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Problem Set No.6 Solutions   Posted: Oct 21, 2008

Problem 1: text problem 4.15
a) The probability of i packets joining the left subset, given k packets in the original set, is given by the binomial distribution

$$\frac{k!2^{-k}}{i!(k-i)!}.$$

b) Assuming $k \geq 2$, the CRP starts with an initial collision that takes one slot. Given that i packets go into the left subset, A_i is the expected number of additional slots required to transmit the left subset and A_{k-i} is the expected number on the right. Taking the expectation over the number i of packets in the left subset, we get the desired result,

$$A_k = 1 + \sum_{i=0}^{k} \frac{k!2^{-k}}{i!(k-i)!}(A_i + A_{k-i}).$$

c) Note that

$$\sum_{i=0}^{k} \frac{k!2^{-k}}{i!(k-i)!}A_i = \sum_{i=0}^{k} \frac{k!2^{-k}}{i!(k-i)!}A_{k-i}.$$

Hence, we have

$$A_k = 1 + 2\sum_{i=0}^{k} \frac{k!2^{-k}}{i!(k-i)!}A_i = 1 + 2^{-k+1}A_k + 2\sum_{i=0}^{k-1} \frac{k!2^{-k}}{i!(k-i)!}A_i.$$

Taking the A_k term to the left side of the equation, we obtain

$$c_{ik} = \frac{k!2^{-k+1}}{i!(k-i)!(1-2^{-k+1})}, \text{for } i < k$$

$$c_{kk} = \frac{1}{1-2^{-k+1}}.$$

Evaluating this numerically, $A_2 = 5$ and $A_3 = 23/3$.

Problem 2: text problem 4.17
a) The joint event $X_L = 0$ and $X_L + X_R \geq 2$ is equivalent to $X_L = 0$ and $X_R \geq 2$, so

$$P[X_L = 0 | X_L + X_R \geq 2] = \frac{P[X_L = 0]P[X_R \geq 2]}{P[X_L + X_R \geq 2]} = \frac{e^{-G}[1-(1+G)e^{-G}]}{1-(1+2G)e^{-2G}}.$$

This uses the fact that X_L and X_R are independent and $X = X_L + X_R$ is Poisson with mean $2G$. The other equalities (in the sequel) also use these facts.

b)

$$P[X_L = 1 | X_L + X_R \geq 2] = \frac{P[X_L = 1]P[X_R \geq 1]}{P[X_L + X_R \geq 2]} = \frac{Ge^{-G}(1 - e^{-G})}{1 - (1 + 2G)e^{-2G}}.$$

c)

$$P[X_L \geq 2 | X_L + X_R \geq 2] = \frac{P[X_L \geq 2]P[X_R \geq 0]}{P[X_L + X_R \geq 2]} = \frac{1 - (1 + G)e^{-G}}{1 - (1 + 2G)e^{-2G}}.$$

d)

$$P[X_R = 1 | X_L = 1, X_L + X_R \geq 2] = \frac{P[X_R = 1]P[X_L = 1]}{P[X_L = 1]P[X_R \geq 1]} = \frac{P[X_R = 1]}{P[X_R \geq 1]} = \frac{Ge^{-G}}{1 - e^{-G}}.$$

Note that this is just $P[X_R = 1 | X_R \geq 1]$.

e)

$$P[X_R = i | X_L = 0, X_L + X_R \geq 2] = \frac{P[X_R = i]P[X_L = 0]}{P[X_L = 0]P[X_R \geq 2]} = \frac{P[X_R = i]}{P[X_R \geq 2]} = \frac{G^i e^{-G}}{i![1 - (1 + G)e^{-G}]}.$$

Note that this is just $P[X_R = i | X_R \geq 2]$.

f)

$$P[X_R = i | X_L \geq 2, X_L + X_R \geq 2] = \frac{P[X_R = i]P[X_L \geq 2]}{P[X_L \geq 2]} = P[X_R = i] = \frac{G^i e^{-G}}{i!}.$$

Problem 3: text problem 4.26

a) The first transmission after a given idle detection will be successful if no other transmission starts within the next β time units. Since the process of initiations is Poisson with rate G, the probability of this is

$$P_{\text{succ}} = e^{-\beta G}$$

.

b) The mean time until the first initiation after an idle detection is $1/G$ (note that all nodes detect the channel as being idle at the same time). If this first initiation is successful, $1 + \beta$ time units are required until the next idle detection; if the initiation is unsuccessful, 2β time units are required. Thus

$$E[\text{time between idle detects}] = G^{-1} + (1 + \beta)e^{-\beta G} + 2\beta(1 - e^{-\beta G})$$

c) The throughput T is the ratio of P_{succ} to the expected time between idle detects. Hence,

$$T = \frac{e^{-\beta G}}{G^{-1} + (1 + \beta)e^{-\beta G} + 2\beta(1 - e^{-\beta G})} = \frac{1}{(G^{-1} + 2\beta)e^{\beta G} + 1 - \beta}.$$

d) We can maximize this by minimizing $1/T$. The function $1/T$ is strictly convex with respect to G, and hence taking the derivative of $1/T$ with respect to G and setting it equal to 0, we find that the minimum of $1/T$ occurs at $\beta G = 1/2$. Substituting this into the expression for T, we get

$$T = \frac{1}{1+\beta(4\sqrt{e}-1)} = \frac{1}{1+5.595\beta}.$$

Problem 4
a) We are given that

$$P(Z > x) \sim x^{-\alpha},$$

$\alpha > 1$. Therefore $\exists$ constants $c > 0$ and $x_0 > 0$ such that $\forall x > x_0$, we have $P(Z > x) < cx^{-\alpha}$. Now,

$$E(Z) = \int_0^\infty P(Z > x)\mathrm{d}x < \int_0^{x_0} P(Z > x)\mathrm{d}x + \int_{x_0}^\infty cx^{-\alpha}\mathrm{d}x < \infty.$$

The last integral is finite since $\alpha > 1$.

b) By the P-K formula, the expected queue occupancy is proportional to $\frac{E(Z^2)}{1-\rho}$. We will show that for certain values of α, $E(Z^2)$ can be infinite, leading to an infinite expected queue occupancy, even though $\rho < 1$. Indeed, reasoning as in part (a), $\exists$ constants $c' > 0$ and $x_0 > 0$ such that $\forall x > x_0$, we have $P(Z > x) > c'x^{-\alpha}$. Thus,

$$E(Z^2) = 2\int_0^\infty xP(Z > x)\mathrm{d}x > \int_0^{x_0} xP(Z > x)\mathrm{d}x + \int_{x_0}^\infty c'x^{1-\alpha}\mathrm{d}x.$$

For $\alpha \leq 2$ the last integral is infinite, and the result follows.

c) This seemingly paradoxical situation is resolved by noting that it is possible for random variables (which are, by definition, finite with probability one) to have an infinite mean. As stated in class, the queue occupancy is finite w.p.1 for *any* G/G/1 queue with $\rho < 1$. However, the *expected* queue occupancy could be infinite, as shown above. When the tail of the job size distribution is heavy, the packets can take on enormous sizes with non-negligible probability. Such a large packet at the head-of-line in an FCFS queue can clog up the entire system, leading to unbounded *expected* queue occupancy, although the queue length remains finite w.p.1.

The above effect is an artifact of the FCFS discipline being choked by enormous head-of-line packets, and can be effectively mitigated by using LCFS or processor sharing disciplines. Indeed, the following general result can be shown for any G/G/1 queue[1]: Under FCFS discipline, the tail of the waiting time distribution is *one degree heavier* than that of the job

[1] J. W. Cohen, Some results on regular variation for distributions in queueing and fluctuation theory. *J. Appl. Prob.*, **10**:343-353, 1973.

size distribution, whereas, under LCFS and processor sharing disciplines, the waiting time and job size distributions are of the *same degree*.

Problem 5

a) The autocorrelation function is given by

$$r(k) = \frac{(k+1)^{2H} - 2k^{2H} + (k-1)^{2H}}{2}.$$

b) Using L'Hopital's rule, we can show that

$$\lim_{k\to\infty} \frac{r(k)}{k^{2H-2}} = H(2H-1),$$

which is a non-zero constant for $H \neq 0.5$.

Therefore, for large k, $r(k)$ decays like $k^{-(2-2H)}$. Note that for $H = 0.5$, the process de-correlates immediately, while for $H > 0.5$, a larger value of the Hurst parameter implies a slower decay rate of correlation.

c) For $0.5 < H < 1$, $r(k)$ decays slower than $\frac{1}{k}$, so that $\sum_k r(k) \to \infty$, and the process exhibits long range dependence.