IAP 2006 Mathematics Lecture Series Lecture 5

"Lie Groups and Differential Equations"

20 **January 2006**

1 Groups

Let X be a set, ϕ: $X \to X$ a one-to-one, onto map (i.e. bijection). We define

$$B(X) = \{\phi : X \to X,\ \phi \text{ a bijection}\}$$

the set of all bijections from X to X. If $\phi, \psi \in B(X)$, then we can define $\phi\psi \in B(X)$ by

$$(\phi\psi)(x) = \phi(\psi(x))$$

Also, for $\phi \in B(X)$, we define $\phi^{-1} \in B(X)$ by

$$\phi^{-1}(\phi(x)) = x$$

Then

$$\phi^{-1}\phi = \phi\phi^{-1} = I$$

where I is the identity map. In addition, for $\phi, \psi, \tau \in B(X)$, we always have

$$\phi(\psi\tau) = (\phi\psi)\tau$$

because $(\phi(\psi\tau))(x) = \phi(\psi(\tau(x))) = \phi\psi(\tau(x)) = ((\phi\psi)\tau)(x)$ for all $x \in B(X)$.
In general

$$\phi\psi \neq \psi\phi$$

For example, let $X = \{1, 2, 3\}$, $\alpha \in B(X)$ a permutation such that

$$\alpha : \begin{pmatrix} 1 & 2 & 3 \\ \alpha(1) & \alpha(2) & \alpha(3) \end{pmatrix} \tag{1}$$

and take

$$\phi : \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},\ \psi : \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

then

$$\phi\psi : \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \neq \psi\phi = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

A group is a set G with the following properties:

- There is binary operation on G such that for any $a, b \in G$, we have $ab \in G$, and $(ab)c = a(bc)$.
- There exist an element $e \in G$ such that $ae = ea$, for all $a \in G$.
- For every $a \in G$, there is $a^{-1} \in G$ such that $aa^{-1} = a^{-1}a = e$.

Observe that $B(X)$ is a group.

A subgroup of G is a subset $H \subset G$, such that if $h, k \in G$, then $hk \in G$, $h^{-1} \in G$. H is a normal subgroup if for all $g \in G$, $h \in H$, we have $ghg^{-1} \in H$.

If H is a normal subgroup, then $G/H = \{gH : g \in G\}$ is a group with the binary operation

$$(g_1 H)(g_2 H) = g_1 g_2 H$$

Here $gH = \{gh : h \in H\}$ are the left cosets of H.

The group G/H is called the factor group.

A group G is called an abelian group if and only if $g_1 g_2 = g_2 g_1$ for all $g_1, g_2 \in G$.

For any group G, let

$$\begin{aligned} G' &= \text{Subgroup generated by all } xyx^{-1}y^{-1},\ x, y \in G \\ &= \text{Smallest subgruop of } G \text{ containing all } xyx^{-1}y^{-1},\ x, y \in G \end{aligned}$$

It can be proved that G/G' is abelian. Obviously

$$G \supset G' \supset (G')' \supset ...$$

G is called solvable if this process ends with $\{e\}$.

The problem of finding roots of an n th order polynomial, (i.e. finding numbers x such that $p(x) = 0$) is solvable by radicals for $n = 1, 2, 3, 4$. That is, up to order 4, we can get closed expressions for the roots in terms of the coefficients of the polynomial. The word radicals refer to expressions involving square (or higher) roots.

However, this is not true when $n \geq 5$. Although, roots of some fifth order polynomials can be located easily by inspection, there is no closed expression for the roots of a *general* fifth order polynomial. And that it is impossible to obtain such an expression is proved by Abel in nineteenth century. Abel showed that roots of a polynomial can be solved by radicals if and only if the group of permutations of those roots is solvable, and it is known that this is not always the case when there are more than five roots involved (more precisely, the permutation groups S_1, S_2, S_3, S_4 are solvable, whereas S_5 is not. Here S_k denotes the permutation group with k elements, for example α in (1) is an element of S_3)

2 Lie's Idea

Consider the differential equation

$$\frac{dy}{dx} = F(x, y) \tag{2}$$

A solution of this equation is a function $u(x)$ such that

$$u'(x) = F(x, u(x))$$

Another name for solutions is "integral curve". This is more relevant when we think of the solutions as curves. In this case, we consider the solution curve (which is called integral curve) traced by the point $(x, u(x))$ in $\mathbb{R}^2$ as x varies in $\mathbb{R}$. Therefore the integral curves live in $\mathbb{R}^2$.

A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ (or $T \in B(\mathbb{R}^2)$) is said to leave the equation (2) stable if (and only if) T permutes the integral curves. That is, if $u(x)$ is a solution, then $T(x, u(x))$ must be an integral curve.

Example: For example, if $F(x,y) = g(x)$, then (2) becomes

$$\frac{dy}{dx} = g(x) \tag{3}$$

which has solutions

$$u(x) = \int_a^x g(z)dz + C \tag{4}$$

where C can be any constant. The integral curves (i.e. solutions) are "parallel" in the sense that the vertical distance between two solutions stays constant. In this case, a transformation $T_t \in B(\mathbb{R}^2)$ which leaves the equation (3) stable is given by

$$T : (x,y) \to (x, y+t)$$

because for every integral curve $(x, u(x)) : x \in \mathbb{R}$,

$$T : (x, u(x)) \to (x, u(x) + t)$$

and $(x, u(x) + t)$ is another integral curve for (2)(because $u(x) + t$ is a solution, here t is constant in x -adding a constant to a solution yields another solution by eqn(4)).

Observe that $T_{t+s} = T_t T_s$, and therefore the set $G = \{T_t : t \in \mathbb{R}\}$ constitutes a group(you can verify this easily). A group G with such properties is called a 1-parameter gruop.

Now, let $t \to \phi_t$ be a differentiable 1-parameter group of bijections of $\mathbb{R}^2$, (in which case $\phi_{t+s} = \phi_t \phi_s$, and ϕ_t is differentiable in t), then for any point $p = (x,y)$, we can define the vector field

$$\Phi_p = \Phi(x,y) = \left\{\frac{d}{dt}(\phi_t(p))\right\}_{t=0} \in \mathbb{R}^2$$

Let us write Φ_p in the form

$$\Phi_p = \xi(x,y).\hat{x} + \eta(x,y).\hat{y}$$

To confuse people (and look more sophisticated), mathematicians use the notation $\frac{\partial}{\partial x}$ for $\hat{x}$, and $\frac{\partial}{\partial y}$ for $\hat{y}$. Using this notation, we can rewrite

$$\Phi_p = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$$

In other contexts, the notation is

$$\begin{aligned} \left(\frac{\partial}{\partial x}\right)_{(x,y)} f &= \left\{\frac{d}{du} f(x+u, y)\right\} \\ \left(\frac{\partial}{\partial y}\right)_{(x,y)} f &= \left\{\frac{d}{du} f(x, y+u)\right\} \end{aligned}$$

which are referred to as partial derivatives.

Theorem1:(Lie1875,NorwegianPeriodical) Consider the differential equation

$$\frac{dy}{dx} = \frac{Y(x,y)}{X(x,y)}$$

and assume that ϕ_t is a 1-parameter group which leaves this equation stable. Then

$$\frac{Xdy - Ydx}{X\eta - Y\xi}$$

is an exact differential $\frac{\partial U}{\partial x}dx+\frac{\partial U}{\partial y}dy$. In other words, there is a function U such that

$$\frac{\partial U}{\partial x}=\frac{-Y}{X\eta-Y\xi},\ \frac{\partial U}{\partial y}=\frac{X}{X\eta-Y\xi} \tag{5}$$

and the equation

$$U\left(x,y\right)=C$$

describes a solution of the equation for every constant C.

The function $\frac{1}{X\eta-Y\xi}$ is called the integrating factor for $Xdy-Ydx$.

The equation system (5) is solved by

$$U\left(x,y\right)=\int_a^x\frac{-Y}{X\eta-Y\xi}dx+h(y)$$

where the second equation in (5) determines $h\left(y\right)$.

Example: Consider

$$\frac{dy}{dx}=\frac{y+x\left(x^2+y^2\right)}{x-y\left(x^2+y^2\right)} \tag{6}$$

This equation can be rewritten, after some algebra, as

$$\frac{\frac{dy}{dx}-\frac{y}{x}}{1+\frac{y}{x}\frac{dy}{dx}}=x^2+y^2$$

Slope of ray from origin to the point (x,y) is $\tan\alpha=\frac{y}{x}$. The slope of the integral curve is $\tan\beta=\frac{dy}{dx}$. Therefore

$$\tan\left(\beta-\alpha\right)=\frac{\tan\beta-\tan\alpha}{1+\tan\alpha\tan\beta}=x^2+y^2=r^2$$

is constant on circles.

Hence, the integral curves are symmetric under rotations about the origin, that is, rotating integral curves should yield us some other integral curves. In symbols,

$$\phi_t:(x,y)\rightarrow(x\cos t-y\sin t,x\sin t+y\cos t) \tag{7}$$

should take integral curves to integral curves(i.e. permutes integral curves). For your information, the transformation (7) is the standard maps which rotates point (x,y) by angle t around the origin. Hence, $\phi_{t+s}=\phi_t\phi_s$.

In this case, we see that $Y=y+x\left(x^2+y^2\right)$, $X=x-y\left(x^2+y^2\right)$, and

$$\frac{d\left(\phi_t p\right)}{dt}|_{t=0}=-y\hat{x}+x\hat{y}=-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}$$

hence $\xi=-y$, $\eta=x$, and therefore integrating factor is

$$\frac{1}{X\eta-Y\xi}=\left[x\left(x-y\left(x^2+y^2\right)\right)-(-y)\left(y+x\left(x^2+y^2\right)\right)\right]^{-1}=\left(x^2+y^2\right)^{-1}$$

The equations (5) becomes

$$\begin{aligned}\frac{\partial U}{\partial x}&=-\left(\frac{y}{x^2+y^2}+x\right)\\ \frac{\partial U}{\partial y}&=-\left(\frac{x}{x^2+y^2}-y\right)\end{aligned}$$

and the potential function U is, by inspection,

$$U(x,y)=\tan^{-1}(y/x)-\frac{x^2+y^2}{2}$$

Therefore the solutions to the original differential equation (6) are given by $U(x,y)=C$, i.e.

$$y=x\tan\left(\frac{1}{2}\left(x^2+y^2\right)+C\right)$$

where C is any constant(note that this is an implicit equation for the solutions).

Exercise1(HW): Consider

$$\frac{dy}{dx}=f(y/x) \tag{8}$$

(i)Show that on each ray emanating from the origin, the tangents to the integral curves at (x,y) and $\left(e^t x, e^t y\right)$ are parallel.

(ii)Deduce that

$$\phi_t:(x,y)\to\left(e^t x,e^t y\right)$$

is a 1-parameter group leaving equation stable.

(iii)Prove that $\Phi_p=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$

(iv)By the theorem, show

$$-f(y/x)\,dx+dy$$

has integrating factor $1/\left(y-x.f(y/x)\right).$

This means that

$$\frac{-f(y/x)}{y-xf(y/x)}dx+\frac{1}{y-x.f(y/x)}dy=\frac{\partial U}{\partial x}dx+\frac{\partial U}{\partial y}dy$$

where the coefficients of dx and dy are meant to be equal.

(v)Verify that, indeed

$$\frac{\partial}{\partial x}\left(\frac{1}{y-xf(y/x)}\right)=\frac{\partial}{\partial y}\left(\frac{-f(y/x)}{y-x.f(y/x)}\right)$$

proving Lie's theorem in this case.

Solution to the original equation (8) is $U(x,y)=C$.

(vi)Prove that the solution is

$$U(x,y)=\int_1^y\frac{dz}{z-x.f(z/x)}+g(x)$$

Here the relation

$$\frac{\partial U}{\partial x}=\frac{-f(y/x)}{y-x.f(y/x)}$$

gives a formula for $g(x)$. Find this $g(x)$ (in terms of expressions involving integrals).

Exercise2(HW): Using method from exercise1, solve the differential equation

$$\frac{dy}{dx}=\left(\frac{y}{x}\right)^2+2\frac{y}{x}$$

and find that the solution is

$$y=\frac{x^2}{e^C-x}$$

3 Higher Parameter Groups

Let $X = \mathbb{R}^2$, and $G \subset B(X)$ be the subgroup preserving distances and orientation. Let $\sigma \in G$. We can call

$$\sigma.O = t.O$$

where O is origin, and t is a translation, i.e. t is translation defined by $t(x) = x + \sigma.O$. Then, we see that

$$t^{-1}\sigma.O = O$$

i.e. $t^{-1}\sigma$ takes the origin to itself, hence it must be a rotation, say it is the rotation by angle $\theta(\sigma)$. Then, as we see $\sigma = t\,(rot.by.\theta(\sigma))$, which is

$$\sigma : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x(\sigma) \\ y(\sigma) \end{pmatrix} + \begin{pmatrix} \cos\theta(\sigma) & \sin\theta(\sigma) \\ -\sin\theta(\sigma) & \cos\theta(\sigma) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

where we used the notation

$$\begin{pmatrix} x(\sigma) \\ y(\sigma) \end{pmatrix} = \sigma.O$$

Then for any $\tau \in G$, we can check that

$$\begin{aligned} x\left(\sigma\tau^{-1}\right) &= x(\sigma) - x(\tau)\cos(\theta(\sigma) - \theta(\tau)) + y(\tau)\sin(\theta(\sigma) - \theta(\tau)) \\ y\left(\sigma\tau^{-1}\right) &= y(\sigma) - x(\tau)\sin(\theta(\sigma) - \theta(\tau)) - y(\tau)\cos(\theta(\sigma) - \theta(\tau)) \\ \theta\left(\sigma\tau^{-1}\right) &= \theta(\sigma) - \theta(\tau) \mod 2\pi \end{aligned} \tag{9}$$

Note that G involves three parameters, $x(\sigma), y(\sigma)$ and $\theta(\sigma)$.

Lie Group: A group whose elements parametrized such that group operations are smooth functions in parameters.

For example, the G described above is a Lie group because the right hand side of (9) is differentiable in x, y, and θ (of both σ and τ).

Lie Algebra: An algebra of functions with a Lie bracket which is antisymmetric and satisfies the Jacobi identity: we now explain.

Consider the transformation group of $\mathbb{R}^n$ depending effectively on r parameters. The elements are

$$T : (x_1, ..., x_n) \to \left(x_1', ..., x_n'\right)$$

where

$$x_i' = f_i(x_1, ..., x_n, t_1, ..., t_r)$$

Assume that, for $S = (s_1, ..., s_r)$, we have

$$TS^{-1} : x_i' = f_i(x_1, ..., x_n, u_1, ..., u_r)$$

where u_i 's are smooth functions of s_i and t_j

We can generalize the vector field Φ_p from before as

$$T_k = \sum_{i=1}^{n} \left(\frac{\partial f_i}{\partial t_k}\right)_{t=0} \frac{\partial}{\partial x_i} \tag{10}$$

which are vector fields on $\mathbb{R}^n$, for $1 \leq k \leq r$.

We can think of the $n-$dimensional vector fields X, Y on $\mathbb{R}^n$ are functions from $\mathbb{R}^n$ to $\mathbb{R}^n$, therefore the composition $X \circ Y$ makes sense. We now define the **lie bracket** of two vector fields by

$$[X, Y] = X \circ Y - Y \circ X$$

then obviously

$$[X, Y] = -[Y, X] \tag{11}$$

Also, by some tedious but straightforward calculation(or arguing by antisymmetry), we can show that

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \tag{12}$$

This is called the Jacobi identity.

A Lie algebra is an algebra of functions with a lie bracket $[,]$ obeying the properties (11) and (12) in addition to linearity

$$[X, Y + Z] = [X, Y] + [X, Z]$$

A lie algebra is abelian if $[X, Y] = 0$ for all X, Y.

Theorem: For the transformations T_k as given in (10), there are constants c_{kl}^m such that

$$[T_k, T_l] = \sum_{m=1}^{r} c_{kl}^m T_m$$

Note that this equation is defined for (say) $x \in \mathbb{R}^n$, and here c_{kl}^m are independent of x.

Let us define

$$\mathfrak{g} = \left\{ \sum_{m=1}^{r} a_m T_m : a_m \in \mathbb{R} \right\}$$

also, let us denote

$$\mathfrak{D}\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] = \{[g_1, g_2] : g_1, g_2 \in \mathfrak{g}\}$$

then, $\mathfrak{g}$ is called solvable if $\mathfrak{D}^s\mathfrak{g} = \{0\}$, for some integer s.

Theorem2: Suppose the system

$$\frac{dy_j}{dx} = f^j(x, y_1, ..., y_r) \;\; 1 \leq j \leq r$$

has a solvable $r-$ dimensional stability group in $(x, y_1, ..., y_r)$ with r -dimensional orbits. Then the solution can be found by repeated quadratures explicitly given by the group. (This is analog to the Galois group theorem)

Example: Consider

$$x^2 \frac{d^2y}{dx^2} = f\left(x, \frac{dy}{dx} - y\right)$$

and put $z = \frac{dy}{dx}$. Then this single equation becomes the $2-$ dimensional system

$$\begin{aligned} x^2 \frac{dz}{dx} &= f(xz - y) \\ \frac{dy}{dx} &= z \end{aligned} \tag{13}$$

Exercise3(HW): Show that

$$T_{s,t} : (x, y, z) \rightarrow \left(sx, y + tx, \frac{z}{s} + \frac{t}{s}\right)$$

leaves the system stable.

(Hint: Put $x_1 = sx, y_1 = y + tx, z_1 = \frac{z}{s} + \frac{t}{s}$ and observe that x_1, y_1, z_1 satisfy the system (13))

Also show that

$$T_{\sigma,\tau} \circ T_{s,t} = T_{\sigma s, t+\tau s}$$

——o—o—o———

Note that $T_{s,0}$ and $T_{1,t}$ are 1-parameter groups corresponding to the vector fields

$$X_1 = x\frac{\partial}{\partial x} - z\frac{\partial}{\partial z},\ X_2 = x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

and

$$[X_1, X_2] = X_2$$

Thus, $\mathfrak{g} = \mathbb{R}X_1 + \mathbb{R}X_2$ is solvable.

By theorem2, the system (13) is solvable by quadratures. We try

$$\int \frac{dw}{f(w)} = \log|x| + C \tag{14}$$

Let us write this as $w = g(x)$. Then

$$\frac{dy}{dx} - \frac{y}{x} = \frac{g(x)}{x}$$

We can solve this by Lie's method by observing that

$$(x, y) \to (x, y + tx)$$

leaves the equation stable. Corresponding vector field is $x\frac{\partial}{\partial y}$. By Lie's first theorem, $1/x$ is an integrating factor for

$$dy - \frac{y + g(x)}{x}dx = 0$$

Thus

$$\frac{dy}{x} - \frac{y + g(x)}{x^2}dx = \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial x}dx$$

and hence from $\frac{\partial U}{\partial y} = \frac{1}{x}$, we get

$$U(x, y) = \frac{y}{x} + h(x)$$

and h is determined from

$$\frac{\partial U}{\partial x} = -\frac{y}{x^2} + h'(x) = -\frac{y + g(x)}{x^2}$$

therefore

$$h(x) = -\int_1^x \frac{g(s)}{s^2}ds + cx$$