MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science
Department of Aeronautics and Astronautics

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Problem Set No.5 Solutions — Posted: Oct 14, 2008

Problem 1: text problem 4.1

a) State n can only be reached from states 0 to $n+1$, and hence the balance equation can be written as

$$p_n = \sum_{i=0}^{n+1} p_i P_{i,n}, 0 \leq n < m.$$

By axiom of probability, we also have

$$\sum_{i=0}^{m} p_i = 1$$

b) Rearranging the above balance equation yields

$$p_{n+1} = \frac{p_n(1 - P_{n,n}) - \sum_{i=0}^{n-1} p_i P_{i,n}}{P_{n+1,n}}.$$

c) Using b), it is easy to obtain

$$p_1 = \frac{p_0(1-P_{0,0})}{P_{1,0}}$$
$$p_2 = \frac{p_1(1-P_{1,1})-p_0P_{0,2}}{P_{2,1}} = p_0\left[\frac{(1-P_{0,0})(1-P_{1,1})}{P_{1,0}P_{2,1}} - \frac{P_{0,2}}{P_{2,1}}\right].$$

d) Combining c) and the second equation in a) yields

$$p_0 = \frac{P_{1,0}P_{2,1}}{P_{1,0}P_{2,1} + (1 - P_{0,0})P_{2,1} + (1 - P_{0,0})(1 - P_{1,1}) - P_{0,2}P_{1,0}}$$

Problem 2: text problem 4.4

a) Let $E\{n\}$ be the expected number of backlogged nodes, averaged over time. Since $m - E\{n\}$ is the expected number of nodes that can accept packets, and q_a is the probability that each receives a packet in a slot, the expected number of accepted arrivals per slot is $E\{N_a\} = q_a(m - E\{n\})$.

b) Since a limited number (i.e. m) arrivals can be in the system at any time, the time average accepted arrival rate must equal the time average departure rate, which is the time average success rate. Thus $E\{P_{\text{success}}\} = E\{N_a\} = q_a(m - E\{n\})$.

c) The expected number of packets in the system, immediately after beginning the slot is the expected backlog plus the expected number of arrivals accepted in the system during the previous slot. Thus $E\{N_{sys}\} = E\{n\} + E\{N_a\} = E\{n\}(1 - q_a) + q_a m$.

d) The accepted arrival rate is $E\{N_a\}$. Thus by Little's theorem we can write

$$T = E\{N_{sys}\}/E\{N_a\} = 1 + E\{n\}/[q_a(m - E\{n\})]$$

e) The above equations express the relevant quantities in terms of $E\{n\}$ and make clear that $E\{N_a\}$ and $E\{P_{\text{success}}\}$ decrease and $E\{N_{sys}\}$ and T increase as n decreases. Thus it makes no difference which of these quantities is optimized; improving one improves others.

Problem 3: text problem 4.7

a) Note that one packet successfully leaves the system each slot in which one or more packets are transmitted. Thus if all waiting packets attempt transmission in every slot, a successful transmission occurs in every slot in which packets are waiting. Since the expected delay is independent of the order in which packets are successfully transmitted, we see that the expected delay is the same as that of a centralized slotted FCFS system. Now compare this policy with an arbitrary policy for transmitting waiting packets; assume any given sequence of arrival times. Each time the arbitrary policy fails to attempt a transmission in a slot with waiting packets, the FCFS system decreases the backlog by 1 while the other policy does not decrease the backlog. Thus the backlog for the arbitrary system is always greater than or equal to that of FCFS system. Thus by Little's theorem, the arbitrary system has an expected delay at least as the FCFS system.

b) This is just slotted FDM system of section 3.5.1 with $m = 1$ (i.e., a slotted M/D/1 queueing system). From Eq. 3.58, the queueing delay is $1/[2(1 - \lambda)]$ slot times. The total delay, including service time, is then $1 + 1/[2(1 - \lambda)]$.

c) The solution to b) can be written as $1 + 1/2 + \lambda/[2(1 - \lambda)]$ where the first term is the transmission time, the second term is the waiting time from an arrival to the beginning of a slot, and the third term is the delay due to collisions with other packets. If each subsequent attempt after an unsuccessful attempt is delayed by k slots, this last term is multiplied by k. Thus the new total delay is $3/2 + k\lambda/[2(1 - \lambda)]$.

Problem 4: text problem 4.8

a) Let X be the time in slots from the beginning of a backlogged slot until the completion of the first success at a given node. Let $q = q_r p$ and note that q is the probability that the node will be successful at any given slot given that it's still backlogged.

$$P\{X = i\} = q(1-q)^{i-1}; i \geq 1$$
$$E\{X\} = \sum_{i=1}^{\infty} iq(1-q)^{i-1} = 1/q$$

And

$$E\{X^2\} = \sum_{i=1}^{\infty} iq(1-q)^{i-1} = (2-q)/q^2 = (2-pq_r)/(pq_r)^2$$

Note that we used the identities for $0 \leq z < 1$:

$$\sum_{i=1}^{\infty} iz^{i-1} = \sum_{i=1}^{\infty} \frac{dz^i}{dz} = \frac{d\sum_{i=1}^{\infty} z^i}{dz} = \frac{d[z/(1-z)]}{dz} = \frac{1}{(1-z)^2}$$

$$\sum_{i=1}^{\infty} i^2 z^{i-1} = \sum_{i=1}^{\infty} \frac{d^2 z^{i+1}}{dz^2} - \sum_{i=1}^{\infty} \frac{dz^i}{dz} = \frac{d^2[z^2/(1-z)]}{dz^2} - \frac{d[z/(1-z)]}{dz} = \frac{1}{(1-z)^2} = \frac{1+z}{(1-z)^3}.$$

b) For an individual node, we have an M/G/1 queue with vacations. The vacations are deterministic with a duration of 1 slot, and the service time has the first and second moments found in part (a). Thus using Eq. 3.55 for the queueing delay and adding an extra service time to get the system delay,

$$T = \frac{(\lambda/m)E\{X^2\}}{2(1-\rho)} + 1/2 + 1/q = \frac{\lambda(2-\rho)}{2q^2(1-\rho)m} + 1/2 + 1/q$$

Since the arrival rate is λ/m and the service rate is q, we have $\rho = \lambda/(mq)$. Substituting this into the above expression for T and simplifying, we have

$$T = \frac{1}{q_r p(1-\rho)} + \frac{1-2\rho}{2(1-\rho)}$$

c) For $p = 1$ and $q_r = 1/m$, we have $\rho = \lambda$, so

$$T = \frac{m}{1-\lambda} + \frac{1-2\lambda}{2(1-\lambda)}$$

In the limit of large m, this is twice the delay of TDM as given in Eq.(3.59).

Problem 5: text problem 4.9

a) Let v be the mean number of packets in the system. Given n packets in the system, with each packet independently transmitted in a slot with probability $1/v$, the probability of an idle slot, $P\{I|n\} = (1-1/v)^n$. The joint probability of an idle slot and n packets in the system is then $P\{I, n\} = P\{I|n\}P\{n\} = \frac{\exp(-v)v^n}{n!}(1-1/v)^n$.

$$P\{I\} = \sum_{n=0} P\{I, n\} = \sum_{n=0} \frac{\exp(-v)(v-1)^n}{n!} = 1/e$$

b) Using the results above, we can find $P\{n|I\}$

$$P\{n|I\} = \frac{P\{n, I\}}{P\{I\}} = \frac{(v-1)^n \exp(-v+1)}{n!}$$

Thus, this probability is Poisson with mean $v-1$.

c) We can find the joint probability of success and n in the system similarly

$$P\{S, n\} = P\{S|n\}P\{n\} = \frac{\exp(-v)v^n}{n!} n(1-1/v)^{n-1}v^{-1} = \frac{\exp(-v)(v-1)^{n-1}}{(n-1)!}$$

$$P\{S\} = \sum_{n=0} \frac{\exp(-v)(v-1)^{n-1}}{(n-1)!} = 1/e$$

d) From this, the probability that there were n packets in the system given a success is

$$P\{n|S\} = \frac{P\{n, S\}}{P\{S\}} = \frac{\exp(-v+1)(v-1)^{n-1}}{(n-1)!}$$

Note that $n-1$ is the number of remaining packets in the system with the successful packet removed, and it is seen from above that this remaining number is Poisson with mean $v-1$.

Problem 6: text problem 4.12

a) Let S_i be the start time of i-th initiation of a k-packet transmission. Then, the start time of the j-th packet in that transmission is given by $S_i + j - 1$, which is the end time of $(j-1)$-th. The j-th packet can be successfully transmitted when the end time of $(i-1)$-th initiation and the start time of i-th initiation do not overlap with the start time and end time of j-th packet, respectively. Let τ_i be the interval between the i-th and $(i+1)$-th initiation, i.e., $\tau_i = S_{i+1} - S_i$. Hence, we have the following:

$$\begin{aligned}
&\Pr[\text{successful transmission of } j\text{-th packet in } i\text{-th initiation}] \\
&= \Pr[S_{i-1} + k \leq S_i + j - 1 \ \ \& \ \ S_i + j \leq S_{i+1}] \\
&= \Pr[S_i - S_{i-1} \geq k - j + 1 \ \ \& \ \ S_{i+1} - S_i \geq j] \\
&= \Pr[\tau_{i-1} \geq k - j + 1 \ \ \& \ \ \tau_i \geq j] \\
&= e^{-G(k-j+1)} e^{-Gj} = e^{-G(k+1)},
\end{aligned}$$

where we use the fact that the inter-arrival time τ_i is exponentially distributed with rate G.

b) Since the group attempt rate is G, the packet attempt rate is Gk. As shown in a), the success rate is $e^{-G(k+1)}$, and hence the throughput can be written as $Gke^{G(k+1)}$. This is maximized when $G = 1/(k+1)$. As a result, the maximum throughput is given by $\frac{k}{e(k+1)}$. This can be made as close to $1/e$ by increasing k, but at the cost of increased delay (as it has to wait for k packets to arrive).