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Problem Set No.4 Due: Oct 7, 2008

Problem 1: text problem 3.30

Problem 2: text problem 3.37

Problem 3: text problem 3.43

Problem 4: revised version of text problem 3.36
A communication line capable of transmitting at a rate of 50 Kbits/sec will be used to accommodate 10 sessions each generating Poisson traffic at a rate 150 packets/min. Assume that 10% of the packets are 100 bits long and the rest are 1500 bits long.

a) For each session, find the average number (N_Q) of packets in queue, the average number (N) in the system, and the average delay (T) per packet when the line is allocated to the sessions by using:

(1) 10 equal-capacity time-division multiplexed channels. (Answer: $N_Q = 0.791$, $N = 1.47$, $T = 0.588$ sec.)

(2) statistical multiplexing. (Answer: $N_Q = 0.0791$, $N = 0.147$, $T = 0.0588$ sec.)

b) Repeat the problem a) for the case where the short packets (N_{Q1}, N_1, T_1) are given nonpreemptive priority over the long packets (N_{Q2}, N_2, T_2). (Answer: (1)-($N_{Q1} = 0.027$, $N_1 = 0.032$, $T_1 = 0.128$, $N_{Q2} = 0.855$, $N_2 = 2.273$, $T_2 = 1.055$), (2)-($N_{Q1} = 0.0027$, $N_1 = 0.0032$, $T_1 = 0.0128$, $N_{Q2} = 0.0855$, $N_2 = 0.2273$, $T_2 = 0.1055$))

Problem 5: Tasting Lyapunov Stability Method
Consider the following single-server queueing system:

- Discrete time-slotted
- Arrival process
  - $A(t)$: the amount of the data arriving during time slot t
  - i.i.d. over time with $E[A(t)] = \lambda, \forall t \geq 0$
- Service process
  - Deterministic with capacity C

  - The server can serve the data size of C for each time slot, possibly merging multiple packets if the size of the packet to be served is less than C

- The arrival and service rates are bounded, i.e., $\lambda < \infty$ and $C < \infty$. Further, we have $A(t) \leq A_{\max}, \forall t \geq 0$, i.e., the maximum amount of the data that can arrive in a slot is $A_{\max} < \infty$.

a) Let $q(t)$ denote the queue length at the end of time slot t. Show that the queue evolution equation obeys:

$$q(t+1) = \max[0, q(t) - C] + A(t). \tag{1}$$

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b) Let $V(q(t)) = \frac{1}{2}q^2(t)$. Show the following inequality:

$$V(q(t+1)) - V(q(t)) \leq \frac{1}{2}(A_{\max}^2 + C^2) + q(t)(A(t) - C). \tag{2}$$

(Hint: Expand $V(q(t+1))$ and use the facts $(\max[0, X])^2 \leq X^2$ and $\max[a, b] \leq 0$ for non-positive numbers a, b)
c) Assume $\lambda < C$. Then, it is obvious that for any such λ, there exists a constant $\epsilon > 0$ such that $\lambda + \epsilon \leq C$. Use this fact to show:

$$\Delta V(t) \leq B - \epsilon q(t), \tag{3}$$

where $\Delta V(t) = E[V(q(t+1)) - V(q(t))|q(t)]$ and $B = \frac{1}{2}(A_{\max}^2 + C^2)$. (Hint: Take conditional expectation $E[\cdot|q(t)]$ on both side of (2))
d) Take expectation on both side of (3) over the distribution of $q(t)$. Sum (3) from $t = 0$ to $t = T - 1$ and use the telescoping to simplify its left hand side. Finally, use nonnegativity of $V(\cdot)$ to show the following inequality:

$$\limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} E[q(t)] \leq \frac{B}{\epsilon}. \tag{4}$$

e) Define $f(Q) = \limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \Pr[q(t) > Q]$, i.e., $f(Q)$ is the overflow probability function. Show the following inequality and explain that this implies the stability of the given queueing system.

$$f(Q) \leq \frac{B}{\epsilon Q}. \tag{5}$$

(Hint: Use the fact $\Pr[q > Q] \leq E[q]/Q$ for nonnegative random variable q, and apply the result (4).)