

# PROJECTION-BASED MODEL REDUCTION ALGORITHMS FOR INVERSE PROBLEMS

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**Abstract.** This paper briefly describes the formulation and implementation of projection-based model reduction for the solution of deterministic and statistical inverse problems. The projection framework is introduced. We highlight the difficulty associated with identifying an efficient basis in the inverse problem setting. The additional challenge of understanding how reduced-order models treat uncertainty in the statistical setting is also presented.

**1. Introduction.** Model reduction algorithms are useful in real-time and multi-query settings. In contaminant source inversion, prediction, and control problems, we utilize reduced-order models in the real-time setting. Using a set of sensor data, we solve an optimization problem for the initial contaminant distribution. The iterative solution requires forward and adjoint solves which are too expensive at full order. Reduced-order models permit real-time inversion. There has been growing interest in the statistical inverse problem recently, especially as it pertains to quantifying uncertainty in the solution. In this setting, we use a Bayesian formulation of the inverse problem. Then, we require many (e.g., hundreds of thousands) forward model evaluations — a task far too costly to be useful in analysis or design. Instead, a reduced-order model can be utilized at far less cost.

**2. Projection framework.** We introduce the basics of the projection framework. Here, we adopt the perspective of a parameterized LTI system in state-space form.<sup>1</sup> Let

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \mathbf{y} = \mathbf{C}\mathbf{x} \tag{2.1}$$

be a state-space system with state  $\mathbf{x} \in \mathbb{R}^N$ , input  $\mathbf{u} \in \mathbb{R}^q$ , and outputs  $\mathbf{y} \in \mathbb{R}^p$ .

A reduced-order model of (2.1) can be created by assuming  $\mathbf{x} \approx \mathbf{V}\mathbf{x}_r$  where  $\mathbf{V} \in \mathbb{R}^{N \times n}$  has a set of trial basis functions as its columns and  $\mathbf{x}_r$  is the reduced state. We may then write the residual of the state equation as  $\mathbf{r} = \mathbf{V}\dot{\mathbf{x}}_r - \mathbf{A}\mathbf{V}\mathbf{x}_r - \mathbf{B}\mathbf{u}$ . If we require this residual to be orthogonal to a set of test functions occupying the columns of  $\mathbf{W} \in \mathbb{R}^{N \times n}$ , we arrive at the reduced-order model

$$\dot{\mathbf{x}}_r = \mathbf{A}_r\mathbf{x}_r + \mathbf{B}_r\mathbf{u}, \quad \mathbf{y}_r = \mathbf{C}_r\mathbf{x}_r \tag{2.2}$$

where we have utilized  $\mathbf{W}^T\mathbf{V} = \mathbf{I}_n$ ,  $\mathbf{A}_r = \mathbf{W}^T\mathbf{A}\mathbf{V}$ ,  $\mathbf{B}_r = \mathbf{W}^T\mathbf{B}$ , and  $\mathbf{C}_r = \mathbf{C}\mathbf{V}$ . We now have a system with  $n \ll N$  states which can be simulated with much greater efficiency. We have purposefully neglected description of the trial and test functions — their construction remains an important application-dependent challenge (see Section 4.1). In the next section, we describe the formulation of a generic inverse problem.

**3. Inverse problems.** An inverse problem consists of determining the indirectly observable parameters of a system given data. In the deterministic setting, the inverse problem is often formulated as a mathematical program whose solution minimizes the  $L^2$ -norm of the difference between model outputs and data, i.e. find parameter  $z^*$  such that

$$z^* = \arg \min_{\mathcal{J}} \int_{\Omega} (y - y_d)^2 \, d\mathbf{x} \tag{3.1}$$

subject to the forward model  $y = y(z)$ . The deterministic solution is a single-point estimate of the underlying parameter.

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<sup>1</sup>Finite element formulations via the weak form are also instructive.

In the statistical setting, we prefer to quantify uncertainty in the solution by casting the inverse problem as one of Bayesian inference. We make two assumptions: (1) a prior probability density over the parameter space, and (2) a probability density over outputs (e.g., normal sensor noise). With these assumptions and Bayes’s rule, we compute the posterior distribution

$$\pi_z(z|y_d) \propto L(y_d|z)\zeta_z(z) \quad (3.2)$$

where  $\zeta_z(z)$  is the prior and  $L(y_d|z)$  is the likelihood. Thus, the posterior expresses the conditional dependence of the parameter on the data. In problems of interest, the posterior distribution is implicit in the parameter, and therefore must be explored by a Monte Carlo technique (e.g., MCMC).

**4. Challenges.** Although there are many challenges for projection-based model reduction, we focus here on two issues which are most closely related to my work: (1) sampling to produce an efficient reduced basis, and (2) performing a rigorous error analysis on the final product, particularly in the statistical setting regarding the treatment of uncertainty.

**4.1. Sampling.** Reduced-order models are often used as surrogates for full-order models when the region of interest in parameter space is well-defined (e.g., control system design). In these cases, constructing the test and trial basis can usually be adequately performed by sampling the parameters of interest.<sup>2</sup> Other approaches based on the Hankel singular values can be shown to be optimal in the Hankel norm.

In inverse problems, however, we do not know *a priori* over which parameters we desire reduced model accuracy. A greedy approach was proposed in [1]. The approach attempts to leverage the governing equations (including model outputs) to select mutually controllable/observable parameters. However, the resulting sampling procedure is apparently suboptimal, very costly for high-dimensional parameter spaces, and has an unclear extension to the statistical setting. The challenge here is to develop a systematic (but cheap) method for sampling parameters to build an efficient reduced-order model in the statistical inverse problem setting. Such a method must have wide-ranging applicability within parameter space to allow for proper characterization of uncertainty.

**4.2. Error analysis.** The analysis of the efficiency of a basis resulting from the sampling process of Section 4.1 and an *a posteriori* error estimate must be readily available to incorporate model reduction techniques into widely used codes for engineering analysis and design.

It is standard practice in projection-based model reduction to demonstrate the trend that your reduced-order model accuracy approaches full-order model accuracy in the limit of increasing number of basis vectors. However, we usually pay little or no attention to the number of basis vectors at which we obtain sufficient accuracy. A thorough understanding of how choice of basis affects reduced model convergence to the full model is necessary to gain insight into the sampling procedures of Section 4.1. A different, but related, question involves *a posteriori* error estimation. Once we have established a reduced model, it would be beneficial to understand the errors introduced by the associated approximation. This question may be particularly challenging in the statistical setting where we must quantify those errors not only for linear estimators but for variance estimators as well.

## REFERENCES

- [1] K. VEROY, C. PRUD’HOMME, D. ROVAS, AND A. PATERA, *A posteriori error bounds for reduced-basis approximation of parametrized noncoercive and nonlinear elliptic partial differential equations*, in Proceedings of the 16th AIAA Computational Fluid Dynamics Conference, Orlando, FL, 2003.

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<sup>2</sup>Though the resulting reduced-order model is quite unlikely to be optimal.