Homework for the IAP lecture

"It's a Matter of Degree"

9 January 2006

1. This function deals with a continuous function $f : S^1 \to S^1$ such that $f^{-1}(P)$ (where P is the point $(1,0) \in S^1$) is given by the points with angle $\theta = 0, \pi/4, \pi/2, \pi$, with signs $+, +, -, +$, in order. Write k for its winding number=degree.

(a) Describe such a function. ("Describe" does not mean, necessarily, "give a formula for.") What is k?

(b) The claim is that f is homotopic to f_k, the map which multiplies polar angle by k. Describe such a homotopy.

(c) The homotopy is a function $h : S^1 \times I \to S^1$, where $I = [0,1]$ is the unit interval. Let's write $h_t(x)$ for the value $h(x,t)$; $x \in S^1$, $t \in I$. $h_0(x) = f(x)$, $h_1(x) = f_k(x)$, and as t moves from 0 to 1, $h_t(x)$ moves from $f(x)$ to $f_k(x)$. For each t, $h_t(P)$ will be some points (with signs) on the circle. If we draw them on the circle on the cylinder $S^1 \times I$ at height t, and let t vary, we get a curve (or a family of curves) on the cylinder. Take your homotopy from **(b)** and draw these curves.

2. Describe the homotopies corresponding to the diagrams of curves on the cylinders drawn below.

3. Suppose $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ is a polynomial. (The coefficients can be real or complex numbers. The "fundamental theorem of algebra" is that, as long as $n > 0$, this polynomial has a complex (which includes the possiblity of real!) root.

Be ready to talk through the following proof.

If $a_0 = 0$, then $z = 0$ is a root. So suppose $a_0 \neq 0$, and assume that there is no complex(or real) root.

Pick R very large (much larger than any of the coefficients) and consider what happens to $p(z)$ as z runs over the circle in the complex plane of radius R and center 0. Since there is no root, we can divide by $|p(z)|$ and get a function from the circle to the circle.

What is its winding number? (This is where you will use the assumption that R is large.)

Now think about what happens to $p(z)$ as z runs over smaller circles. For very small radius, $p(z)$ stays near $p(0) = a_0$. Obtain a contradiction!

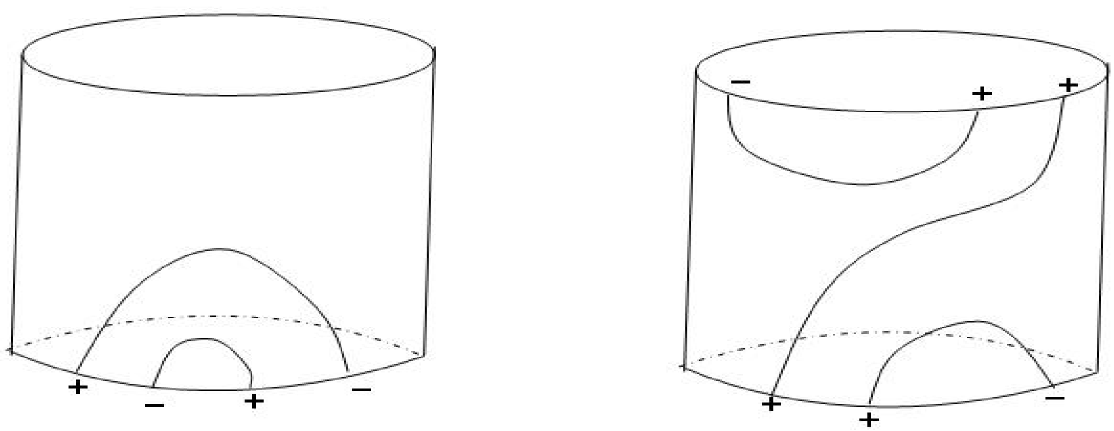

Figure 1: Problem 2 asks the corresponding homotopies to these diagrams.