Given: $\sigma : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ (quadratic form)

A : real, n-by-n

B : real, n-by-m

when is it true that

(*) $\sigma^H(x,u) \gg 0$ forall $j\omega x = Ax + Bu$, $x \in \mathbb{C}^m$, $u \in \mathbb{C}^n$, $\omega \in \mathbb{R}$?

$\{ \exists \varepsilon > 0 : \sigma^H(x,u) - \varepsilon(|x|^2 + |u|^2) \geq 0 \ \forall x, u, \omega : j\omega x = Ax + Bu \}$

Th. 1 $\Leftrightarrow \exists P = P' : \sigma(x,u) + 2x'P(Ax+Bu) \gg 0$

Th. 2 $\Leftrightarrow \sigma(0,u) \gg 0$, and the 2n-by-2n matrix of the ODE

"Hamiltonian system" $\begin{cases} \dfrac{\partial x}{\partial t} = -\left(\dfrac{\partial H}{\partial \psi}\right)' \\ \dfrac{\partial \psi}{\partial t} = \left(\dfrac{\partial H}{\partial x}\right)' \\ \dfrac{\partial H}{\partial u} = 0 \end{cases}$ $\quad H = \sigma(x,u) - \psi'(Ax+Bu)$

has no $j\omega$-axis eigenvalues

Ex. 1: checking that $\sup_\omega |C(j\omega I - A)^{-1}B| < \gamma$

$\sigma(x,u) = \frac{1}{2}[\gamma^2 |u|^2 - |Cx|^2]$

$H = \frac{1}{2}[\gamma^2 |u|^2 - |Cx|^2] - \psi'(Ax+Bu)$

H.S.: $\begin{cases} \dot{x} = Ax + Bu \\ \dot{\psi} = -C'Cx - A'\psi \\ B'\psi = \gamma^2 u \end{cases}$ $\quad H_0 = \begin{bmatrix} A & \gamma^{-2}BB' \\ -C'C & -A' \end{bmatrix}$

! $\det(j\omega - H_0) = \text{const} \cdot \det(\gamma^2 I - G(j\omega)'G(j\omega)) \cdot |\det(j\omega I - A)|^2$

where $G(s) = C(sI-A)^{-1}B$