hp Adaptive Discontinuous Galerkin Modeling of MBX

J. C. Wiley, P. M. Valanju, S. M. Mahajan
Institute for Fusion Studies,
The University of Texas at Austin, Austin, TX 78712

The discontinuous Galerkin (DG) method has received a great deal of attention [1] recently particularly in CFD. The reason is that it is able to utilize much of the theoretical and practical experience from the last couple of decades in finite difference, finite volume, finite element, and spectral methods while retaining some of the best features of each. The DG method was originally a technique for solving systems of hyperbolic conservation laws, however, it has been extended to include higher order derivatives [2]. It can be considered as a finite element method in which the approximation space relaxes the continuity requirement. This relaxation frees the finite element method from many practical and mathematical restrictions while retaining the most important advantages of being variational and able to represent geometrically complex boundaries.

We describe a particular high-order nodal-hp DG implementation for the simulation of the Magneto-Bernoulli experiment (MBX, see Fig. 1) by the solution of the Hall MHD equations. By hp we mean that the triangular grid (the initial implementation is two dimensional, 3D is planned as a natural extension) can be adaptively locally refined (h), and that the polynomial order of the basis functions in each element can be adaptively adjusted (p), currently up to tenth order. Within each element the functions are approximated by a Lagrange basis [3] constructed from tensor products of Jacobi polynomials [4]. The Lagrange representation largely eliminates the need for transforms from coefficient to function space.

The DG method is locally conserving as are finite volume methods in contrast to usual finite element methods. In finite elements, the continuity constraint couples each element and leads to the solution of large linear systems. In contrast, the DG method couples elements only through the element edge fluxes and leads to a largely matrix free method. Linear system size is at most that of the number of unknowns in each element. With explicit time integration, there are no global matrices to solve, greatly simplifying the communication needed for parallel implementations as each step only requires communication of edge fluxes to nearest neighbors. Certain choices of basis functions lead to diagonal mass matrices which lead to matrix free methods [5], but that choice has not been made in this implementation.

The DG method begins by expanding each function in terms of local basis functions. A projection of the differential equation on to the local approximation space followed by an integration by parts leads to the usual DG formulation. A second integration by parts [6], recovers the original differential operator, plus an edge flux integral, which can be considered a penalty term. The key point is that the flux in this term is not defined, as the flux can have a different value as the edge is approached from within or from outside the element. The difficulty is resolved by defining a numerical flux function, which is a

function of both interior and exterior values. Suitable flux functions have been studied for quite some time in the context of TVD finite volume methods, and the required properties and a number of functional forms are well known. The key properties that a numerical flux function must have are that it must be consistent, conservative, and locally Lipshitz continuous. Since the problem at the edge is essentially one dimensional, solutions of the Riemann problem provide the key to building numerical flux functions. The Riemann solvers also provide a direct handle on the local waves [7] and the fundamental physics. The resulting ODEs are integrated with an explicit Runga-Kutta solver.

The DG method has previously been applied to compressional MHD [8]. For problems with smooth solutions, spectral convergence was observed with increasing order as expected. Since typically, the $\nabla \cdot B=0$ is not exactly satisfied, various experiments have been done to investigate the effects since at least for equilibrium problems an error in $\nabla \cdot B$ will eventually cause distortion and loss of equilibrium [9]. It has been shown [10] for time domain Maxwell's equations, the growth in the error of $\nabla \cdot B$ is linear in time, and that this estimate is sharp. The initial value can be made small by using high order basis functions and small h, so whether the problem is of concern for a particular simulation depends on the relative time scales. A number of "divergence cleaning methods" [11] have been implemented in the code.

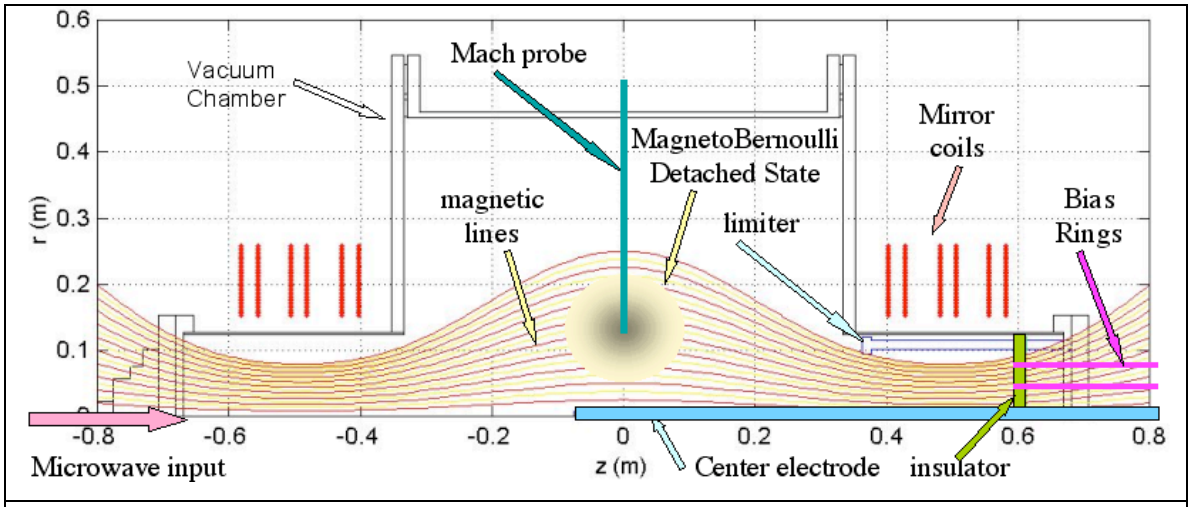

Fig. 1a. Schematic of MBX experiment.

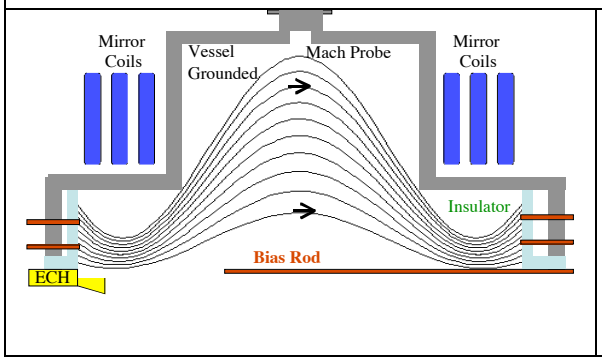

Fig. 1b. Rotating Mirror showing pre-relaxed centrifugally confined plasma.

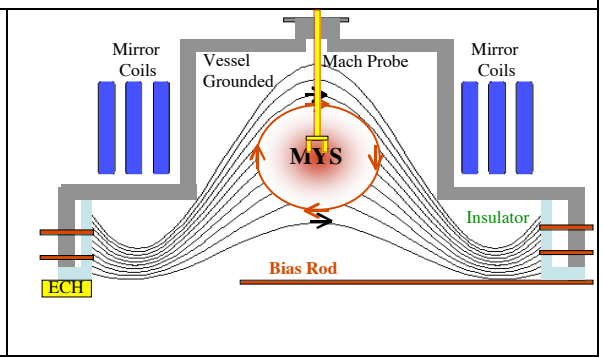

Fig. 1c. Relaxation leads to formation of detached, toroidal magneto-Bernoulli state.

The MBX experiment is a small mirror machine with segmented electrode rings at the mirror throat. These electrodes are biased to create a radial electric field which through E×B rotation produces fast rotation which leads to centrifugal confinement [12]. For velocities on the order of the Alfvèn velocity and for certain initial conditions, a bifurcation into self-organized magneto Bernoulli states (see Fig.1) is expected [13].

The goal of MBX is to create and observe these novel magnetofluid relaxed plasma configurations in which: a) large sheared flow, strongly coupled to the magnetic field, leads to relaxed states with good equilibrium, stability, and confinement, b) confinement is via the Bernoulli mechanism that does not need external symmetry, rather than the standard paradigm of nested magnetic surfaces, c) multiple scale lengths can exist for flow and magnetic fields, and d) efficient conversion between flow and magnetic fields occurs during relaxation. MBX creates centrifugally confined mirror plasmas as precursors to spontaneous or induced relaxation to detached magnetofluid states. In doing so, the expectation is to verify the predictions of centrifugal confinement, and then to delineate the regions of the parameter space of ideal constants of motion (total energy and helicities) where the centrifugally confined plasma can and cannot relax to a magnetofluid state.

Creation and observation of these novel magnetofluid relaxed plasma configurations is of fundamental and practical interest because: a) the magnetofluid states differ qualitatively from those accessible to either normal fluids or to conventional MHD plasmas, b) the strong diamagnetism of Magneto-Bernoulli confinement resembles superconductors and makes efficient use of both flow and magnetic fields, c) the strongly coupled magneto-fluid regime is expected to lead to new mechanisms for the inter-conversion of fluid and magnetic energy (dynamo and reverse dynamo action), d) because magnetofluid confinement does not depend on symmetry and flux surfaces, it may provide a better paradigm for confinement in most astrophysical and other natural environments, and e) theoretical research in plasmas with high flows is attracting attention in astrophysics and fusion. Further, since centrifugally confined mirror plasmas are a possible candidate for an inexpensive fusion device, additional knowledge of centrifugal confinement could be beneficial.

The simulation goal is to model the experiment including coupling the external drive through the boundary conditions through plasma spin up decay of the three invariants: energy, helicity, and magnetic enstrophy [14] as the plasma creates a self-organized state. The internal, self-organized states are supported by externally imposed voltages and currents. Therefore, it is important that the two-fluid simulations include the external boundary conditions at the material walls and electrodes. Simulation of the transition region between the externally driven flows and currents, and the internal, self-organized state is facilitated by the adaptive grid refinement and the use of higher order basis functions. We are also exploring the possibility of calculating the critical effects of charge-exchange drag using the same adaptive grid.

References:

[1] Cockburn, B., Shu C.-W., "Runge-Kutta Discontinuous Galerkin Methods for Convection-Dominated Problems," J Sci. Comp. **16**, 3, 173-261 (2001).

[2] J. Yan, C. Shu, "A Local Discontinuous Galerkin Method for KdV Type Equations," SIAM J. Numer. Anal. **40** 2, 769-791 (2002).

[3] J. S. Hesthaven, "From electrostatics to almost optimal nodal sets for polynomial interpolation in a simplex," SIAM J. Numer. Anal. **35** (2), 655-676 (1998), J. S. Hesthaven, C. H. Teng, "Stable spectral methods on tetrahedral elements," SIAM J. Sci. Comput. **21** (6), 2352-2380 (2000).

[4] T. C. Warburton, S. J. Sherwin, G. E. Karniadakis, "Basis functions for triangular and quadrilateral high-order elements," SIAM J. Sci. Comput. **20** (5) 1671-1695 (1999).

[5] J. Remacle, J. E. Flaherty, M. S. Shepard, "An Adaptive Discontinuous Galerkin Technique with an Orthogonal Basis Applied to Compressible Flow Problems," SIAM Review **45** 1, 53-72 (2003).

[6] F. X. Giraldo, J. S. Hesthaven, T. Warburton, "Nodal high-order discontinuous Galerkin methods for the spherical shallow water equations," J. Comp. Phys. **181** (), 499-525 (2002).

[7] M. Wesenberg, "Efficient MHD Riemann solvers for simulations on unstructured triangular grids," J. Numer. Math. **10**, 1, 37-71,(2002).

[8] T. C. Warburton, G. E. Karniadakis, "A discontinuous Galerkin method for the viscous MHD equations", J. Comp. Phys. **152**, 608-641 (1999).

[9] J. U. Brackbill, D. C. Barnes, "Note: The effect on nonzero $\nabla\cdot{\bf B}$ on the numerical solution of the magnetohydrodynamics equations," J.Comp. Phys. **35**, 426 (1980).

[10] J. S. Hesthaven, T. Warburton, "Nodal High-Order Methods on Unstructured Grids," J. Comp. Phys. **181**, 186-221 (2002).

[11] A. Dedner, F. Kemm, D. Kroner, C.-D. Munz, T. Schnitzer, and M. Wesenberg, "Hyperbolic Divergence Cleaning for the MHD Equations," J. Comp. Phys. **175**, 645-673 (2002).

[12] R.F. Ellis, A. B. Hassam, S. Messer, B. R. Osborn, "An experiment to test centrifugal confinement for fusion," Physics of Plasmas, 8, 2057 (2001)

[13] S. Mahajan and Z. Yoshida, "Double curl Beltrami flow: diamagnetic structures," Phys. Rev. Lett. **81**, 4863 (1998).

[14] Z. Yoshida, S. M. Mahajan, "Variational principles and self-organization in two-fluid plasma," PRL, **88** 9 (2002).