Problem 1: Birkhoff-von Neumann Switch

a) We are given the following decomposition:

$$\tilde{R} = \sum_k \phi_k P_k.$$  

Multiplying both side by a column vector $e$ of all its elements being 1, we obtain

$$e = \tilde{R}e = \sum_k \phi_k P_k e = \left(\sum_k \phi_k\right) e,$$

The first equality follows from the fact that $\tilde{R}$ is doubly stochastic, and the last equality follows from the fact that $P_k$ is a permutation matrix for every $k$. This proves that $\sum_k \phi_k = 1$.

b) It is easy to see that in class $k$, there are $l$ tokens whose virtual finishing times are not greater than $F^l_k$. In class $j \neq k$, if the number of such tokens is denoted by $l_j$, then it has to satisfies

$$\frac{l_j}{\phi_j} = F^l_j \leq F^l_k = \frac{l}{\phi_k}$$

$$\Rightarrow l_j \leq \frac{\phi_j l}{\phi_k}$$

$$\Rightarrow l_j = \left\lfloor \frac{\phi_j l}{\phi_k} \right\rfloor, \forall j \neq k$$

because $l_j$ is an integer. Summing up the above equality together with $l$ yields the desired result.

c) In worst case, the $l$-th token of class $k$ will be served after all the tokens have been served. Hence, the time slot $\tau^l_k$ satisfies

$$\tau^l_k \leq l + \sum_{j \neq k} \left(\frac{l \phi_j}{\phi_k}\right)$$

$$\leq l + \sum_{j \neq k} \frac{l \phi_j}{\phi_k}$$

$$= l \frac{\phi_k}{\phi_k} + \sum_{j \neq k} \frac{l \phi_j}{\phi_k}$$

$$= \frac{\phi_k}{\phi_k} \sum_j \phi_j$$

$$= \phi_k$$

$$= F^l_k.$$
d) Because $\tau_k^l \leq F_k^l$, we have
\[ D_k(t) = \sup \{ l : \tau_k^l \leq t \} \geq \sup \{ l : F_k^l \leq t \} = \sup \{ l : l \leq \phi_k t \} = [\phi_k t]. \]

e) We have
\[ \sum_{k \in E_{ij}} D_k(t) = t - \sum_{k \not\in E_{ij}} D_k(t) \leq t - \sum_{k \not\in E_{ij}} [\phi_k t] \leq t - \sum_{k \not\in E_{ij}} (\phi_k t - 1) = \sum_{k \in E_{ij}} \phi_k t - \sum_{k \not\in E_{ij}} \phi_k t + (K - |E_{ij}|) = \sum_{k \in E_{ij}} \phi_k t + (K - |E_{ij}|), \]
where we have used the fact that $K = |E_{ij}| + |E_{ij}^C|$ (where $A^C$ is the complement of set $A$) and $\sum_k \phi_k = 1$.

f) Applying the result in d) for $C_{ij}(t)$ and the result in e) for $C_{ij}(s)$, we obtain
\[ C_{ij}(t) - C_{ij}(s) \geq \sum_{k \in E_{ij}} [\phi_k t] - \sum_{k \in E_{ij}} \phi_k s - (K - |E_{ij}|) \geq \sum_{k \in E_{ij}} (\phi_k t - 1) - \sum_{k \in E_{ij}} \phi_k s - (K - |E_{ij}|) = \sum_{k \in E_{ij}} \phi_k (t - s) - K. \]

g) If the arrival rate matrix $R$ satisfy the constraints ((6) and (7) in PS8) of doubly substochastic matrix with strict inequalities, then it is obvious that there exists a stochastic matrix $\tilde{R}$ such that $\tilde{r}_{ij} > r_{ij}, \forall(i, j)$. To see this, if the constraints (6) and (7) are satisfied, it follows that $\max_{(i, j)} r_{ij} < 1$. Then, there exists a constant $\epsilon > 0$ such that $r_{ij} + \epsilon < 1$. Hence, adding $\epsilon/N$ to each coordinate of $R$ yields another doubly substochastic matrix $R^\epsilon$ satisfying $r^\epsilon_{ij} > r_{ij}, \forall(i, j)$. By von Neumann’s Theorem, there exists a doubly stochastic matrix $\tilde{R}$ such that $\tilde{r}_{ij} \geq r^\epsilon_{ij}, \forall(i, j)$. This proves that there exists a doubly stochastic matrix $\tilde{R}$ that dominates $R$ in component-wise.

On the other hand, let $s = 0$, divide both side of the inequality in f), and take $\lim \inf_{t \to \infty}$, then we obtain
\[ \lim \inf_{t \to \infty} \frac{C_{ij}(t)}{t} \geq \sum_{k \in E_{ij}} \phi_k = \tilde{r}_{ij}, \forall(i, j), \]
where the equality follows from Birkhoff’s Theorem. Because $\tilde{r}_{ij} > r_{ij}, \forall(i, j)$, the above inequality implies that the service rate is greater than the arrival rate, and this proves the
stability of the algorithm.

**NOTE:** Refer to the following paper for more details: Cheng-Shang Chang, Wen-Jyh Chen and Hsiang-Yi Huang, "On service guarantees for input-buffered crossbar switches: capacity decomposition approach by Birkhoff and von Neumann," IWQoS 1999.