Model reduction for fast approximations to nonlinear continuum mechanics

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1 Projection-based model reduction

Full simulations are simulations without reduction. We express deformable simulations as the following (high-dimensional) second order system of ODEs:

$$\ddot{q} = F(q, \dot{q}, t) + \overline{B}u. \quad (1)$$

Here, $q \in \mathbb{R}^n$ is the state vector (n is typically at least several thousands), $F(q, \dot{q}, t) \in \mathbb{R}^n$ is some (nonlinear) function specifying the system's internal dynamics, $\overline{B} \in \mathbb{R}^{n \times m}$ is a constant *control matrix*, and $u \in \mathbb{R}^m$ is the *control vector*. State vector q consists of displacements of the vertices of a 3D volumetric mesh. The Equation 1 can model, for example, nonlinear FEM deformable objects or mass spring systems. The nonlinearity arises due to large deformations, and due to any additional material nonlinearities.

Reduced simulations are obtained by projecting Equation 1 onto a r-dimensional *subspace*, spanned by columns of some basis matrix $U \in \mathbb{R}^{n \times r}$ (typically, r is in the range of 10 to 40 in our simulations). The full state is approximated as $q = Uz$, where $z \in \mathbb{R}^r$ is the *reduced state*. The resulting low-dimensional system of ODEs

$$\ddot{z} = \tilde{F}(z, \dot{z}, t) + Bw, \quad \text{for} \quad \tilde{F}(z, \dot{z}, t) = U^T F(Uz, U\dot{z}, t), \quad (2)$$

approximates the high-dimensional system provided that the true solution states q are well-captured by the chosen basis U. Here, $B \in \mathbb{R}^{r,s}$ is a constant matrix, and $w \in \mathbb{R}^s$ is the *reduced control vector*. With fluids, equations are first order and the velocity field must be divergence-free. However, projection-based model reduction can be applied in the same way as with solids.

2 Fast deformations

With geometrically nonlinear FEM deformable objects, there exists an efficient cubic polynomial formula for $\tilde{F}(z, \dot{z}, t)$ which can be used to significantly accelerate the simulation [Barbič and James 2005]. The basis U can be selected by running full simulations, and employing Principal Component Analysis to reduce the problem dimensionality (POD). We also demonstrated that one can build an *a priori* basis by extending linear vibrational modes with their natural "nonlinear extension", the *modal derivatives*. This gives large deformation models around a specific selected configuration (typically the rest pose). The approach is automatic in that it does not require pre-simulation. Details are available in [Barbič 2007].

3 Fast and convergent control

Using controlled simulations to design physically plausible motion easily and interactively is a difficult, long-standing problem in computer animation. During my post-doctoral work at MIT, I analyzed the problem of whether model reduction can be used not only for fast simulation, but also for interactive **control** of deformable models and fluids. Control of such systems is difficult due to the nonlinearities and because of the high-dimensional controller space.

We demonstrated that model reduction, which greatly decreases the number of simulated degrees of freedom, can accelerate a standard controller such as a Linear Quadratic Regulator to real-time rates.

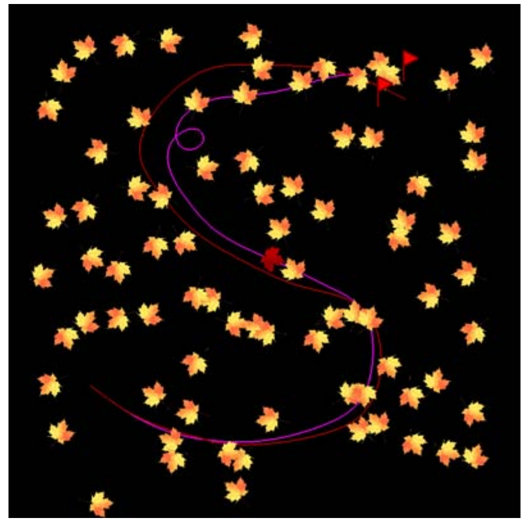

Figure 1: Controlled simulation with variety: *the red leaf in this fluid simulation is controlled to follow the S curve, while fighting random external forces (Perlin noise), resulting in different trajectories for each simulation run.*

In particular, we presented a real-time optimal controller that finds minimal control forces to track given pre-existing motion of deformable objects and fluids, while simultaneously providing proper physical response to any real-time user forces, randomly sampled external forces, or other disturbances [Barbič and Popović 2008] (see Figure 1). Such a system can be used in computer games to enforce desired simulation outcomes, while simultaneously making every play different. I am currently working on using reduction as a keyframing tool for interactive authoring of nonlinear deformable object simulations. The user specifies a set of keyframes, and the system then generates the in-between motion by solving a space-time optimization problem interactively. Such a system could accelerate the process of generating deformable animations from scratch, or editing existing animations. The general underlying goal, however, is that of completing a nonlinear transient simulation specified only by giving a small set of snapshots in time.

References

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