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Problem Set No.7 Solutions — Posted: Oct 28, 2008

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**Problem 1: text problem 4.31**

**a)** A simultaneous transmission on links 1 and 3 causes a collision for the transmission no link 1; similarly, a collision on link 2 occurs if 2 and 3 are used simultaneously. Finally, simultaneous transmission on links 1 and 2 cause a collision for both transmission. Thus, at most one link can be used successfully at a time and $f_1+f_2+f_3 \leq 1$. To view this in terms of Eq. (4.75), we let $x_1$, $x_2$, and $x_3$ be the collision free vectors (100), (010), and (001) respectively, and we let $x_4$ be the trivial CFV (000). Then, for $i \leq i \leq 3$, $f_i$ corresponds to $a_i$ in Eq. (4.75) and the constraints $a_i \geq 0$ and $a_1+a_2+a_3+a_4 = 1$ is equivalent to $f_1+f_2+f_3 \leq 1$.

**b)** Based on the arguments in a) and Eq. (4.77), we have

$$\begin{aligned} p_1 &= (1-q_3)(1-q_2) \\ p_2 &= (1-q_3)(1-q_1) \\ p_3 &= 1 \end{aligned}$$

Eq. (4.78) then gives us the fractional utilizations

$$\begin{aligned} f = f_1 &= q_1(1-q_3)(1-q_2) \\ f = f_2 &= q_2(1-q_3)(1-q_1) \\ 2f = f_3 &= q_3. \end{aligned}$$

By equating the first and second inequalities, we obtain $q_2(1-q_1) = q_1(1-q_2)$, which implies $q_1 = q_2$.

**c)** Using $q_1 = q_2$ and $q_3 = 2f$, we have $f = q_1(1-2f)(1-q_1)$. Thus,

$$\frac{f}{1-2f} = q_1(1-q_1) \leq \frac{1}{4}.$$

It immediately follows from the above inequality that $f \leq \frac{1}{6}$.

**Problem 2: Throughput Analysis of Simplified 802.11**

**a)**

$$\begin{aligned} &P[i,k|i,k+1] = 1, && k \in [0, W_i - 2], \quad i \in [0,m] \\ &P[0,k|i,0] = (1-p)/W_0, && k \in [0, W_0 - 1], \quad i \in [0,m] \\ &P[i,k|i-1,0] = p/W_i, && k \in [0, W_i - 1], \quad i \in [1,m] \\ &P[m,k|m,0] = p/W_m, && k \in [0, W_m - 1] \quad , \end{aligned}$$

The first transition probability is obvious because the backoff counter decreases every time slot by 1, until it reaches 0. When the backoff counter is 0, it attempts to transmit and it is successful with probability $1-p$. In the event of successful transmission, the window stage is reset to 0 and the backoff counter is uniformly randomly selected from $[0, W_0 - 1]$. Hence, the second transition probability holds. The transition probability upon a collision can be similarly obtained.

**b)** It turns out that it suffices to prove the last relationship for $k \in [0, W_i - 1]$. First consider the state $(0, W_0 - 1)$. The outgoing transition probability is 1, and each of the states $(j, 0), j = 0, 1, ..., m$ makes a transition to $(0, W_0 - 1)$ with probability $(1-p)/W_0$. Hence the balance equation can be written as

$$p_{0,W_0-1} = (1-p)/W_0 \sum_{j=0}^{m} q_{j,0}.$$

Consider now the state $(0, W_0 - 2)$. The outgoing transition probability is also 1 in this case. But, it has one incoming transition compared to the previous case; namely, from $(0, W_0 - 1)$ and from $(j, 0), j = 0, 1, ..., m$. Hence, the balance equation can be written as

$$\begin{aligned} p_{0,W_0-2} &= (1-p)/W_0 \sum_{j=0}^{m} q_{j,0} + p_{0,W_0-1} \\ &= (1-p)/W_0 \sum_{j=0}^{m} q_{j,0} + (1-p)/W_0 \sum_{j=0}^{m} q_{j,0} \\ &= (2/W_0)(1-p) \sum_{j=0}^{m} q_{j,0}. \end{aligned}$$

Continuing this process yields the relationship for $q_{i,k}$ for $i = 0$ and $k \in [0, W_i - 1]$. Note that the state $(0, 0)$ has multiple outgoing transitions, but as in the other states, the total outgoing transition probability is 1. This fact was used here and will also be used in the sequel.

For $0 < i < m$, we similarly obtain the result from the rightmost in the Markov chain. First, the outgoing transition probability of state $(i, W_i - 1)$ is 1. It has a single incoming transition from $(i-1, 0)$, with probability $p/W_i$. The balance equation is then written as

$$q_{i,W_i-1} = p/W_i q_{i-1,0}.$$

Similarly, the balance equation at state $(i, W_i - 2)$ can be written as

$$\begin{aligned} q_{i,W_0-2} &= p/W_i q_{i-1,0} + q_{i,W_i-1} \\ &= p/W_i q_{i-1,0} + p/W_i q_{i-1,0} \\ &= (2/W_i) p q_{i-1,0}. \end{aligned}$$

Continuing this process yields the relationship for $q_{i,k}$ for $0 < i < m$ and $k \in [0, W_i - 1]$.

For $i = m$, the balance equation at state $(m, W_m - 1)$ can be written as

$$\begin{aligned} q_{m,W_m-1} &= p/W_m q_{m-1,0} + p/W_m q_{m,0} \\ &= p/W_m(q_{m-1,0} + q_{m,0}). \end{aligned}$$

Similarly for state $(m, W_m - 2)$, we have

$$\begin{aligned} q_{m,W_m-2} &= p/W_m q_{m-1,0} + p/W_m q_{m,0} + q_{m,W_m-1} \\ &= p/W_m q_{m-1,0} + p/W_m q_{m,0} + p/W_m q_{m-1,0} + p/W_m q_{m,0} \\ &= 2p/W_m(q_{m-1,0} + q_{m,0}). \end{aligned}$$

Continuing this process yields the relationship for $q_{i,k}$ for $i = m$ and $k \in [0, W_m - 1]$.

The first two relationships then readily follow.

**c)** For $i = 0$, putting $\sum_{i=0}^{m} q_{i,0} = q_{0,0}/(1-p)$ into the equation yields. For $0 < i < m$, first note that the relation $q_{i,0} = p^i q_{0,0}, 0 < i < m$ in b) implies $q_{i,0} = pq_{i-1,0}$. Applying this equation to the case of $0 < i < m$ yields the desired result. For $i = m$, the relationship $q_{m,0} = \frac{p^m}{1-p} q_{0,0}$ implies $q_{m-1,0}p = (1-p)q_{m,0}$. Substituting this into the case of $i = m$ yields the desired result.

**d)** Using the results in b) and c), we obtain the following

$$\begin{aligned} \sum_{i=0}^{m} \sum_{k=0}^{W_i - 1} q_{i,k} &= \sum_{i=0}^{m} \sum_{k=0}^{W_i-1} \tfrac{W_i - k}{W_i} q_{i,0} \\ &= \sum_{i=0}^{m} q_{i,0} \tfrac{W_i+1}{2} \\ &= \tfrac{q_{0,0}}{2} \sum_{i=0}^{m} p^i (W_i + 1) \\ &= \tfrac{q_{0,0}}{2} \left\{ \tfrac{1-p^{m+1}}{1-p} + W_0 \sum_{i=0}^{m} j(2p)^i \right\} \\ &= \tfrac{q_{0,0}}{2} \left\{ \tfrac{1-p^{m+1}}{1-p} + W_0 \tfrac{1-(2p)^{m+1}}{1-2p} \right\}. \end{aligned}$$

By the axiom of probability, the above value should be 1, and rearranging the equation yields the desired result.

**e)** The probability that a user transmits in a slot is given by $\sum_{i=0}^{m} q_{i,0}$. Using the fact $\sum_{i=0}^{m} q_{i,0} = q_{0,0}/(1-p)$ and the results in d), we have

$$\tau = \sum_{i=0}^{m} q_{i,0} = q_{0,0}/(1-p) = \frac{2(1-2p)}{(1-2p)(W_0+1) + pW_0(1-(2p)^m)}.$$

**f)** The probability that a successful transmission occurs is given by

$$n\tau(1-\tau)^{n-1},$$

and this value is maximized when $\tau = 1/n$. Hence, we have

$$\frac{1}{n} = \frac{2(1-2p)}{(1-2p)(W_0+1) + pW_0(1-(2p)^m)}.$$

Solving this equation for $W_0$ yields the desired result.

**NOTE:** This is the result in the landmark paper (in the analysis of IEEE 802.11) by G. Bianchi. See the following paper for more details: G. Bianchi, "Performance Analysis of the IEEE 802.11 Distributed Coordination Function," IEEE Journal on Selected Areas in Communications, Vol. 18, No. 3, 2000.