MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science
Department of Aeronautics and Astronautics

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Problem Set No.4 Solutions — Posted: Oct 7, 2008

Problem 1: text problem 3.30
From Little's theorem, we have that $P\{\text{the system is busy}\} = \lambda E\{X\}$.

Therefore, $P\{\text{the system is empty}\} = 1 - \lambda E\{X\}$.

The length of an idle period is the interarrival time between two typical customer arrivals. Therefore it has an exponential distribution with parameter λ, and its average length is $1/\lambda$.

Let B be the average length of a busy period and let I be the average length of an idle period. By expressing the proportion of time the system is busy as $B/(I + B)$ and also as $\lambda E\{X\}$, we obtain

$$B = E\{X\}/(1 - \lambda E\{X\})$$

From this, the expression $1/(1 - \lambda E\{X\})$ for the average number of customers served in a busy period is evident.

Problem 2: text problem 3.37

a)

$$\begin{aligned} \lambda &= 1/60 \\ E\{X\} &= 16.5 \\ E\{X^2\} &= 346.5 \\ W &= \lambda E\{X^2\}/(2(1 - \lambda E\{X\})) = 3.98 \end{aligned}$$

b) In the following, subscript 1 and 2 will imply the quantities for priority 1 and 2 customers, respectively.

$$\begin{aligned}
\lambda &= 1/60, \lambda_1 = 1/300, \lambda_2 = 1/75 \\
E\{X\} &= 16.5, E\{X_1\} = 4.5, E\{X_2\} = 19.5 \\
E\{X^2\} &= 346.5 \\
R &= .5\lambda E\{X^2\} = 2.88 \\
\rho_1 &= \lambda_1 E\{X_1\} = .015 \\
\rho_2 &= \lambda_2 E\{X_2\} = .26 \\
W_1 &= R/(1-\rho_1) = 2.93 \\
W_2 &= R/(1-\rho_2) = 4.043 \\
W &= \frac{\lambda_1 W_1 + \lambda_2 W_2}{\lambda}
\end{aligned}$$

Problem 3: text problem 3.43

We have

$$W = R + \rho W + W_B, \tag{1}$$

where the mean residual service time is

$$R = \frac{\lambda \bar{n} \bar{X^2}}{2} \tag{2}$$

We derive the average waiting time of a customer for other customers that arrived in the same batch

$$W_B = \sum_j r_j E\{W_B | \text{batch has size} j\}$$

where P_j = probability a batch has size j. r_j = Proportion of customers arriving in a batch of size j. We have $r_j = \frac{jP_j}{\bar{n}}$.

Since the customer is equally likely to be in any position within the batch,

$$E\{W_B | \text{batch has size} j\} = \sum_{k=1}^{j} (k-1)\bar{X}/j = \frac{j-1}{2}\bar{X}$$

Therefore

$$E\{W_B\} = \sum_j \frac{jP_j(j-1)\bar{X}}{2\bar{n}} = \frac{\bar{X}(\bar{n^2} - \bar{n})}{2\bar{n}} \tag{3}$$

By substituting (2) and (3) in (1), we obtain.

Problem 4: revised version of text problem 3.36

a).(1) For each session, the arrival rates, average transmission times and utilization factors for the short packets (class 1), and the long packets (class 2) are

$$\lambda_1 = 0.25 \text{ packets/sec}, \quad \frac{1}{\mu_1} = 0.02 \text{ secs}, \quad \rho_1 = 0.005$$
$$\lambda_2 = 2.25 \text{ packets/sec}, \quad \frac{1}{\mu_2} = 0.3 \text{ secs}, \quad \rho_2 = 0.675$$

The corresponding second moments of transmission time are

$$E[X_1^2] = 0.0004 \qquad E[X_2^2] = 0.09$$

The total arrival rate for each session is $\lambda = 2.5$ packets/sec. The overall 1st and 2nd moments of the transmission time, and overall utilization factors are given by

$$1/\mu = 0.1 * (1/\mu_1) + 0.9 * (1/\mu_2) = 0.272$$
$$E[X^2] = 0.1 * E[X_1^2] + 0.9 * E[X_2^2] = 0.081$$
$$\rho = \lambda/\mu = 2.5 * 0.272 = 0.68$$

We obtain the average time in queue W via the P-K formula $W = (\lambda E[X^2])/(2 * (1 - \rho)) = 0.3164$. The average time in the system is $T = 1/\mu + W = 0.588$. The average number in queue and in the system are $N_Q = \lambda W = 0.791$, and $N = \lambda T = 1.47$.

a).(2) The arrival rate increases by a factor of 10, and the service rate increases by a factor 10. Consequently, the load factor remains the same. In addition, the second moment decreases by a factor of 100. Combining all these together, we obtain $N_Q = 0.791$, $N = 1.47$, and $T = 0.588$ sec. Because these is the system backlogs and delay, the backlogs and delay of each session will decrease by a factor of 10, i.e., $N_Q = 0.0791$, $N = 0.147$, $T = 0.0588$ sec.

b).(1) In the nonpreemptive priority case, we obtain using the corresponding formulas

$$W_1 = (\lambda_1 E[X_1^2] + \lambda_2 E[X_2^2])/(2 * (1 - \rho_1)) = 0.108$$
$$W_2 = (\lambda_1 E[X_1^2] + \lambda_2 E[X_2^2])/(2 * (1 - \rho_1) * (1 - \rho_1 - \rho_2)) = 0.38$$
$$T_1 = 1/\mu_1 + W_1 = 0.128 \qquad T_2 = 1/\mu_2 + W_2 = 0.68$$
$$N_{Q1} = \lambda_1 * W_1 = 0.027 \qquad N_{Q2} = \lambda_2 * W_2 = 0.855$$
$$N_1 = \lambda_1 * T_1 = 0.032 \qquad N_2 = \lambda_2 * T_2 = 1.53$$

b).(2) As in a).(2), all the values (N_{Q1}, N_1, T_1) and (N_{Q2}, N_2, T_2) decrease by a factor of 10.

Problem 5: Tasting Lyapunov Stability Method

a) $q(t) - C$ is the remained amount of previously backlogged data at the end of time slot t, but because the backlog cannot be negative, it should be $\max[0, q(t) - C]$. Meanwhile, the data $A(t)$ arrives during slot t, but it cannot be immediately served. Hence, we have the following queue update equation:

$$q(t+1) = \max[0, q(t) - C] + A(t). \tag{4}$$

b) It follows from the queue update equation that

$$\begin{aligned}
&V(q(t+1)) - V(q(t)) \\
&= \tfrac{1}{2}q^2(t+1) - \tfrac{1}{2}q^2(t) \\
&= \tfrac{1}{2}\left(q(t+1) + q(t)\right)\left(q(t+1) - q(t)\right) \\
&= \tfrac{1}{2}\left(\max\left[q(t) - C, 0\right] + A(t) + q(t)\right)\left(\max\left[q(t) - C, 0\right] + A(t) - q(t)\right) \\
&= \tfrac{1}{2}A^2(t) + A(t)\max\left[q(t) - C, 0\right] + \tfrac{1}{2}\max^2\left[q(t) - C, 0\right] - \tfrac{1}{2}q^2(t) \\
&\leq \tfrac{1}{2}A^2(t) + A(t)\max\left[q(t) - C, 0\right] + \tfrac{1}{2}\left(q(t) - C\right)^2 - \tfrac{1}{2}q^2(t) \\
&= \tfrac{1}{2}A^2(t) + A(t)q(t) + \max\left[-A(t)C, -A(t)q(t)\right] + \tfrac{1}{2}C^2 - q(t)C \\
&\leq \tfrac{1}{2}A^2(t) + \tfrac{1}{2}C^2 + q(t)\left(A(t) - C\right) \\
&\leq \tfrac{1}{2}A_{\max}^2 + \tfrac{1}{2}C^2 + q(t)(A(t) - C).
\end{aligned}$$

The first inequality follows from the fact $(\max[0, X]) \leq X^2$, and the second inequality follows from the fact $\max[a, b] \leq 0$ for non-positive numbers a and b. The last inequality is obtained by using the boundedness of arrival and service rates.

c) Taking conditional expectation on the above inequality yields

$$\begin{aligned}
\Delta V(t) \leq &\; B + q(t)(E[A(t)|q(t)] - C) \\
= &\; B + q(t)(\lambda - C) \\
\leq &\; B + q(t)(C - \epsilon - C) \\
= &\; B - \epsilon q(t),
\end{aligned}$$

where we use the i.i.d. property (first equality) and the condition $\lambda < C (\Rightarrow \lambda + \epsilon \leq C)$.

d) Taking conditional expectation on the above inequality over the distribution of $q(t)$ yields the following:

$$E[V(q(t+1))] - E[V(q(t))] \leq B - \epsilon E[q(t)].$$

Summing the above inequality from $t = 0$ to $t = T - 1$ and applying the telescoping on the left hand side yield

$$E[V(q(T))] - E[V(q(0))] \leq BT - \epsilon \sum_{t=0}^{T-1} E[q(t)].$$

Because $V \geq 0$, the above equation leads to

$$\epsilon \sum_{t=0}^{T-1} E[q(t)] \leq BT + E[V(q(0))].$$

Dividing both side by ϵT and taking $\limsup_{T\to\infty}$ obtain

$$\limsup_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} E[q(t)] \leq \frac{B}{\epsilon}.$$

e) By Markov's inequality $\Pr[q > Q] \leq E[q]/Q$, we have the bound on $f(Q)$ as

$$\begin{aligned} f(Q) &= \limsup_{T\to\infty} \tfrac{1}{T}\sum_{t=0}^{T-1} \Pr[q(t) > Q] \\ &\leq \limsup_{T\to\infty} \tfrac{1}{T}\sum_{t=0}^{T-1} E[q(t)]/Q \\ &\leq \tfrac{B}{\epsilon Q}. \end{aligned}$$

As Q increases to infinity, the probability decreases to zero. This implies that the queue remains bounded with probability 1.