A Work-Efficient Parallel Breadth-First Search Algorithm (or How to Cope with the Nondeterminism of Reducer Hyperobjects)

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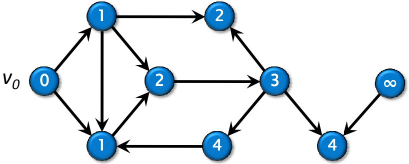
Breadth-first search (BFS)



Problem: Given an unweighted graph $G = (V, E)$ and a designated starting vertex v_0, find the shortest path distance from v_0 to all other $u \in V$.

- Guarantee that the vertices are visited in ***breadth-first order***: For all distances d, all vertices that are d away from v_0 must be visited before any vertex of distance $d + 1$.
- We want a parallel algorithm to solve this problem.

Serial BFS

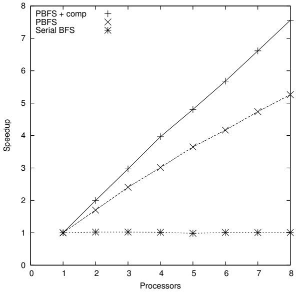
```
SERIAL-BFS(G = (V, E), v0)
1   for each vertex u ∈ V − {v0}
2       u.dist = ∞
3   v0.dist = 0
4   Q = {v0}
5   while Q ≠ ∅
6       u = DEQUEUE(Q)
7       for each v ∈ V such that (u, v) ∈ E
8           if v.dist == ∞
9               v.dist = u.dist + 1
10              ENQUEUE(Q, v)
```

- The queue Q is a FIFO queue.
- The distance of vertex $u = \text{DEQUEUE}(Q)$ in line 6 is monotonically increasing.
- Consequently, vertices are visited in breadth-first order.

This algorithm does not parallelize well.

- FIFO queue is a serial bottleneck.
- Parallelizing the **for** loops gives $O(E/V)$ parallelism, which is puny for sparse graphs.

Summary of results

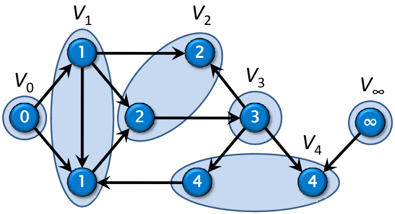


- We have designed a parallel breadth-first search algorithm, called PBFS, and we have implemented PBFS using Cilk++.
- PBFS obtains $5\times$ to $6\times$ speedup on eight processing cores on many real-world benchmark graphs.
- When run serially, PBFS is competitive with SERIAL-BFS.
- The theoretical running time of PBFS on P processors is $O((V + E)/P + D \lg^3(V/D))$.

Outline

Outline

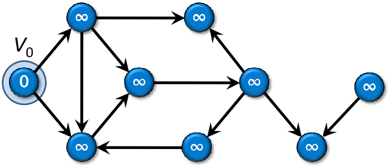
Strategy for parallelizing breadth-first search



Strategy: consider the graph in layers.

- The dth ***layer*** of G is the set V_d of vertices that are all at distance d from v_0.
- Breadth-first ordering: all vertices in V_d are visited before any vertex in V_{d+1}.
- We shall examine the layers V_d serially, but
- For each layer V_d, we shall process all vertices in V_d *in parallel*.

Strategy for parallelizing breadth-first search



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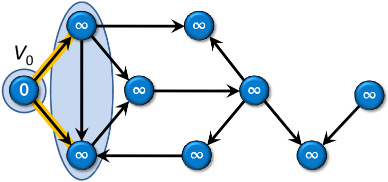

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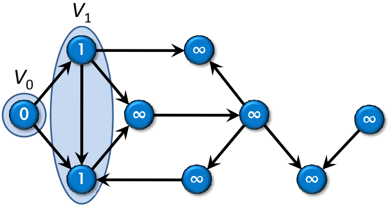

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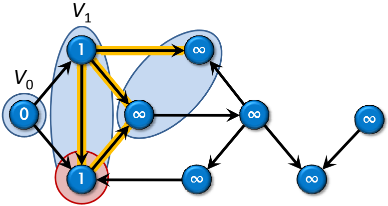

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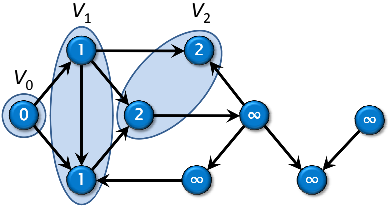

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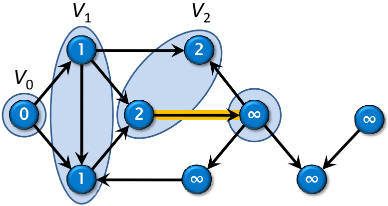

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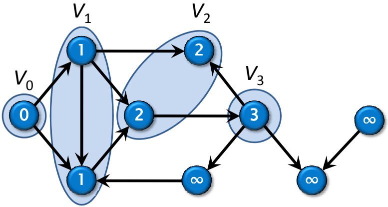

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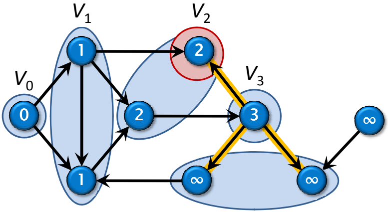

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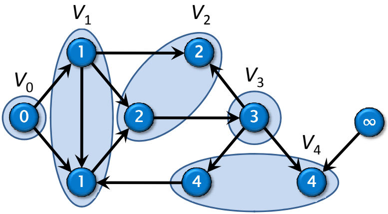

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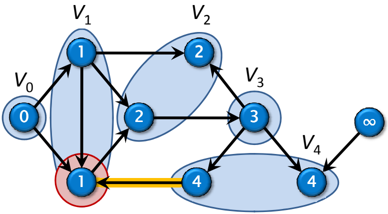

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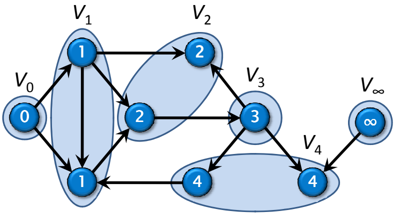

Outline

Storing a layer of the graph

Problem: We need a data structure to handle a single layer. Specifically, we need a data structure that does the following:

- It must store an unordered set of elements.
- It must support efficient parallel traversal of the stored elements.
- It must allow parallel workers to add elements simultaneously to the same structure.

Storing a layer of the graph

Solution: Use a ***bag*** — a multi-set data structure, which supports the following operations:

BAG-CREATE	Create a new, empty bag.
BAG-INSERT	Add an element to a bag.
BAG-SPLIT	Divide a bag into two equal-sized bags.
BAG-UNION	Combine the contents of two bags into a single bag.

Processing a layer

```
PROCESS-LAYER(in-bag, out-bag, d)
if BAG-SIZE(in-bag) < GRAINSIZE
    for each u ∈ in-bag
        parallel for each v ∈ Adj[u]
            if v.dist == ∞
                v.dist = d + 1        // benign race
                BAG-INSERT(out-bag, v)
    return
new-bag = BAG-SPLIT(in-bag)
spawn PROCESS-LAYER(new-bag, out-bag, d)
PROCESS-LAYER(in-bag, out-bag, d)
sync
```

Processing a layer

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                                BAG-INSERT(out-bag, v)  // malignant race
        return
new-bag = BAG-SPLIT(in-bag)
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sync
```

Cilk++ reducers

Cilk++ supports a type of parallel data structure, called a ***reducer***.

```
x = 10
x++
x += 3
x += -2
x += 6
x--
x += 4
x += 3
x++
x += -9
```

```
x = 10
x++
x += 3
x += -2
x += 6
x' = 0
x'--
x' += 4
x' += 3
x'++
x' += -9
x += x'
```

```
x = 10
x++
x += 3
x' = 0
x' += -2
x' += 6
x'--
x'' = 0
x'' += 4
x'' += 3
x''++
x'' += -9
x += x'
x += x''
```

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1 $x = 10$
2 $x{+}{+}$
3 $x \mathrel{+}= 3$
4 $x \mathrel{+}= -2$
5 $x \mathrel{+}= 6$
6 $x{-}{-}$
7 $x \mathrel{+}= 4$
8 $x \mathrel{+}= 3$
9 $x{+}{+}$
10 $x \mathrel{+}= -9$

1 $x = 10$
2 $x{+}{+}$
3 $x \mathrel{+}= 3$
4 $x \mathrel{+}= -2$
5 $x \mathrel{+}= 6$
$x' = 0$
6 $x'{-}{-}$
7 $x' \mathrel{+}= 4$
8 $x' \mathrel{+}= 3$
9 $x'{+}{+}$
10 $x' \mathrel{+}= -9$
$x \mathrel{+}= x'$

1 $x = 10$
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3 $x \mathrel{+}= 3$
$x' = 0$
4 $x' \mathrel{+}= -2$
5 $x' \mathrel{+}= 6$
6 $x'{-}{-}$
$x'' = 0$
7 $x'' \mathrel{+}= 4$
8 $x'' \mathrel{+}= 3$
9 $x''{+}{+}$
10 $x'' \mathrel{+}= -9$
$x \mathrel{+}= x'$
$x \mathrel{+}= x''$

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$x'{-}{-}$
$x'' = 0$
$x'' \mathrel{+}= 4$
$x'' \mathrel{+}= 3$
$x''{+}{+}$
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$x \mathrel{+}= x'$
$x \mathrel{+}= x''$

Cilk++ reducers

We use the bag as a Cilk++ reducer to solve our malignant race.

- After stealing a task, a worker starts executing the task with a local, "identity" copy of a new reducer.
- Each worker freely manipulates its local copy with write-only update operations.
- As tasks return, the workers' local copies are combined together into a single data structure using REDUCE operations.
- If REDUCE is associative, then the program has serial semantics.

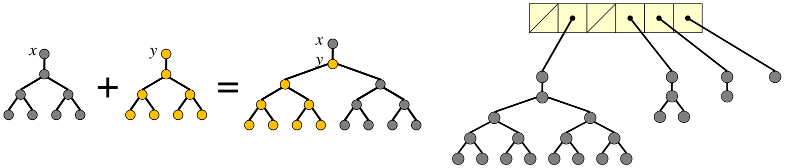
Cilk++ reducers

We use the bag as a Cilk++ reducer to solve our malignant race.

- After stealing a task, a worker starts executing the task with a local, “identity” copy of a new reducer.
  - For bags, the identity is an empty bag.
- Each worker freely manipulates its local copy with write-only update operations.
  - For bags, the update operation is BAG-INSERT.
- As tasks return, the workers’ local copies are combined together into a single data structure using REDUCE operations.
  - For bags, REDUCE = BAG-UNION.
- If REDUCE is associative, then the program has serial semantics.
  - For bags, BAG-UNION is not strictly associative, since the order of elements within a bag is nondeterministic. Bags have a notion of “logical associativity,” which is sufficient for PBFS’s correctness.

Outline

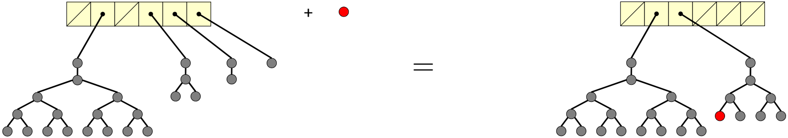
The bag data structure



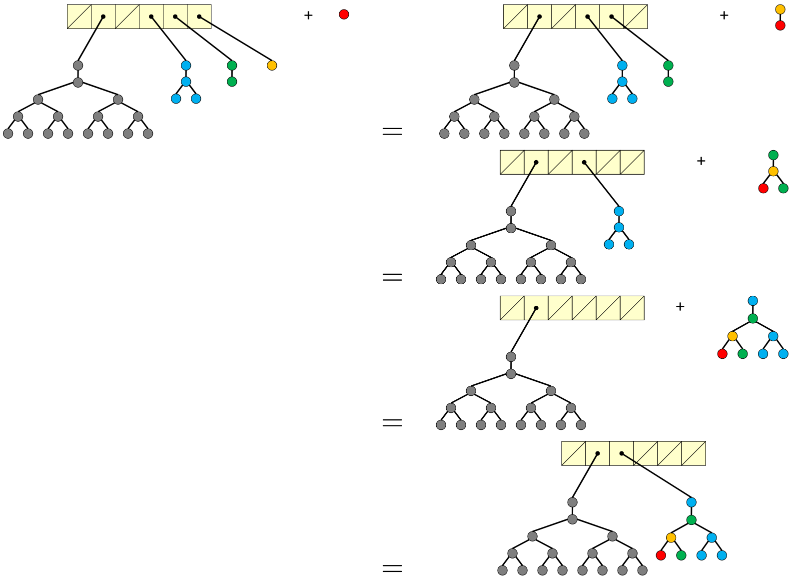
- A bag is made up of ***pennants*** — complete binary trees with extra root nodes, which store the elements.
  - Pennants may be split and combined in $O(1)$ time by changing pointers.
  - A pennant is only combined with another pennant of the same size.
- A bag is an array of pointers to pennants.
  - For all i, the ith entry in the array is either null or points to a pennant of size 2^i.
  - Intuitively, a bag acts much like a binary number.

The bag data structure — BAG-INSERT

Inserting an element works similarly to incrementing a binary number.

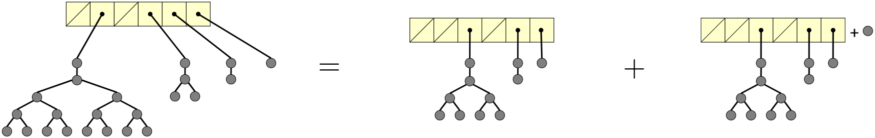


BAG-INSERT runs in $O(1)$ amortized time and $O(\lg n)$ worst-case time.



The bag data structure — BAG-SPLIT

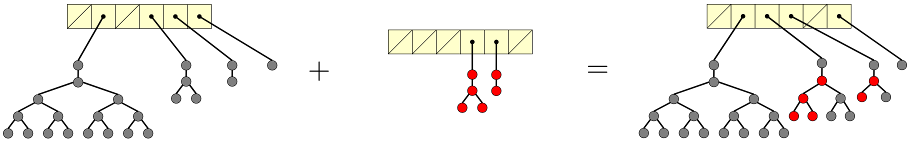
Splitting a bag works similarly to an arithmetic right shift.



BAG-SPLIT runs in $O(\lg n)$ time.

The bag data structure — BAG-UNION

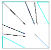
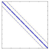
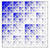
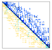
Unioning two bags is works similarly to adding two binary numbers.



BAG-UNION works in $O(\lg n)$ time.

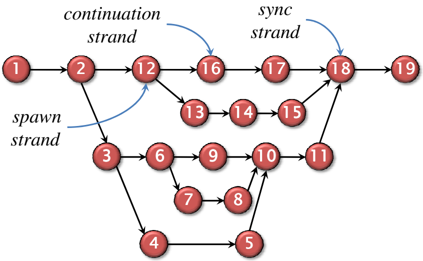
Outline

Empirical Results

Name *Description*	*Spy Plot*	$\lvert V \rvert$ $\lvert E \rvert$ D	Parallelism	$\frac{\text{PBFS } T_1}{\text{SERIAL BFS } T_1}$	PBFS T_1 / T_8
Kkt_power Optimal power flow, nonlinear opt.		2.05 M 12.76 M 31	104.09	0.705	6.102
Freescale1 Circuit simulation		3.43 M 17.1 M 128	153.06	1.120	5.145
Cage14 DNA electrophoresis		1.51 M 27.1 M 43	246.35	1.060	5.442
Wikipedia Links between Wikipedia pages		2.4 M 41.9 M 460	179.02	0.804	6.833
Grid3D200 3D 7-point finite-diff mesh		8 M 55.8 M 598	79.27	0.747	4.902
RMat23 Scale-free graph model		2.3 M 77.9 M 8	93.22	0.835	6.794
Cage15 DNA electrophoresis		5.15 M 99.2 M 50	675.22	1.058	5.486
Nlpkkt160 Nonlinear optimization		8.35 M 225.4 M 163	331.57	1.138	6.096

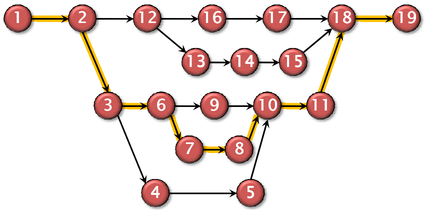
Outline

The dag model of computation



- We can model a Cilk++ program with a dag (directed acyclic graph) *A*.
- Each vertex in *A* corresponds to a ***strand*** — a sequence of serially executed instructions.
- Edges in *A* describe control dependencies between strands.

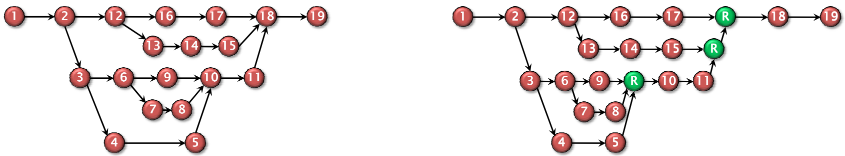
The dag model of computation



- ***Work*** $W(A)$: The sum of the lengths of all of the strands in A.
- ***Span*** $S(A)$: The length of the longest path in A.
- The Cilk++ scheduler guarantees that A runs in $T_P(A) \leq W(A)/P + O(S(A))$.
- ***Parallelism*** of A: $W(A)/S(A)$.
- This model does not accurately represent runtime system operations on reducers.

Outline

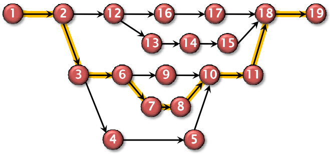
Modeling reducers

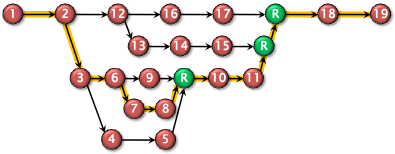


For a computation A:

- Consider the ***user dag*** User(A) — a dag where runtime system operations on reducers are not represented.
  - This dag models the computation as the user understands it.
- Insert REDUCE strands performed by the runtime system into User(A) before the sync strand that requires its completion to get a ***performance dag*** Perf(A).
- A delay-sequence argument proves that the performance of A is $T_P(A) \le W(\mathrm{Perf}(A))/P + O(S(\mathrm{Perf}(A)))$.
- We want a performance bound in terms of User(A).

Modeling reducers





Let τ be the worst-case running time of any REDUCE operation. In the paper, we show that:

- $S(\text{Perf}(A)) = O(\tau \cdot S(\text{User}(A)))$ and
- $W(\text{Perf}(A)) = W(\text{User}(A)) + O(\tau^2 P \cdot S(\text{User}(A)))$.

Outline

Analysis of PBFS

- PBFS's user dag has $O(V + E)$ work and $O(D \lg(V/D))$ span for bounded-degree input graphs.
- The worst-case cost of any BAG-UNION in PBFS is $O(\lg(V/D))$.
- Relating the user and performance dags for PBFS, we have $S(\text{PBFS}) = O(D \lg^2(V/D))$ and $W(\text{PBFS}) = O(V + E) + O(PD \lg^3(V/D))$.
- Consequently, we have $T_P(\text{PBFS}) = O((V + E)/P + D \lg^3(V/D))$.
- If $O((V + E)/P) \gg O(D \lg^3(V/D))$, then we expect linear speedup from PBFS. We define the ***effective parallelism*** of PBFS to be $O\left(\frac{V+E}{D \lg^3(V/D)}\right) \approx O\left(\frac{E}{D}\right)$.

Conclusion

- We have seen a parallel breadth-first search algorithm implemented in Cilk++, which uses a novel Cilk reducer for unordered sets.
- Future work includes:
  - Comparing the performance of PBFS versus an implementation that uses thread-local storage instead of reducers.
  - Augmenting PBFS to return a deterministic BFS tree.
  - Parallelizing other graph algorithms, such as weighted SSSP, max flow, or min-cost flow.
  - Parallelizing other non-numeric algorithms using Cilk technologies.

Thank you

Questions?

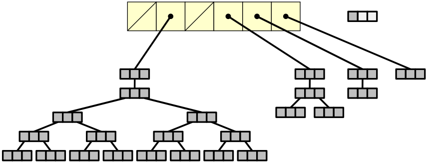
Cilk++ reducers

Reducers are implemented in Cilk++ with one additional optimization.

- After stealing a task, a worker starts executing the task with a local *null* copy of a reducer.
  - For example, this null copy may be a NULL pointer to a bag.
- The first time the worker tries to manipulate its local copy of the reducer after a steal, the runtime system initializes the reducer using a CREATE-IDENTITY operation.
  - For bags, CREATE-IDENTITY = BAG-CREATE.
- Modeling CREATE-IDENTITY calls in theory is similar (and simpler) than modeling REDUCE calls.

Optimizing the bag data structure

We can improve the real-world efficiency of the bag by storing an array of data at each node.



- Each node in a pennant stores a fixed-size array of data, which is guaranteed to be full.
- The bag stores an extra fixed-size array of data, called the ***hopper***, which may not be full.
- Inserts first attempt to insert into the hopper. Once the hopper is full, a new, empty hopper is created while the old hopper is inserted into the bag using the original algorithm.

With this optimization, the common case for BAG-INSERT is exactly like enqueueing a vertex in a FIFO queue.

Locked PBFS

To simplify theoretical analysis, we analyze a locked version of PBFS.

```
PROCESS-LAYER(in-bag, out-bag, d)
if BAG-SIZE(in-bag) < GRAINSIZE
    for each u ∈ in-bag
        parallel for each v ∈ Adj[u]
            if v.dist == ∞
                if TRY-LOCK(v)
                    if v.dist == ∞
                        v.dist = d + 1
                        BAG-INSERT(out-bag, v)
                    RELEASE-LOCK(v)
    return
new-bag = BAG-SPLIT(in-bag)
spawn PROCESS-LAYER(new-bag, out-bag, d)
PROCESS-LAYER(in-bag, out-bag, d)
sync
```

Relating the user and performance dags

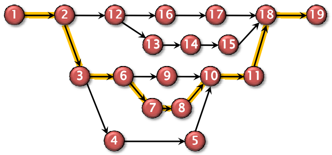
Lemma

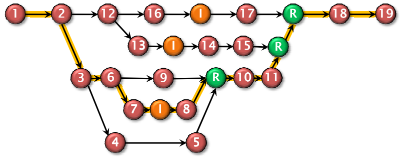
Consider a computation A, and let τ be the worst-case running time of any REDUCE *or* CREATE-IDENTITY *operation in A. We have* $S(\text{Perf}(A)) = O(\tau \cdot S(\text{User}(A)))$ *in expectation.*

Proof.

- Every successful steal may force a CREATE-IDENTITY operation.
- Every successful steal may force a REDUCE operation.

Relating the user and performance dags





Proof cont'd.

- Consider a critical path p in Perf(A).
- This path p corresponds to some path q in User(A), which has length at most $S(\text{User}(A))$.
- Since at most every node in q corresponds to a steal, the length of p is $O(\tau \cdot S(\text{User}(A)))$ in expectation.

□

Relating the user and performance dags

Lemma

Consider a computation A, and let τ be the worst-case running time of any REDUCE *or* CREATE-IDENTITY *operation in A. We have* $W(\mathrm{Perf}(A)) = W(\mathrm{User}(A)) + O(\tau^2 P \cdot S(\mathrm{User}(A)))$ *in expectation.*

Proof.

- The computation A contains all of the strands in $\mathrm{User}(A)$.
- At most $O(P \cdot S(\mathrm{Perf}(A)))$ steals occur during A's execution.
- Consequently, REDUCE and CREATE-IDENTITY strands contribute $O(\tau P \cdot S(\mathrm{Perf}(A)))$ additional work to $\mathrm{User}(A)$.
- From the previous lemma, we have $S(\mathrm{Perf}(A)) = O(\tau \cdot S(\mathrm{User}(A)))$.
- Therefore, we have $W(\mathrm{Perf}(A)) = W(\mathrm{User}(A)) + O(\tau^2 P \cdot S(\mathrm{User}(A)))$ in expectation.

$\square$