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Problem Set No.8 Solutions — Posted: Nov 4, 2008

Problem 1: Birkhoff-von Neumann Switch

a) We are given the following decomposition:

$$\tilde{R} = \sum_k \phi_k P_k.$$

Multiplying both side by a column vector e of all its elements being 1, we obtain

$$e = \tilde{R}e = \sum_k \phi_k P_k e = \left(\sum_k \phi_k \right) e,$$

The first equality follows from the fact that $\tilde{R}$ is doubly stochastic, and the last equality follows from the fact that P_k is a permutation matrix for every k. This proves that $\sum_k \phi_k = 1$.

b) It is easy to see that in class k, there are l tokens whose virtual finishing times are not greater than F_k^l. In class $j \neq k$, if the number of such tokens is denoted by l_j, then it has to satisfies

$$\begin{aligned} & \tfrac{l_j}{\phi_j} = F_j^{l_j} \leq F_k^l = \tfrac{l}{\phi_k} \\ \Rightarrow \quad & l_j \leq \tfrac{\phi_j}{\phi_k} l \\ \Rightarrow \quad & l_j = \lfloor \tfrac{\phi_j}{\phi_k} l \rfloor, \forall j \neq k \end{aligned}$$

because l_j is an integer. Summing up the above equality together with l yields the desired result.

c) In worst case, the l-th token of class k will be served after all the tokens have been served. Hence, the time slot τ_k^l satisfies

$$\begin{aligned} \tau_k^l &\leq l + \textstyle\sum_{j \neq k} \lfloor (l\phi_j)/\phi_k \rfloor \\ &\leq l + \textstyle\sum_{j \neq k} (l\phi_j)/\phi_k \\ &= l\tfrac{\phi_k}{\phi_k} + \textstyle\sum_{j \neq k} (l\phi_j)/\phi_k \\ &= \tfrac{l}{\phi_k} \textstyle\sum_j \phi_j \\ &= \tfrac{l}{\phi_k} \\ &= F_k^l. \end{aligned}$$

d) Because $\tau_k^l \leq F_k^l$, we have

$$\begin{aligned} D_k(t) &= \sup\{l : \tau_k^l \leq t\} \\ &\geq \sup\{l : F_k^l \leq t\} \\ &= \sup\{l : l \leq \phi_k t\} \\ &= \lfloor \phi_k t \rfloor. \end{aligned}$$

e) We have

$$\begin{aligned} \textstyle\sum_{k\in E_{ij}} D_k(t) &= t - \textstyle\sum_{k\notin E_{ij}} D_k(t) \\ &\leq t - \textstyle\sum_{k\notin E_{ij}} \lfloor \phi_k t \rfloor \\ &\leq t - \textstyle\sum_{k\notin E_{ij}} (\phi_k t - 1) \\ &= \textstyle\sum_k \phi_k t - \sum_{k\notin E_{ij}} \phi_k t + (K - |E_{ij}|) \\ &= \textstyle\sum_{k\in E_{ij}} \phi_k t + (K - |E_{ij}|), \end{aligned}$$

where we have used the fact that $K = |E_{ij}| + |E_{ij}^C|$ (where A^C is the complement of set A) and $\sum_k \phi_k = 1$.

f) Applying the result in d) for $C_{ij}(t)$ and the result in e) for $C_{ij}(s)$, we obtain

$$\begin{aligned} C_{ij}(t) - C_{ij}(s) &\geq \sum_{k\in E_{ij}} \lfloor \phi_k t \rfloor - \sum_{k\in E_{ij}} \phi_k s - (K - |E_{ij}|) \\ &\geq \sum_{k\in E_{ij}} (\phi_k t - 1) - \sum_{k\in E_{ij}} \phi_k s - (K - |E_{ij}|) \\ &= \sum_{k\in E_{ij}} \phi_k (t-s) - K. \end{aligned}$$

g) If the arrival rate matrix R satisfy the constraints ((6) and (7) in PS8) of doubly substochastic matrix with strict inequalities, then it is obvious that there exists a stochastic matrix $\tilde{R}$ such that $\tilde{r}_{ij} > r_{ij}, \forall (i,j)$. To see this, if the constraints (6) and (7) are satisfied, it follows that $\max_{\forall (i,j)} r_{ij} < 1$. Then, there exists a constant $\epsilon > 0$ such that $r_{ij} + \epsilon < 1$. Hence, adding ϵ / N to each coordinate of R yields another doubly substochastic matrix R^ϵ satisfying $r_{ij}^\epsilon > r_{ij}, \forall (i,j)$. By von Neumann's Theorem, there exists a doubly stochastic matrix $\tilde{R}$ such that $\tilde{r}_{ij} \geq r_{ij}^\epsilon, \forall (i,j)$. This proves that there exists a doubly stochastic matrix $\tilde{R}$ that dominates R in component-wise.

On the other hand, let $s = 0$, divide both side of the inequality in f), and take $\liminf_{t\to\infty}$, then we obtain

$$\liminf_{t\to\infty} \frac{C_{ij}(t)}{t} \geq \sum_{k\in E_{ij}} \phi_k = \tilde{r}_{ij}, \forall (i,j),$$

where the equality follows from Birkhoff's Theorem. Because $\tilde{r}_{ij} > r_{ij}, \forall (i,j)$, the above inequality implies that the service rate is greater than the arrival rate, and this proves the stability of the algorithm.

NOTE: Refer to the following paper for more details: Cheng-Shang Chang, Wen-Jyh Chen and Hsiang-Yi Huang, "On service guarantees for input-buffered crossbar switches: acapacity decomposition approach by Birkhoff and von Neumann," IWQoS 1999.