\
&= \frac{(\alpha t)^n}{n!} e^{-\alpha t},
\end{aligned}$$

which shows that the packet arrival follows the Poisson process with rate α.