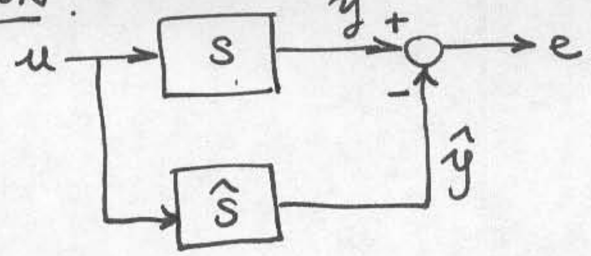
Motivational question:

NOTATION.



STABILITY.

"L2 gain" $\max \sqrt{\dfrac{\text{power}(e)}{\text{power}(u)}}$

ASSUMPTION. S & $\hat{S}$ LTI systems

$$\text{L2 gain} = \| G - \hat{G} \|_\infty = \sup_\omega \sigma_{max}\left[G(j\omega) - \hat{G}(j\omega) \right]$$

"absolute measure"

Theorem 1:

given quadratic form $\sigma : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$
and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n \times m}$

then, (a) $\exists$ $\varepsilon > 0$ such that $\sigma^H(x,u) \geq \varepsilon(|x|^2 + |u|^2)$
$j\omega x = Ax + Bu$, $\omega \in \mathbb{R}$, $x \in \mathbb{C}^m$, $u \in \mathbb{C}^m$

(b) $\sigma(0,u) >> 0$

$$\boxed{\begin{aligned}\sigma &= x_1 x_2 \\ \sigma^H &= Re(\bar{x}_1 x_2)\end{aligned}}$$

"Hamiltonian System" $\begin{cases} \dfrac{\partial x}{\partial t} = -\left(\dfrac{\partial H}{\partial \psi}\right)' \\ \dfrac{\partial \psi}{\partial t} = \left(\dfrac{\partial H}{\partial x}\right)' \\ \dfrac{\partial H}{\partial u} = 0 \end{cases}$ where $H = \sigma(x,u) - \psi'(Ax + Bu)$

∘ want show H has no eigenvalues on $j\omega$-axis

Establish bound on norm...

$\| C(j\omega I - A)^{-1} B \|_\infty < \gamma$

$\sigma(x,u) = \gamma^2 |u|^2 - |Cx|^2$

$Cx = C(j\omega I - A)^{-1} Bu$ when $j\omega x = Ax + Bu$.

Hamiltonian system ...

$\dot{x} = Ax + Bu$, $\dot{\psi} = C'Cx - A'\psi$, $B'\psi = \gamma^2 u$

$$\left.\begin{aligned} \dot{x} &= Ax + \gamma^{-2}BB'\psi \\ \dot{\psi} &= C'Cx - A'\psi \end{aligned}\right\} \text{ eliminate } u.$$

$$H_0 = \begin{bmatrix} A & \gamma^{-2}BB' \\ C'C & -A' \end{bmatrix} \in \mathbb{R}^{2n \times 2n}$$

eigenvalues of H_0 symmetric
w.r.t. real <u>and</u> <u>imag</u> axes.

$$\underline{\det(j\omega I - H_0)} = \text{const} \cdot \underbrace{\det(\gamma^2 I - G(j\omega)'G(j\omega))}_{\text{zero of}} \cdot \underbrace{|\det(j\omega I - A)|^2}_{\substack{>0 \text{ for} \\ A \text{ stable}}}$$

implies zero of

$\blacksquare$

From Theorem 1...
equivalent condition

(c) $\exists\ P = P' > 0$ s.t. $\sigma(x,u) + \underbrace{2x'P(Ax+Bu)}_{\frac{\partial}{\partial t}(x'Px)} \gg 0$

$$\int_0^T \gamma^2|u|^2 - |Cx|^2 + x'Px\big|_T - x'Px\big|_0 \geq 0 .$$

$$P = \begin{bmatrix} P_{oo} & P_{or} \\ P_{ro} & P_{rr} \end{bmatrix}$$

P_{oo} from energy functional from original system