Counting III

Today we'll briefly review some facts you derived in recitation on Friday and then turn to some applications of counting.

1   The Bookkeeper Rule

In recitation you learned that the number of ways to rearrange the letters in the word BOOKKEEPER is:

$$\frac{\overbrace{10!}^{\text{total letters}}}{\underbrace{1!}_{\text{B's}}\ \underbrace{2!}_{\text{O's}}\ \underbrace{2!}_{\text{K's}}\ \underbrace{3!}_{\text{E's}}\ \underbrace{1!}_{\text{P's}}\ \underbrace{1!}_{\text{R's}}}$$

This is a special case of an exceptionally useful counting principle.

Rule 1 (Bookkeeper Rule). *The number of sequences with n_1 copies of l_1, n_2 copies of l_2, ..., and n_k copies of l_k is*

$$\frac{(n_1 + n_2 + \ldots + n_k)!}{n_1!\, n_2!\ \ldots\ n_k!}$$

provided $l_1, \ldots, l_k$ are distinct.

Let's review some applications and implications of the Bookkeeper Rule.

1.1   20-Mile Walks

I'm planning a 20 miles walk, which should include 5 northward miles, 5 eastward miles, 5 southward miles, and 5 westward miles. How many different walks are possible?

There is a bijection between such walks and sequences with 5 N's, 5 E's, 5 S's, and 5 W's. By the Bookkeeper Rule, the number of such sequences is:

$$\frac{20!}{5!^4}$$

1.2   Bit Sequences

How many n-bit sequences contain exactly k ones?

Each such sequence also contains $n - k$ zeroes, so there are

$$\frac{n!}{k!\,(n-k)!}$$

by the Bookkeeper Rule.

1.3 k-element Subsets of an n-element Set

How many k-elements subsets of an n-element set are there? This question arises all the time in various guises:

- In how many ways can I select 5 books from my collection of 100 to bring on vacation?

- How many different 13-card Bridge hands can be dealt from a 52-card deck?

- In how many ways can I select 5 toppings for my pizza if there are 14 available?

There is a natural bijection between k-element subsets of an n-element set and n-bit sequences with exactly k ones. For example, here is a 3-element subset of $\{x_1, x_2, \ldots, x_8\}$ and the associated 8-bit sequence with exactly 3 ones:

$$\begin{array}{ccccccccc} \{ & x_1, & & & x_4, & x_5 & & & \} \\ (& 1, & 0, & 0, & 1, & 1, & 0, & 0, & 0 \quad) \end{array}$$

Therefore, the answer to this problem is the same as the answer to the earlier question about bit sequences.

Rule 2 (Subset Rule). *The number of k-element subsets of an n-element set is:*

$$\frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$

The factorial expression in the Subset Rule comes up so often that there is a shorthand, $\binom{n}{k}$. This is read "n choose k" since it denotes the number of ways to choose k items from among n. We can immediately knock off all three questions above using the Sum Rule:

- I can select 5 books from 100 in $\binom{100}{5}$ ways.

- There are $\binom{52}{13}$ different Bridge hands.

- There are $\binom{14}{5}$ different 5-topping pizzas, if 14 toppings are available.

The k-element subsets of an n-element set are sometimes called k-***combinations***. There are a great many similar-sounding terms: permutations, r-permutations, permutations with repetition, combinations with repetition, permutations with indistinguishable objects, and so on. For example, the Bookkeeper rule is elsewhere called the "formula for permutations with indistinguishable objects". We won't drag you through all this terminology other than to say that, broadly speaking, all the terms mentioning *permutations* concern sequences and all terms mentioning *combinations* concern subsets.

1.4   An Alternative Derivation

Let's derive the Subset Rule another way to gain an alternative perspective. The number of *sequences* consisting of k distinct elements drawn from an n-element set is

$$n \cdot (n-1) \cdot (n-2) \cdots (n-k+1) = \frac{n!}{(n-k)!}$$

by the Generalized Product Rule. Now suppose we map each sequence to the set of elements it contains. For example:

$$(x_1, x_2, x_3) \qquad \rightarrow \qquad \{x_1, x_2, x_3\}$$

This is a $k!$-to-1 mapping since each k-element set is mapped to by all of its $k!$ permutations. Thus, by the Quotient Rule, the number of k-element subsets of an n-element set is:

$$\frac{n!}{k! \, (n-k)!} = \binom{n}{k}$$

1.5   Word of Caution

Someday you might refer to the Bookkeeper Rule in front of a roomful of colleagues and discover that they're all staring back at you blankly. This is not because they're dumb, but rather because we just made up the name "Bookkeeper Rule". However, the rule is excellent and the name is apt, so we suggest that you play through: "You know? The Bookkeeper Rule? Don't you guys know *anything???*"

2   Binomial Theorem

Counting gives insight into one of the basic theorems of algebra. A *binomial* is a sum of two terms, such as $a + b$. Now let's consider a postive, integral power of a binomial:

$$(a + b)^n$$

Suppose we multiply out this expression completely for, say, $n = 4$:

$$\begin{aligned}
(a+b)^4 = {}& aaaa + aaab + aaba + aabb \\
& + abaa + abab + abba + abbb \\
& + baaa + baab + baba + babb \\
& + bbaa + bbab + bbba + bbbb
\end{aligned}$$

Notice that there is one term for every sequence of a's and b's. Therefore, the number of terms with k copies of b and $n - k$ copies of a is:

$$\frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$

by the Bookkeeper Rule. Now let's group equivalent terms, such as $aaab = aaba = abaa = baaa$. Then the coefficient of $a^{n-k}b^k$ is $\binom{n}{k}$. So for $n = 4$, this means:

$$(a+b)^4 = \binom{4}{0} \cdot a^4 b^0 + \binom{4}{1} \cdot a^3 b^1 + \binom{4}{2} \cdot a^2 b^2 + \binom{4}{3} \cdot a^1 b^3 + \binom{4}{4} \cdot a^0 b^4$$

In general, this reasoning gives the Binomial Theorem:

Theorem 1 (Binomial Theorem). *For all $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$:*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

The expression $\binom{n}{k}$ is often called a "binomial coefficient" in honor of its appearance here.

3   Poker Hands

There are 52 cards in a deck. Each card has a *suit* and a *value*. There are four suits:

spades	hearts	clubs	diamonds
♠	♡	♣	◇

And there are 13 values:

$$2\,,\ 3\,,\ 4\,,\ 5\,,\ 6\,,\ 7\,,\ 8\,,\ 9\,,\ \overset{\text{jack}}{J}\,,\ \overset{\text{queen}}{Q}\,,\ \overset{\text{king}}{K}\,,\ \overset{\text{ace}}{A}$$

Thus, for example, $8\heartsuit$ is the 8 of hearts and $A\spadesuit$ is the ace of spades. Values farther to the right in this list are considered "higher" and values to the left are "lower".

Five-Card Draw is a card game in which each player is initially dealt a *hand*, a subset of 5 cards. (Then the game gets complicated, but let's not worry about that.) The number of different hands in Five-Card Draw is the number of 5-element subsets of a 52-element set, which is 52 choose 5:

$$\text{total \# of hands} = \binom{52}{5} = 2,598,960$$

Let's get some counting practice by working out the number of hands with various special properties.

3.1   Hands with a Four-of-a-Kind

A *Four-of-a-Kind* is a set of four cards with the same value. How many different hands contain a Four-of-a-Kind? Here a couple examples:

$$\{ \quad 8\spadesuit, \quad 8\diamondsuit, \quad Q\heartsuit, \quad 8\heartsuit, \quad 8\clubsuit \quad \}$$
$$\{ \quad A\clubsuit, \quad 2\clubsuit, \quad 2\heartsuit, \quad 2\diamondsuit, \quad 2\spadesuit \quad \}$$

As usual, the first step is to map this question to a sequence-counting problem. A hand with a Four-of-a-Kind is completely described by a sequence specifying:

1. The value of the four cards.

2. The value of the extra card.

3. The suit of the extra card.

Thus, there is a bijection between hands with a Four-of-a-Kind and sequences consisting of two distinct values followed by a suit. For example, the three hands above are associated with the following sequences:

$$(8, Q, \heartsuit) \quad \leftrightarrow \quad \{ \quad 8\spadesuit, \quad 8\diamondsuit, \quad 8\heartsuit, \quad 8\clubsuit, \quad Q\heartsuit \quad \}$$
$$(2, A, \clubsuit) \quad \leftrightarrow \quad \{ \quad 2\clubsuit, \quad 2\heartsuit, \quad 2\diamondsuit, \quad 2\spadesuit, \quad A\clubsuit \quad \}$$

Now we need only count the sequences. There are 13 ways to choose the first value, 12 ways to choose the second value, and 4 ways to choose the suit. Thus, by the Generalized Product Rule, there are $13 \cdot 12 \cdot 4 = 624$ hands with a Four-of-a-Kind. This means that only 1 hand in about 4165 has a Four-of-a-Kind; not surprisingly, this is considered a very good poker hand!

3.2   Hands with a Full House

A *Full House* is a hand with three cards of one value and two cards of another value. Here are some examples:
$$\{ \quad 2\spadesuit, \quad 2\clubsuit, \quad 2\diamondsuit, \quad J\clubsuit, \quad J\diamondsuit \quad \}$$
$$\{ \quad 5\diamondsuit, \quad 5\clubsuit, \quad 5\heartsuit, \quad 7\heartsuit, \quad 7\clubsuit \quad \}$$
Again, we shift to a problem about sequences. There is a bijection between Full Houses and sequences specifying:

1. The value of the triple, which can be chosen in 13 ways.

2. The suits of the triple, which can be selected in $\binom{4}{3}$ ways.

3. The value of the pair, which can be chosen in 12 ways.

4. The suits of the pair, which can be selected in $\binom{4}{2}$ ways.

The example hands correspond to sequences as shown below:

$$(2, \{\spadesuit, \clubsuit, \diamondsuit\}, J, \{\clubsuit, \diamondsuit\}) \leftrightarrow \{ \ 2\spadesuit, \ 2\clubsuit, \ 2\diamondsuit, \ J\clubsuit, \ J\diamondsuit \ \}$$
$$(5, \{\diamondsuit, \clubsuit, \heartsuit\}, 7, \{\heartsuit, \clubsuit\}) \leftrightarrow \{ \ 5\diamondsuit, \ 5\clubsuit, \ 5\heartsuit, \ 7\heartsuit, \ 7\clubsuit \ \}$$

By the Generalized Product Rule, the number of Full Houses is:

$$13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2}$$

We're on a roll— but we're about to hit a speedbump.

3.3 Hands with Two Pairs

How many hands have *Two Pairs*; that is, two cards of one value, two cards of another value, and one card of a third value? Here are examples:

$$\{ \ 3\diamondsuit, \ 3\spadesuit, \ Q\diamondsuit, \ Q\heartsuit, \ A\clubsuit \ \}$$
$$\{ \ 9\heartsuit, \ 9\diamondsuit, \ 5\heartsuit, \ 5\clubsuit, \ K\spadesuit \ \}$$

Each hand with Two Pairs is described by a sequence consisting of:

1. The value of the first pair, which can be chosen in 13 ways.

2. The suits of the first pair, which can be selected $\binom{4}{2}$ ways.

3. The value of the second pair, which can be chosen in 12 ways.

4. The suits of the second pair, which can be selected in $\binom{4}{2}$ ways.

5. The value of the extra card, which can be chosen in 11 ways.

6. The suit of the extra card, which can be selected in $\binom{4}{1} = 4$ ways.

Thus, it might appear that the number of hands with Two Pairs is:

$$13 \cdot \binom{4}{2} \cdot 12 \cdot \binom{4}{2} \cdot 11 \cdot 4$$

Wrong answer! The problem is that there is *not* a bijection from such sequences to hands with Two Pairs. This is actually a 2-to-1 mapping. For example, here are the pairs of sequences that map to the hands given above:

$$(3, \{\diamondsuit, \spadesuit\}, Q, \{\diamondsuit, \heartsuit\}, A, \clubsuit) \searrow$$
$$\phantom{(3, \{\diamondsuit, \spadesuit\}, Q, \{\diamondsuit, \heartsuit\}, A, \clubsuit)} \quad \{ \ 3\diamondsuit, \ 3\spadesuit, \ Q\diamondsuit, \ Q\heartsuit, \ A\clubsuit \ \}$$
$$(Q, \{\diamondsuit, \heartsuit\}, 3, \{\diamondsuit, \spadesuit\}, A, \clubsuit) \nearrow$$

$$(9, \{\heartsuit, \diamondsuit\}, 5, \{\heartsuit, \clubsuit\}, K, \spadesuit) \searrow$$
$$\phantom{(9, \{\heartsuit, \diamondsuit\}, 5, \{\heartsuit, \clubsuit\}, K, \spadesuit)} \quad \{ \ 9\heartsuit, \ 9\diamondsuit, \ 5\heartsuit, \ 5\clubsuit, \ K\spadesuit \ \}$$
$$(5, \{\heartsuit, \clubsuit\}, 9, \{\heartsuit, \diamondsuit\}, K, \spadesuit) \nearrow$$

The problem is that nothing distinguishes the first pair from the second. A pair of 5's and a pair of 9's is the same as a pair of 9's and a pair of 5's. We avoided this difficulty in counting Full Houses because, for example, a pair of 6's and a triple of kings is different from a pair of kings and a triple of 6's.

We ran into precisely this difficulty last time, when we went from counting arrangements of *different* pieces on a chessboard to counting arrangements of two *identical* rooks. The solution then was to apply the Division Rule, and we can do the same here. In this case, the Division rule says there are twice as many sequences and hands, so the number of hands with Two Pairs is actually:

$$\frac{13 \cdot \binom{4}{2} \cdot 12 \cdot \binom{4}{2} \cdot 11 \cdot 4}{2}$$

Another Approach

The preceding example was disturbing! One could easily overlook the fact that the mapping was 2-to-1 on an exam, fail the course, and turn to a life of crime. You can make the world a safer place in two ways:

1. Whenever you use a mapping $f : A \rightarrow B$ to translate one counting problem to another, check the number elements in A that are mapped to each element in B. This determines the size of A relative to B. You can then apply the Division Rule with the appropriate correction factor.

2. As an extra check, try solving the same problem in a different way. Multiple approaches are often available— and all had better give the same answer! (Sometimes different approaches give answers that *look* different, but turn out to be the same after some algebra.)

We already used the first method; let's try the second. There is a bijection between hands with two pairs and sequences that specify:

1. The values of the two pairs, which can be chosen in $\binom{13}{2}$ ways.

2. The suits of the lower-value pair, which can be selected in $\binom{4}{2}$ ways.

3. The suits of the higher-value pair, which can be selected in $\binom{4}{2}$ ways.

4. The value of the extra card, which can be chosen in 11 ways.

5. The suit of the extra card, which can be selected in $\binom{4}{1} = 4$ ways.

For example, the following sequences and hands correspond:

$$(\{3, Q\}, \{\diamondsuit, \spadesuit\}, \{\diamondsuit, \heartsuit\}, A, \clubsuit) \quad \leftrightarrow \quad \{\quad 3\diamondsuit, \quad 3\spadesuit, \quad Q\diamondsuit, \quad Q\heartsuit, \quad A\clubsuit \quad \}$$
$$(\{9, 5\}, \{\heartsuit, \clubsuit\}, \{\heartsuit, \diamondsuit\}, K, \spadesuit) \quad \leftrightarrow \quad \{\quad 9\heartsuit, \quad 9\diamondsuit, \quad 5\heartsuit, \quad 5\clubsuit, \quad K\spadesuit \quad \}$$

Thus, the number of hands with two pairs is:

$$\binom{13}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 11 \cdot 4$$

This is the same answer we got before, though in a slightly different form.

3.4   Hands with Every Suit

How many hands contain at least one card from every suit? Here is an example of such a hand:

$$\{\quad 7\diamondsuit, \quad K\clubsuit, \quad 3\diamondsuit, \quad A\heartsuit, \quad 2\spadesuit \quad \}$$

Each such hand is described by a sequence that specifies:

1. The values of the diamond, the club, the heart, and the spade, which can be selected in $13 \cdot 13 \cdot 13 \cdot 13 = 13^4$ ways.

2. The suit of the extra card, which can be selected in 4 ways.

3. The value of the extra card, which can be selected in 12 ways.

For example, the hand above is described by the sequence:

$$(7, K, A, 2, \diamondsuit, 3) \quad \leftrightarrow \quad \{\quad 7\diamondsuit, \quad K\clubsuit, \quad A\heartsuit, \quad 2\spadesuit, \quad 3\diamondsuit \quad \}$$

Are there other sequences that correspond to the same hand? There is one more! We could equally well regard either the $3\diamondsuit$ or the $7\diamondsuit$ as the extra card, so this is actually a 2-to-1 mapping. Here are the two sequences corresponding to the example hand:

$$(7, K, A, 2, \diamondsuit, 3) \quad \searrow$$
$$\qquad\qquad\qquad\qquad \{\quad 7\diamondsuit, \quad K\clubsuit, \quad A\heartsuit, \quad 2\spadesuit, \quad 3\diamondsuit \quad \}$$
$$(3, K, A, 2, \diamondsuit, 7) \quad \nearrow$$

Therefore, the number of hands with every suit is:

$$\frac{13^4 \cdot 4 \cdot 12}{2}$$

4 Magic Trick

There is a Magician and an Assistant. The Assistant goes into the audience with a deck of 52 cards while the Magician looks away. Five audience members each select one card from the deck. The Assistant then gathers up the five cards and reveals four of them to the Magician, one at a time. The Magician concentrates for a short time and then correctly names the secret, fifth card!

4.1 The Secret

The Assistant somehow communicated the secret card to the Magician just by naming the other four cards. In particular, the Assistant has two ways to communicate:

1. He can announce the four cards in any order. The number of orderings of four cards is $4! = 24$, so this alone is insufficient to identify which of the remaining 48 cards is the secret one.

2. The Assistant can choose which four of the five cards to reveal. Of course, the Magician can not determine which of these five possibilities the Assistant selected since he does not know the secret card.

Nevertheless, these two forms of communication allow the Assistant to covertly reveal the secret card to the Magician.

We'll let you think about how this might be done, and you will see the answer in recitation tomorrow!.

4.2 Same Trick with Four Cards?

Suppose that the audience selects only *four* cards and the Assistant reviews a sequence of *three* to the Magician. Can the Magician determine the fourth card?

Let X be all the sets of four cards that the audience might select, and let Y be all the sequences of three cards that the Assistant might reveal. Now, one on hand, we have

$$|X| = \binom{52}{4} = 270,725$$

by the Subset Rule. On the other hand, we have

$$|Y| = 52 \cdot 51 \cdot 50 = 132,600$$

by the Generalized Product Rule. Thus, by the Pigeonhole Principle, the Assistant must reveal the *same* sequence of three cards for some two *different* sets of four. This is bad news for the Magician: if he hears that sequence of three, then there are at least two possibilities for the fourth card which he cannot distinguish!

5   Combinatorial Proof

Suppose you have n different T-shirts only want to keep k. You could equally well select the k shirts you want to keep or select the complementary set of $n - k$ shirts you want to throw out. Thus, the number of ways to select k shirts from among n must be equal to the number of ways to select $n - k$ shirts from among n. Therefore:

$$\binom{n}{k} = \binom{n}{n-k}$$

This is easy to prove algebraically, since both sides are equal to:

$$\frac{n!}{k!\,(n-k)!}$$

But we didn't really have to resort to algebra; we just used counting principles.

Hmm.

5.1   Boxing

Swastik, famed 6.042 TA, has decided to try out for the US Olympic boxing team. After all, he's watched all of the *Rocky* movies and spent hours in front of a mirror sneering, "Yo, you wanna piece a' *me*?!" Swastik figures that n people (including himself) are competing for spots on the team and only k will be selected. Thus, there are two cases to consider:

- Swastik is selected for the team, and his $k - 1$ teammates are selected from among the other $n - 1$ competitors. The number of different teams that be formed in this way is:
$$\binom{n-1}{k-1}$$

- Swastik is not selected for the team, and all k team members are selected from among the other $n - 1$ competitors. The number of teams that can be formed this way is:
$$\binom{n-1}{k}$$

All teams of the first type contain Swastik, and no team of the second type does; therefore, the two sets of teams are disjoint. Thus, by the Sum Rule, the total number of possible Olympic boxing teams is:

$$\binom{n-1}{k-1} + \binom{n-1}{k}$$

Amy, equally-famed 6.042 TA, thinks Swastik isn't so tough and so she might as well try out also. She reasons that n people (including herself) are trying out for k spots. Thus, the number of ways to select the team is simply:

$$\binom{n}{k}$$

Amy and Swastik each correctly counted the number of possibly boxing teams; thus, their answers must be equal. So we know:

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

This is called **Pascal's Identity**. And we proved it *without any algebra!* Instead, we relied purely on counting techniques.

5.2   Combinatorial Proof

A **combinatorial proof** is an argument that establishes an algebraic fact by relying on counting principles. Many such proofs follow the same basic outline:

1. Define a set S.

2. Show that $|S| = n$ by counting one way.

3. Show that $|S| = m$ by counting another way.

4. Conclude that $n = m$.

In the preceding example, S was the set of all possible United States Olympic boxing teams. Swastik computed $|S| = \binom{n-1}{k-1} + \binom{n-1}{k}$ by counting one way, and Amy computed $|S| = \binom{n}{k}$ by counting another. Equating these two expressions gave Pascal's Identity.

More typically, the set S is defined in terms of simple sequences or sets rather than an elaborate, invented story. (You probably realized this was invention; after all, Swastik is not a U.S. citizen and thus wouldn't be *eligible* for the US team.) Here is less-colorful example of a combinatorial argument.

Theorem 2.
$$\sum_{r=0}^{n} \binom{n}{r}\binom{2n}{n-r} = \binom{3n}{n}$$

Proof. We give a combinatorial proof. Let S be all n-card hands that can be dealt from a deck containing n red cards (numbered $1, \ldots, n$) and $2n$ black cards (numbered $1, \ldots, 2n$). First, note that every $3n$-element set has

$$|S| = \binom{3n}{n}$$

n-element subsets.

From another perspective, the number of hands with exactly r red cards is

$$\binom{n}{r}\binom{2n}{n-r}$$

since there are $\binom{n}{r}$ ways to choose the r red cards and $\binom{2n}{n-r}$ ways to choose the $n-r$ black cards. Since the number of red cards can be anywhere from 0 to n, the total number of n-card hands is:

$$|S| = \sum_{r=0}^{n}\binom{n}{r}\binom{2n}{n-r}$$

Equating these two expressions for $|S|$ proves the theorem. $\square$

Combinatorial proofs are almost magical. Theorem 2 looks pretty scary, but we proved it without any algebraic manipulations at all. The key to constructing a combinatorial proof is choosing the set S properly, which can be tricky. Generally, the simpler side of the equation should provide some guidance. For example, the right side of Theorem 2 is $\binom{3n}{n}$, which suggests choosing S to be all n-element subsets of some $3n$-element set.