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Crosscutting Areas

On Revenue Management with Strategic Customers Choosing When and What to Buy

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Abstract. Consider a network revenue management model in which a seller offers multiple products, which consume capacitated resources. The seller uses an anonymous posted-price policy, and arriving customers strategize on (a) when and (b) which product to purchase to maximize their utility, based on heterogeneous product valuations. Such models, whereby customers are both forward looking and choose what to buy, have not yet been amenable to analysis, mainly because their associated dynamic mechanism design counterparts are multidimensional; that is, they involve constraints with multivariate private information (the product valuations). Within the context of the aforementioned model, we present a novel decomposition approach that enables us to deal with the underlying multidimensional mechanism design problem. Using this approach, we derive for all nonanticipating dynamic pricing policies an upper bound to expected revenues. We use our bound to conduct theoretical and numerical performance analyses of static pricing policies. In our theoretical analysis, we derive guarantees for the performance of static pricing, for the classical fluid-type regime where inventory and demand grow large. Our numerical analysis shows static pricing to be able capture at least 75%–90% of maximum possible expected revenue under a wide range of realistic problem parameters.

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Keywords: network revenue management • mechanism design • strategic customers • discrete choice model • approximation algorithms

1. Introduction

1.1. Motivation

Dynamic pricing is by now considered a mature technology that has enabled firms to boost their revenues and extract greater value from their customers, notably in the retail, airline, and hospitality industries (Talluri and Van Ryzin 2006, chapter 10). For customers, however, it is well understood that interacting with a firm using dynamic pricing over some period of time raises their awareness of and familiarity with its pricing policies. In turn, this enables them to strategize their purchasing decisions.

Customers usually strategize along two dimensions: (1) *when to buy* and (2) *what to buy*. That is, they might delay a purchase in anticipation of a future price reduction, and they might purchase a cheaper substitute (see the empirical studies by Li et al. 2014 and Moon et al. 2017 on the retail and airline industries, respectively). Such strategizing is today abetted by an abundance of online tools explicitly designed to facilitate *best responses* to dynamic pricing.[1]

What pricing policy to follow in the face of customers who strategize on when and what to buy is an unexplored and open question. In particular, papers in the literature have thus far dealt exclusively with either one of the two extremes whereby one of the two aforementioned strategic dimensions is missing. For example, a large body of work has studied forward-looking customers strategizing on *when to buy* but has done so merely within single-product settings or models in which customers have no alternatives to choose from. Conversely, another large body of work on assortment optimization and choice modeling has studied customers who choose *what to buy* from multiple product offerings but acting myopically. Whereas there are plenty of applications that could fit in either of these two extreme cases, at the same time, however, there is a vast swatch of practical settings in which customers strategize *both* on their time of purchase and preferred product, as supported by empirical evidence within the retail industry.

That no paper has thus far dealt with customers who choose both when and what to buy attests to the technical challenges that combining these two dimensions of strategic behavior represents. In particular, the first dimension—*choice over when to buy*—

introduces game theoretic interactions, which make deriving an optimal, that is, revenue-maximizing, dynamic pricing policy intractable in general, even in a single-product model (Pai and Vohra 2013). Nonetheless, researchers have been able to leverage, among others, mechanism design techniques to provide approximation analyses and characterizations of optimal policies under certain conditions. Notably, such techniques require customers' private information to be univariate, allowing to capture just their valuation of a single preferred product. Simultaneously considering the second dimension of strategic behavior—*choice over what to buy*—requires multivariate private information to allow for multiple product valuations and therefore invalidates all extant approaches.

1.2. Technical Advancements and Contributions

This paper makes a first step toward analyzing pricing in the face of strategic customers who choose both when and what to buy in a network revenue management (NRM) context. The analysis is made possible through a novel decomposition approach that enables us to deal with dynamic mechanism design problems that involve multivariate private information. In particular, we illustrate in a dynamic mechanism design problem for NRM, how customers' incentive compatibility and individual rationality constraints that involve multiple product valuations can be decomposed to constraints that involve single product valuations in a meaningful way that enables a characterization of the optimal mechanism.

Building on this technical advancement, we derive an upper bound on the expected revenues generated by any dynamic pricing policy in a canonical NRM model extended to allow customer product choice and forward-looking behavior. Establishing an upper bound is important as it can serve to benchmark the performance of proposed dynamic pricing policies in such a setting. Because our bound is obtained as the optimal value of a simple optimization problem, it could potentially enable both theoretical and numerical performance analyses of pricing policies.

Indeed, using our upper bound, we provide a theoretical performance analysis of static pricing in NRM with strategic customers choosing what and when to buy. In particular, we derive lower bounds on the expected revenues static pricing can generate relative to those generated by any dynamic pricing policy, including the optimal revenues, under all possible problem instances. In an asymptotic, fluid-type regime we derive a constant-factor guarantee, showing that static pricing can always capture at least 1/4 of optimal revenues. For settings in which inventory is scarce, the guarantee is further improved to a factor that relates to the ratio of unserved customers to total number of customers. Specifically, as available inventory relative to demand goes to zero, almost all customers are unserved, and our guarantee converges to 1. Guarantees of the type we provide have assumed a place of prominence in a number of operational problems ranging from revenue management to inventory and supply chain management. We interpret these guarantees as a strong indicator of the robustness of static prices across parameter regimes.

Finally, we provide a numerical performance analysis of static pricing in our NRM model, using our upper bound and simulation. The analysis shows that revenues under static pricing are within 75%–90% of our bound under a wide range of realistic problem instances. That the simplest pricing policy captures such a high percentage of our bound consistently across all experiments provides some evidence about the bound's strength and practical relevance and also supports our aforementioned claim about the robustness of static prices across parameter regimes. We further explore, for various parameter choices, how performance of static pricing responds to product proliferation, product differentiation, product complexity, and load factor, relative to dynamic pricing.

1.3. Literature Review

Our paper lies at the intersection of two very broad streams of work in revenue management (RM). The first one, usually referred to as RM with forward-looking, or strategic, customers, deals with customers interested in purchasing a single product and choosing when to do so. The second stream, usually referred to RM with customer choice, deals with myopic customers choosing among offered product assortments. Here, we make no attempt to survey either of these streams, but rather focus on the papers that are the closest to ours.

RM with strategic customers: Among the papers in this stream, closest to ours are papers using dynamic mechanism design. One of the earliest papers in this area by Vulcano et al. (2002) considers the sale of a single type good via sequences of auctions (as opposed to posting prices). Gallien (2006) studies a discounted infinite horizon extension of the classical single-product RM model with customers who are impatient as their product valuation decays exponentially over time. By assuming that, for all customers, this decay occurs at a constant rate that is common knowledge, the author applies the Myersonian approach (Myerson 1981) to analyze the dynamic mechanism problem and how it can be implemented via a pricing policy. Said (2012) considers the same setting and models customers' impatience in a similar fashion. However, customers cannot conceal their presence and depart from the system at some exogenous rate. The author shows that the decision maker can implement the efficient allocation using a sequence of ascending auctions. Li (2009) and Board and Skrzypacz (2016) design efficient and optimal mechanisms, respectively, in finite-horizon variants of the model considered by Gallien (2006) and Said (2012). Pai and Vohra (2013) also study a similar single-product finite-horizon RM setting but model the customers' impatience differently. In particular, customers have heterogenous deadlines, as opposed to time-decaying valuations. The authors argue that tractability in their model hinges on these deadlines being common knowledge. Chen and Farias (2018) introduce a model that allows for heterogeneity in customers' impatience to be private information and derive a 29% guarantee for a class of dynamic policies.

All these papers study RM with a single product. Gao et al. (2019) consider two competing firms: one using a fixed price policy and the other running an auction. They find that customers with either high or low waiting cost choose to obtain service from the bid-based firm, whereas those with moderate waiting cost choose the fixed-price firm.

To the best of our knowledge, the paper by Chen et al. (2018) is perhaps the first paper that uses the dynamic mechanism design approach to study customer forward-looking behaviors in an NRM setting with multiple products that are produced with multiple resources. The authors show that the optimal pricing policy in a corresponding fluid myopic customer setting is optimal in an asymptotical regime in the stochastic forward-looking customer setting. We note that both that and our present paper use dynamic mechanism design approaches to study NRM problems with forward-looking customers. However, these two papers have the following fundamental distinctions:

1. From a modeling perspective, Chen et al. (2018) assume that each customer is exogenously linked to a single product and is restricted to purchase this product regardless of the availability and the prices of other products. Therefore, each customer's decision is only *when* to buy their (exogenously determined) desired product. In contrast, we allow each customer to choose among all available products. Therefore, customers' decisions are *when* to buy and *what* product to buy.

2. From a methodologic perspective, in Chen et al. (2018), customer private information on the product valuation is one-dimensional. Therefore, by forcing a customer to truthfully report her nonvaluation private information, the Myersonian approach (Myerson 1981, a standard approach in solving mechanism design problems wherein customer private information is one-dimensional) can be directly applied to solve these problems. In contrast, in the mechanism design problem in the present paper, customer private information on the product valuation is multidimensional. Therefore, we cannot directly apply the Myersonian approach to analyze our model. We propose a novel decomposition approach that allows us to cast a relaxation of the mechanism design problem with multidimensional customer private information on product valuations as a series of mechanism design problems wherein customer private information on product valuation is one dimensional.

RM with customer choice: Papers in this stream study the problem of selling multiple (substitutable) products, which are produced from multiple capacitated resource types, to customers arriving over a finite horizon. The decision maker's decision is to dynamically determine either the set of products to offer to customers, given some exogenously determined product prices (quantity-based RM), or the prices to post for each product (price-based RM). When facing substitutable products, arriving customers usually purchase at most one product unit among those offered. As before, we make no attempt to survey this vast literature. For some recent work on quantity-based RM with customer choice, we refer the reader to Liu and Van Ryzin (2008), Zhang (2011), Jasin and Kumar (2012), Li et al. (2015), Jagabathula (2016), Jagabathula and Rusmevichientong (2016), Kunnumkal and Talluri (2016), and Wang and Wang (2016). For price-based RM with customer choice, see Maglaras and Meissner (2006), Song and Xue (2007), Aydin and Porteus (2008), Dong et al. (2009), Akcay et al. (2010), Besbes and Zeevi (2012), Jasin (2014), Li et al. (2015), Chen et al. (2016), Jagabathula and Rusmevichientong (2016), Li and Huh (2011), Yan et al. (2016), Gao et al. (2017), and Ferreira et al. (2018). All papers that study RM with customer choice that we are aware of, however, assume that customers are myopic. In contrast, our paper allows for forward-looking customers.

2. Model

Consider a decision maker (DM) in charge of selling n different products to customers in continuous time over a fixed season $[0, T]$. Products, indexed by $j \in \{1, \ldots, n\}$, correspond to bundles of m resources, indexed by $i \in \{1, \ldots, m\}$. In particular, the jth product is a bundle including $A_{ij} \in \{0, 1\}$ units of the ith resource. Both resources and products are indivisible. In the beginning of the season, the DM has some given inventory of resources $\boldsymbol{x}_0 \in \mathbb{R}^m$ available, with no replenishment opportunity thereafter. That is, at $t = 0$ the DM possesses x_0^i units of the ith resource.[2]

The DM implements an anonymous posted price mechanism by dynamically posting prices for each product. Let $\boldsymbol{\pi}_t \in \mathbb{R}^n$ be the posted prices at time t. In case a customer decides to purchase a unit of product j at time t, the DM generates revenues of π_t^j and her ith resource inventory is reduced by A_{ij}. We denote the resource inventory the DM possesses at time t by $\boldsymbol{X}_t$.

Customers arrive according to a Poisson process of rate λ. Each customer values each product differently, and her valuations decay at an exponential rate throughout time. That is, if a customer's valuation for the jth product at time t is v^j, then at time $\tau \geq t$ it is $v^j e^{-d^j(\tau - t)}$, where d^j is a decay rate. To introduce some notation, we denote all distinguishing characteristics of a customer by the tuple $\phi \triangleq (t_\phi, \boldsymbol{v}_\phi, \boldsymbol{d}_\phi)$, where $t_\phi \in [0, T]$ is her arrival time, $\boldsymbol{v}_\phi \in \mathbb{R}^n_+$ is the vector of her product valuations on arrival, and $\boldsymbol{d}_\phi \in \mathbb{R}^n_+$ her associated valuation decay rates. For simplicity, we refer to ϕ as the customer's *type* and denote by Φ the set of all possible types. We assume that customer ϕ's valuation v^j_ϕ for any product j is independent from all her other characteristics t_ϕ, $\boldsymbol{v}^{-j}_\phi$, $\boldsymbol{d}_\phi$. Also, customer valuations for each product j are independent and identically distributed, with f_j, F_j and $\bar{F}_j$ denoting the corresponding probability density function (p.d.f.), cumulative distribution function (c.d.f.), and complementary cumulative distribution function (c.c.d.f.), all assumed to be continuous.

At any time $t \geq t_\phi$ after her arrival, customer ϕ chooses to either stay in the system, or permanently exit by purchasing at most one unit of a product type among the n alternatives. Let $\tau_\phi \in [t_\phi, T]$ be the time that customer ϕ chooses to exit, and $a^j_\phi \in \{0, 1\}$ the indicator of whether she chooses to purchase one unit of product j at that time. We denote by p_ϕ the corresponding payment that customer ϕ makes to the DM at time τ_ϕ, that is, $p_\phi \triangleq \boldsymbol{\pi}^\top_{\tau_\phi} \boldsymbol{a}_\phi$. The tuple $y_\phi \triangleq (\tau_\phi, \boldsymbol{a}_\phi, p_\phi)$ characterizes all actions taken by customer ϕ. Customers derive utility

$$U(\phi, y_\phi) \triangleq \sum_{j=1}^{n} v^j_\phi e^{-d^j_\phi(\tau_\phi - t_\phi)} a^j_\phi - p_\phi.$$

This utility model is in line with other papers in the literature by assuming time-value-of-money considerations to be negligible, therefore discounting only valuations (Desiraju and Shugan 1999, Aviv and Pazgal 2008). This valuation discounting goes beyond time-value-of-money considerations, and could be, for example, because of seasonal or fashionable products becoming less desirable.

The type of each customer is her private information. That is, the DM does not observe customers who have arrived but delay their (potential) purchases. Instead, the customer-related information accessible to the DM at some time t pertains to historical sales and is given by $\mathcal{S}_t \triangleq \{(\tau_\phi, \boldsymbol{a}_\phi) : \tau_\phi \leq t, \mathbf{1}^\top \boldsymbol{a}_\phi = 1\}$, where $\mathbf{1}$ is the vector of ones. We denote the filtration induced by this information with $\mathcal{F}_t \triangleq \sigma(\mathcal{S}_{t-})$. Customers observe posted-price information $\mathcal{P}_t \triangleq \{\pi_s : s \leq t\}$; let $\mathcal{C}_t \triangleq \sigma(\mathcal{P}_t)$ be the associated filtration.

We assume that the DM has the power to commit to a pricing policy, which she announces at the beginning of the game. In this context, the DM and the customers are playing a dynamic game. Specifically, at $t = 0$, the DM chooses a dynamic pricing policy $\pi = \{\pi^j_t : t \in [0, T], j \in \{1, \ldots, n\}\}$ that she commits to post prices with and that is common knowledge to all customers. The set of admissible pricing policies, denoted by Π, consists of policies such that, for all $t \in [0, T]$ and $j \in \{1, \ldots, n\}$, (a) π^j_t is $\mathcal{F}_t$-progressive and that (b) an infinite price is posted for all products that include a resource whose inventory has been depleted, that is, $\pi^j_t = \infty$ if there exists an $i \in \{1, \ldots, m\}$ such that $X^i_{t-} = 0$ and $A_{ij} = 1$, for all $t \in [0, T]$ and $j \in \{1, \ldots, n\}$.

In response, customers are forward looking and seek to maximize their expected derived utilities, using (symmetric) stopping and purchasing rules contingent on their types that constitute a symmetric Markov perfect equilibrium. In particular, under policy $\pi \in \Pi$, customer ϕ follows actions $y^\pi_\phi \triangleq (\tau^\pi_\phi, \boldsymbol{a}^\pi_\phi, p^\pi_\phi)$, where $(\tau^\pi_\phi, \boldsymbol{a}^\pi_\phi)$ are $\mathcal{C}_t$-progressive stopping and purchasing rules that solve[3] the optimal stopping problem

$$\begin{aligned} &\text{maximize} && \mathsf{E}\Big[U(\phi, y_\phi) \,\big|\, \mathcal{C}_{t_\phi}\Big] \\ &\text{subject to} && \tau_\phi \in \big[t_\phi, T\big],\ \boldsymbol{a}_\phi \in \{0,1\}^n,\ \mathbf{1}^\top \boldsymbol{a}_\phi \leq 1, \end{aligned}$$

with the expectation above assuming that other customers use symmetric stopping and purchasing rules, and $p^\pi_\phi \triangleq \boldsymbol{\pi}^\top_{\tau^\pi_\phi} \boldsymbol{a}^\pi_\phi$. For any pricing policy $\pi \in \Pi$, we denote the set of all customers' actions it induces with $y^\pi \triangleq \{y^\pi_\phi : \phi \in \Phi\}$, its expected revenues with $J^\pi \triangleq \mathsf{E}[\sum_{\phi \in \mathcal{H}_T} p^\pi_\phi]$, where $\mathcal{H}_t \triangleq \{\phi : t_\phi \leq t\}$ is the set of customer types that arrive up to time t, for all $t \in [0, T]$.

We shall refer to an admissible pricing policy that maximizes expected revenues as *revenue-maximizing* and to the expected revenues it generates as *optimal revenues*. Let $J^\star$ denote the optimal revenues, that is,

$$J^\star \triangleq \sup_{\pi \in \Pi} J^\pi.$$

3. Upper Bound on Optimal Revenues

We derive an upper bound on $J^\star$ that could serve as a useful benchmark in theoretical or numerical analyses of pricing policies for our fairly general model. To this end, we first present some useful properties of customers' best response in Section 3.1. Then, in Section 3.2, we consider a suitably constructed dynamic mechanism design problem. Our analysis introduces a novel decomposition approach for multivariate incentive compatibility and individual rationality constraints, thereby enabling us to arrive at a tractable formulation that yields a useful upper bound on $J^\star$.

3.1. Properties of Customers' Best Responses

We show that under any admissible policy, customers act so that they derive nonnegative utility almost surely and that they choose not to purchase a product if customers with higher valuation for it choose not to purchase, ceteris paribus. For any $\phi \in \Phi$, $j \in \{1, \ldots, n\}$, let $\phi_{j,w} \triangleq (t_\phi, v_\phi^1, \dots, v_\phi^{j-1}, w, v_\phi^{j+}1, \dots, v_\phi^n, \boldsymbol{d}_\phi)$ be customer ϕ whose valuation for the jth product is w.

Proposition 1. *Under any policy $\pi \in \Pi$, any customer $\phi \in \Phi$ acts so that (a) $U(\phi, y_\phi^\pi) \geq 0$ and (b) $(1 - a_\phi^{\pi,j})a_{\phi_{j,v'}}^{\pi,j} = 0$, for all $j \in \{1, \dots, n\}$ and $v' \leq v_\phi^j$.*

Proof of Proposition 1. Fix a policy $\pi \in \Pi$ and a customer $\phi \in \Phi$. At time τ_ϕ^π, we have

$$\sum_{j=1}^n a_\phi^{\pi,j} = \begin{cases} 1 & \text{if } \max_{j\in\{1,\dots,n\}} v_\phi^j e^{-d_\phi^j\left(\tau_\phi^\pi - t_\phi\right)} - \pi_{\tau_\phi^\pi}^j \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

yielding (a) $U(\phi, y_\phi^\pi) \geq 0$. To show (b), fix any $j \in \{1, \dots, n\}$ and any $v' \leq v_\phi^j$. Note that $a_\phi^{\pi,j}, a_{\phi_{j,v'}}^{\pi,j} \in \{0, 1\}$. Therefore, it is equivalent to prove that if $a_\phi^{\pi,j} = 0$, then $a_{\phi_{j,v'}}^{\pi,j} = 0$. To prove this result, we derive two useful properties.

First, we prove that if $a_\phi^{\pi,j} = 0$, then $\tau_{\phi_{j,v'}}^\pi \in [t_\phi, \tau_\phi^\pi]$. To this end, consider any stopping rule τ and purchasing rule $\boldsymbol{a}$ such that $\tau \in [\tau_\phi^\pi, T]$, $\boldsymbol{a} \in \{0,1\}^n$, and $\mathbf{1}^\mathsf{T}\boldsymbol{a} \leq 1$. Then,

$$\begin{aligned}
U\left(\phi_{j,v'}, y_\phi^\pi\right) &= U\left(\phi, y_\phi^\pi\right) \\
&\geq \mathsf{E}\left[\sum_{j'=1}^n v_\phi^{j'} e^{-d_\phi^{j'}(\tau - t_\phi)} a^{j'} - \sum_{j'=1}^n \pi_\tau^{j'} a^{j'} \middle| \mathcal{C}_{\tau_\phi^\pi}\right] \\
&= \sum_{j'\neq j} \mathsf{E}\left[v_\phi^{j'} e^{-d_\phi^{j'}(\tau - t_\phi)} - \pi_\tau^{j'} \middle| \mathcal{C}_{\tau_\phi^\pi}, a^{j'} = 1\right] \\
&\quad \times \mathsf{P}\left(a^{j'} = 1 \middle| \mathcal{C}_{\tau_\phi^\pi}\right) \\
&\quad + \mathsf{E}\left[v_\phi^j e^{-d_\phi^j(\tau - t_\phi)} - \pi_\tau^j \middle| \mathcal{C}_{\tau_\phi^\pi}, a^j = 1\right] \\
&\quad \times \mathsf{P}\left(a^j = 1 \middle| \mathcal{C}_{\tau_\phi^\pi}\right) \\
&\geq \sum_{j'\neq j} \mathsf{E}\left[v_\phi^{j'} e^{-d_\phi^{j'}(\tau - t_\phi)} - \pi_\tau^{j'} \middle| \mathcal{C}_{\tau_\phi^\pi}, a^{j'} = 1\right] \\
&\quad \times \mathsf{P}\left(a^{j'} = 1 \middle| \mathcal{C}_{\tau_\phi^\pi}\right) \\
&\quad + \mathsf{E}\left[v' e^{-d_\phi^j(\tau - t_\phi)} - \pi_\tau^j \middle| \mathcal{C}_{\tau_\phi^\pi}, a^j = 1\right] \\
&\quad \times \mathsf{P}\left(a^j = 1 \middle| \mathcal{C}_{\tau_\phi^\pi}\right) \\
&= \mathsf{E}\left[\sum_{j'=1}^n v_{\phi_{j,v'}}^{j'} e^{-d_\phi^{j'}(\tau - t_\phi)} a^{j'} - \sum_{j'=1}^n \pi_\tau^{j'} a^{j'} \middle| \mathcal{C}_{\tau_\phi^\pi}\right],
\end{aligned}$$

where the first equality follows from $a_\phi^{\pi,j} = 0$ and $v_{\phi_{j,v'}}^{j'} = v_\phi^{j'}$ for all $j' \neq j$, the first inequality follows from $(\tau_\phi^\pi, \boldsymbol{a}_\phi^\pi)$ being customer ϕ's best response, and the second inequality follows from the property that $v' \leq v_\phi^j$. Thus, we established that the utility that customer of type $\phi_{j,v'}$ would derive by following $(\tau_\phi^\pi, \boldsymbol{a}_\phi^\pi)$ is greater than or equal to the expected utility he would have derived in case he had followed any other stopping and purchasing rules $(\tau, \boldsymbol{a})$ at time τ_ϕ^π. Formally, if we let $y \triangleq (\tau, \boldsymbol{a}, \boldsymbol{\pi}_\tau^\mathsf{T}\boldsymbol{a})$, we have

$$\left(\tau_\phi^\pi, \boldsymbol{a}_\phi^\pi\right) \in \mathop{\arg\max}_{\substack{\tau\in\left[\tau_\phi^\pi, T\right] \\ \boldsymbol{a}\in\{0,1\}^n \\ \mathbf{1}^\mathsf{T}\boldsymbol{a}\leq 1}} \mathsf{E}\left[U\left(\phi_{j,v'}, y\right) \middle| \mathcal{C}_{\tau_\phi^\pi}, a_\phi^{\pi,j} = 0\right].$$

In other words, if a customer of type $\phi_{j,v'}$ has not stopped by time τ_ϕ^π and $a_\phi^{\pi,j} = 0$, then it is optimal to stop at time τ_ϕ^π. This implies that if $a_\phi^{\pi,j} = 0$, then $\tau_{\phi_{j,v'}}^\pi \in [t_\phi, \tau_\phi^\pi]$.

Second, we prove that if $\tau_\phi^\pi \geq \tau_{\phi_{j,v'}}^\pi$ and $a_{\phi_{j,v'}}^{\pi,j} = 1$, then $a_\phi^{\pi,j} = 1$. Consider any stopping and purchasing rules such that $\tau \in [\tau_{\phi_{j,v'}}^\pi, T]$, $\boldsymbol{a} \in \{0,1\}^n$, and $\mathbf{1}^\mathsf{T}\boldsymbol{a} \leq 1$. Let $y \triangleq (\tau, \boldsymbol{a}, \boldsymbol{\pi}_\tau^\mathsf{T}\boldsymbol{a})$. We have

$$\begin{aligned}
0 &\leq U\left(\phi_{j,v'}, y_{\phi_{j,v'}}^\pi\right) - \mathsf{E}\left[U\left(\phi_{j,v'}, y\right) \middle| \mathcal{C}_{\tau_{\phi_{j,v'}}^\pi}\right] \\
&= v' e^{-d_\phi^j\left(\tau_{\phi_{j,v'}}^\pi - t_\phi\right)} - \pi_{\tau_{\phi_{j,v'}}^\pi}^j - \mathsf{E}\left[U\left(\phi_{j,v'}, y\right) \middle| \mathcal{C}_{\tau_{\phi_{j,v'}}^\pi}\right] \\
&= v' e^{-d_\phi^j\left(\tau_{\phi_{j,v'}}^\pi - t_\phi\right)} - \pi_{\tau_{\phi_{j,v'}}^\pi}^j \\
&\quad - \sum_{j'\neq j} \mathsf{E}\left[v_\phi^{j'} e^{-d_\phi^{j'}(\tau - t_\phi)} - \pi_\tau^{j'} \middle| \mathcal{C}_{\tau_{\phi_{j,v'}}^\pi}, a^{j'} = 1\right] \\
&\quad \times \mathsf{P}\left(a^{j'} = 1 \middle| \mathcal{C}_{\tau_{\phi_{j,v'}}^\pi}\right) \\
&\quad - \mathsf{E}\left[v' e^{-d_\phi^j(\tau - t_\phi)} - \pi_\tau^j \middle| \mathcal{C}_{\tau_{\phi_{j,v'}}^\pi}, a^j = 1\right] \\
&\quad \times \mathsf{P}\left(a^j = 1 \middle| \mathcal{C}_{\tau_{\phi_{j,v'}}^\pi}\right) \\
&= v' \underbrace{\left(\begin{array}{c} e^{-d_\phi^j\left(\tau_{\phi_{j,v'}}^\pi - t_\phi\right)} - \mathsf{E}\left[e^{-d_\phi^j(\tau - t_\phi)} \middle| \mathcal{C}_{\tau_{\phi_{j,v'}}^\pi}, a^j = 1\right] \\ \mathsf{P}\left(a^j = 1 \middle| \mathcal{C}_{\tau_{\phi_{j,v'}}^\pi}\right) \end{array}\right)}_{A} \\
&\quad - \pi_{\tau_{\phi_{j,v'}}^\pi}^j + \mathsf{E}\left[\pi_\tau^j \middle| \mathcal{C}_{\tau_{\phi_{j,v'}}^\pi}, a^j = 1\right] \mathsf{P}\left(a^j = 1 \middle| \mathcal{C}_{\tau_{\phi_{j,v'}}^\pi}\right) \\
&\quad - \sum_{j'\neq j} \mathsf{E}\left[v_\phi^{j'} e^{-d_\phi^{j'}(\tau - t_\phi)} - \pi_\tau^{j'} \middle| \mathcal{C}_{\tau_{\phi_{j,v'}}^\pi}, a^{j'} = 1\right] \\
&\quad \times \mathsf{P}\left(a^{j'} = 1 \middle| \mathcal{C}_{\tau_{\phi_{j,v'}}^\pi}\right) \\
&\leq v_\phi^j A - \pi_{\tau_{\phi_{j,v'}}^\pi}^j + \mathsf{E}\left[\pi_\tau^j \middle| \mathcal{C}_{\tau_{\phi_{j,v'}}^\pi}, a^j = 1\right] \mathsf{P}\left(a^j = 1 \middle| \mathcal{C}_{\tau_{\phi_{j,v'}}^\pi}\right) \\
&\quad - \sum_{j'\neq j} \mathsf{E}\left[v_\phi^{j'} e^{-d_\phi^{j'}(\tau - t_\phi)} - \pi_\tau^{j'} \middle| \mathcal{C}_{\tau_{\phi_{j,v'}}^\pi}, a^{j'} = 1\right] \mathsf{P}\left(a^{j'} = 1 \middle| \mathcal{C}_{\tau_{\phi_{j,v'}}^\pi}\right)
\end{aligned}$$

$$
\begin{aligned}
&= v^j_\phi e^{-d^j_\phi\left(\tau^\pi_{\phi_{j,v'}} - t_\phi\right)} - \pi^j_{\tau^\pi_{\phi_{j,v'}}} \\
&\quad - \sum_{j'\neq j} \mathsf{E}\left[v^{j'}_\phi e^{-d^{j'}_\phi(\tau - t_\phi)} - \pi^{j'}_\tau \middle| \mathcal{C}_{\tau^\pi_{\phi_{j,v'}}}, a^{j'} = 1\right] \\
&\quad \times \mathsf{P}\left(a^{j'} = 1 \middle| \mathcal{C}_{\tau^\pi_{\phi_{j,v'}}}\right) \\
&\quad - \mathsf{E}\left[v^{j}_\phi e^{-d^{j}_\phi(\tau - t_\phi)} - \pi^{j}_\tau \middle| \mathcal{C}_{\tau^\pi_{\phi_{j,v'}}}, a^{j} = 1\right] \\
&\quad \times \mathsf{P}\left(a^{j} = 1 \middle| \mathcal{C}_{\tau^\pi_{\phi_{j,v'}}}\right) \\
&= v^j_\phi e^{-d^j_\phi\left(\tau^\pi_{\phi_{j,v'}} - t_\phi\right)} - \pi^j_{\tau^\pi_{\phi_{j,v'}}} - \mathsf{E}\left[U(\phi, y) \middle| \mathcal{C}_{\tau^\pi_{\phi_{j,v'}}}\right] \\
&= U\left(\phi, y^\pi_{\phi_{j,v'}}\right) - \mathsf{E}\left[U(\phi, y) \middle| \mathcal{C}_{\tau^\pi_{\phi_{j,v'}}}\right].
\end{aligned}
$$

The first and the sixth equalities follow from the property that $a^{\pi,j}_{\phi_{j,v'}} = 1$. The first inequality follows from the property that $y^\pi_{\phi_{j,v'}}$ is customer $\phi_{j,v'}$'s best response. The second inequality follows from the property that $v' \le v^j_\phi$ and the property that $\tau \ge \tau^\pi_{\phi_{j,v'}}$ and $\mathsf{P}(\cdot) \le 1$ imply $A \ge 0$. Therefore, if we let $y \triangleq (\tau, \boldsymbol{a}, \pi^{\mathsf{T}}_\tau \boldsymbol{a})$, then we have

$$
\left(\tau^\pi_{\phi_{j,v'}}, \boldsymbol{a}^\pi_{\phi_{j,v'}}\right) \in \underset{\substack{\tau \in \left[\tau^\pi_{\phi_{j,v'}}, T\right] \\ \boldsymbol{a} \in \{0,1\}^n \\ \mathbf{1}^{\mathsf{T}} \boldsymbol{a} \le 1}}{\arg\max}\, \mathsf{E}\left[U(\phi, y) \middle| \mathcal{C}_{\tau^\pi_{\phi_{j,v'}}}, a^{\pi,j}_{\phi_{j,v'}} = 1\right].
$$

This implies that if $\tau^\pi_\phi \ge \tau^\pi_{\phi_{j,v'}}$ and $a^{\pi,j}_{\phi_{j,v'}} = 1$, then $a^{\pi,j}_\phi = 1$.

From the two properties that we proved, it follows that $a^{\pi,j}_\phi = 0$ implies $a^{\pi,j}_{\phi_{j,v'}} = 0$ for all $v' \le v_\phi$. To see this, suppose $a^{\pi,j}_\phi = 0$ and $a^{\pi,j}_{\phi_{j,v'}} = 1$. Then $a^{\pi,j}_\phi = 0$ and the first property implies $\tau_{\phi_{j,v'}} \le \tau_\phi$. In addition, $a^{\pi,j}_{\phi_{j,v'}} = 1$, $\tau_{\phi_{j,v'}} \le \tau_\phi$ and the second property implies $a^{\pi,j}_\phi = 1$, a contradiction. ■

3.2. A Dynamic Mechanism Design Problem

Given the complexity of the model in Section 2, we are going to instead study a suitable dynamic mechanism design problem, with the DM (the principal) choosing a mechanism that assigns products at certain prices to arriving customers (the agents), depending on how they choose to report their type, which includes their arrival time and product valuation.

The mechanism design problem will be formulated to closely mimic the interactions of the DM and the customers in our original model while being amendable to analysis. Therefore, for brevity, we simply emphasize below the ways the two models differ. Also, for exposition purposes we overload notation to make the connections between the two models clearer.

Transactions are governed by a mechanism that assigns products to customers, after they report their types.[4] A customer who reported her type to be $\phi \in \Phi$ is assigned $y_\phi \triangleq (\tau_\phi, \boldsymbol{a}_\phi, p_\phi)$, where $\tau_\phi \in [t_\phi, T]$ is the time that the DM decides to sell precisely one product unit to that customer or refuse to sell any product to him, $a^j_\phi \in \{0, 1\}$ is an indicator for whether ϕ is allocated with one unit of product $j \in \{1, \ldots, n\}$, and p_ϕ is the price paid by the customer. We denote such a mechanism, that is, the set of all these allocation and payment rules, with $y \triangleq \{y_\phi : \phi \in \Phi\}$.

The information set of the DM at time t is given by $\mathcal{G}_t \triangleq \sigma(\mathcal{H}_t)$, where $\mathcal{H}_t \triangleq \{\phi : t_\phi \le t\}$ is the set of customer reports made up to time t. We shall say that a mechanism y is *feasible* if y_ϕ satisfies the following properties for all $\phi \in \Phi$: (we denote the class of all feasible mechanisms with $\mathcal{Y}$)

1. Causality: τ_ϕ is a $\mathcal{G}_t$-measurable stopping time and $(\boldsymbol{a}_\phi, p_\phi)$ are $\mathcal{G}_{\tau_\phi}$-measurable.
2. Substitutability: The DM allocates at most one product to a customer, that is, $\mathbf{1}^{\mathsf{T}} \boldsymbol{a}_\phi \le 1$, a.s.
3. Limited Inventory: The DM does not run out of inventory, that is,

$$
\sum_{\phi \in \mathcal{H}_T} \sum_{j=1}^{n} A_{ij} a^j_\phi \le x^i_0, \forall i \in \{1, \ldots, m\}, \text{ a.s.} \tag{1}
$$

4. Nonnegative prices: The DM charges prices that are nonnegative, that is, $p_\phi \ge 0$, a.s.

Under some feasible mechanism y, customer ϕ derives utility $U(\phi, y_\phi)$, when truthfully reporting her type. Otherwise, if misreporting her type as $\hat{\phi}$, she derives utility $U(\phi, y_{\hat{\phi}})$. When choosing to misreport her type, customer ϕ can only reveal her arrival time no earlier than her true arrival and no later than the end of the season, that is, $t_{\hat{\phi}} \in [t_\phi, T]$.

By choosing mechanism y, the DM collects expected revenue $V(y) \triangleq \mathsf{E}[\sum_{\phi \in \mathcal{H}_T} p_\phi]$. Consider the following optimization problem for choosing y,

$$
\begin{aligned}
\text{maximize} \quad & V(y) \\
\text{subject to} \quad & y \in \mathcal{Y} \\
& \mathsf{E}_{-\phi}[U(\phi, y_\phi)] \ge \mathsf{E}_{-\phi}\left[U\left(\phi, y_{\hat{\phi}}\right)\right], \\
& \forall\ \phi, \hat{\phi} : t_{\hat{\phi}} \in \left[t_\phi, T\right] \quad \text{(IC)} \\
& U(\phi, y_\phi) \ge 0, \quad \forall\ \phi \quad \text{(IR)} \\
& \left(1 - a^j_\phi\right) a^j_{\phi_{j,v'}} = 0, \quad \forall\ \phi, j, v' \le v^j_\phi.
\end{aligned} \tag{2}
$$

That is, the DM chooses a feasible mechanism to maximize her expected revenues, subject to the incentive compatibility (IC) and individual rationality (IR) constraints. The previous formulation deviates from classical mechanism design problems by requiring that (i) the (IR) constraint holds almost surely and that (ii) a customer is not assigned a product if it is not assigned to another customer who has a higher valuation on it. These deviations ensure that the mechanism induces similar properties to the ones we derived in Section 3.1.

3.2.1. Analysis. Let J^{md} be the optimal value of the mechanism design problem (2). We first argue that it provides an upper bound to the expected revenues in our original model in Section 2, under any admissible pricing policy she might follow (proofs of Lemmata are included in the e-companion).

Lemma 1. *For any pricing policy* $\pi \in \Pi$, $J^{\pi} \le J^{\mathrm{md}}$, *and* $J^{\star} \le J^{\mathrm{md}}$.

The mechanism design problem (2) involves customers who have multivariate, heterogeneous valuations for the different products. The (IC) and (IR) constraints in the problem then involve utilities that couple all this private information. This coupling represents a major technical challenge as it precludes the application of the standard Myersonian solution approach (Myerson 1981). To overcome this challenge, we consider decomposing the utility of each customer as follows: we consider separate utility components associated with each product, that is,

$$U_j(\phi, y_\phi) \triangleq v^j_\phi e^{-d^j_\phi(\tau_\phi - t_\phi)} a^j_\phi - p^j_\phi, \tag{3}$$

where $p^j_\phi \triangleq p_\phi a^j_\phi$ is the price that customer ϕ pays for product j. We then replace the (IC) and (IR) constraints in Problem (2) with the constraints associated with each product in a decoupled fashion. The resulting (IC) constraint associated with the jth product, for example, would be

$$\mathsf{E}_{-\phi}[U_j(\phi, y_\phi)] \ge \mathsf{E}_{-\phi}\Big[U_j\Big(\phi, y_{\hat\phi}\Big)\Big], \forall\ \phi,\ \hat\phi\ :\ t_{\hat\phi} \in [t_\phi, T].$$

Reformulating Problem (2) by decoupling the original (IC) and (IR) constraints would be a step closer to enabling the application of the Myersonian approach. Unfortunately, such a reformulation would yield a stricter, rather than a relaxed reformulation, because it would shrink the set of feasible mechanisms. To see this, note that mechanisms satisfying the decoupled constraints also satisfy the original (IC) and (IR) constraints, but not vice versa.

To counterbalance this, we restrict agents by allowing them to misreport only their product valuations, which provides a relaxation. We are able to show that the resulting reformulation, with the decoupled (IC) and (IR) constraints and the aforementioned restriction, yields a *net relaxation* to Problem (2). Formally, consider the following problem and let $\hat J$ be its optimal value.

$$\begin{aligned}
\text{maximize}\quad & \mathsf{E}\Bigg[\sum_{\phi\in\mathcal{H}_T}\sum_{j=1}^{n} p^j_\phi\Bigg] \qquad (4)\\
\text{subject to}\quad & \mathsf{E}_{-\phi}[U_j(\phi, y_\phi)] \ge \mathsf{E}_{-\phi}\Big[U_j\Big(\phi, y_{\phi_{j,w}}\Big)\Big],\\
& \forall\ j,\ \phi,\ w \ge 0 \quad (\mathrm{IC}_j)\\
& \mathsf{E}_{-\phi}[U_j(\phi, y_\phi)] \ge 0, \quad \forall\ j,\ \phi \quad (\mathrm{IR}_j)\\
& \mathbf{1}^{\mathsf{T}}\boldsymbol{a}_\phi \le 1, \quad \forall\ \phi\\
& \sum_{\phi\in\mathcal{H}_T}\sum_{j=1}^{n} A_{ij} a^j_\phi \le x^i_0, \quad \forall\ i\\
& \boldsymbol{a}_\phi \in \{0,1\}^n, \quad \forall\ \phi.
\end{aligned}$$

Proposition 2. The optimal value of Problem (4) provides an upper bound to the optimal value of Problem (2), that is $J^{\mathrm{md}} \le \hat J$.

Proof of Proposition 2. In this proof, for notational clarity, we define $b^j_\phi \triangleq e^{-d^j_\phi(\tau_\phi - t_\phi)} a^j_\phi$. Consider any $y \in \mathcal{Y}$ that is feasible to optimization problem (2). We use y to construct a feasible solution to optimization problem (4), denoted by $\{\hat y^j_\phi\}$. In particular, for all j and ϕ, let

$$\hat a^j_\phi \triangleq a^j_\phi, \quad \hat p^j_\phi \triangleq v^j_\phi b^j_\phi - \int_{v'=0}^{v^j_\phi} b^j_{\phi_{j,v'}} dv', \quad \hat\tau_\phi \triangleq \tau_\phi.$$

To show feasibility of $\{\hat y^j_\phi\}$, we first derive two useful properties. First, we use the envelope theorem to express the utility of a ϕ-customer under y_ϕ as

$$\begin{aligned}
&\mathsf{E}_{-\phi}[U(\phi, y_\phi)]\\
&\quad = \mathsf{E}_{-\phi}\Bigg[\int_{v'=0}^{v^j_\phi} \frac{\partial U}{\partial v^j_\phi}\Big(\phi_{j,v'}, y_{\phi_{j,v'}}\Big)dv' + U\Big(\phi_{j,0}, y_{\phi_{j,0}}\Big)\Bigg]\\
&\quad = \mathsf{E}_{-\phi}\Bigg[\int_{v'=0}^{v^j_\phi} b^j_{\phi_{j,v'}} dv' + U\Big(\phi_{j,0}, y_{\phi_{j,0}}\Big)\Bigg].
\end{aligned}$$

Second, the utility of a ϕ-customer when reporting valuation w for the jth product can be expressed

$$\begin{aligned}
\mathsf{E}_{-\phi}\Big[U\Big(\phi, y_{\phi_{j,w}}\Big)\Big] &= \mathsf{E}_{-\phi}\Bigg[\sum_{j=1}^{n} v^j_\phi b^j_{\phi_{j,w}} - p_{\phi_{j,w}}\Bigg]\\
&= \mathsf{E}_{-\phi}\Bigg[\sum_{j=1}^{n} v^j_{\phi_{j,w}} b^j_{\phi_{j,w}} - p_{\phi_{j,w}} + \Big(v^j_\phi - w\Big)b^j_{\phi_{j,w}}\Bigg]\\
&= \mathsf{E}_{-\phi}\Big[U\Big(\phi_{j,w}, y_{\phi_{j,w}}\Big) + \Big(v^j_\phi - w\Big)b^j_{\phi_{j,w}}\Big]\\
&= \mathsf{E}_{-\phi}\Bigg[\int_{v'=0}^{w} b^j_{\phi_{j,v'}} dv' + U\Big(\phi_{j,0}, y_{\phi_{j,0}}\Big)\\
&\qquad + \Big(v^j_\phi - w\Big)b^j_{\phi_{j,w}}\Bigg],
\end{aligned}$$

where for the last equality we used the previous property.

Now, for all $j \in \{1,\ldots,n\}$, $\phi \in \Phi$, $w \geq 0$ we have that

$$
\begin{aligned}
&\mathsf{E}_{-\phi}\Big[U_j(\phi,\hat{y}_\phi) - U_j\Big(\phi,\hat{y}_{\phi_{j,w}}\Big)\Big]\\
&\quad = \mathsf{E}_{-\phi}\left[\int_{v'=0}^{v_\phi^j} b^j_{\phi_{j,v'}}\,dv' - \Big(v^j_\phi - w\Big)b^j_{\phi_{j,w}} - \int_{v'=0}^{w} b^j_{\phi_{j,v'}}\,dv'\right]\\
&\quad = \mathsf{E}_{-\phi}\Big[U(\phi,y_\phi) - U\Big(\phi,y_{\phi_{j,w}}\Big)\Big] \geq 0,
\end{aligned}
$$

where the first equality follows by substituting for $\hat{y}^j_\phi$, the second equality from the two properties above, and the inequality follows the (IC) constraint that (the feasible) y satisfies in optimization problem (2). Therefore, $\{\hat{y}^j_\phi\}$ satisfies (IC_j) in optimization problem (4).

Furthermore, for any $\phi \in \Phi$, $j \in \{1,\ldots,n\}$, $\mathsf{E}_{-\phi}[U_j(\phi,\hat{y}_\phi)] = \mathsf{E}_{-\phi}[\int_{v'=0}^{v^j_\phi} b^j_{\phi_{j,v'}}\,dv'] \geq 0$. Therefore, $\{\hat{y}^j_\phi\}$ satisfies (IR_j) in (4). The remaining constraints are also satisfied given that $y \in \mathcal{Y}$.

To complete the proof, we show that the objective of (4) evaluated at $\{\hat{y}^j_\phi\}$ is greater than or equal to the objective of (2) evaluated at y:

$$
\begin{aligned}
&\mathsf{E}\left[\sum_{\phi\in\mathcal{H}_T}\sum_{j=1}^{n}\hat{p}^j_\phi\right]\\
&= \mathsf{E}\left[\sum_{\phi\in\mathcal{H}_T}\sum_{j=1}^{n}\mathsf{E}_{-\phi}\Big[\hat{p}^j_\phi\Big]\right]\\
&= \mathsf{E}\left[\sum_{\phi\in\mathcal{H}_T}\sum_{j=1}^{n}\mathsf{E}_{-\phi}\left[v^j_\phi b^j_\phi - \int_{v'=0}^{v^j_\phi} b^j_{\phi_{j,v'}}\,dv'\right]\right]\\
&= \mathsf{E}\left[\sum_{\phi\in\mathcal{H}_T}\sum_{j=1}^{n}\mathsf{E}_{-\phi}\left[\left(v^j_\phi b^j_\phi - \int_{v'=0}^{v^j_\phi} b^j_{\phi_{j,v'}}\,dv'\right)a^j_\phi\right]\right]\\
&= \mathsf{E}\left[\sum_{\phi\in\mathcal{H}_T}\sum_{j=1}^{n}\mathsf{E}_{-\phi}\left[\Big(v^j_\phi b^j_\phi - U(\phi,y_\phi) + U\Big(\phi_{j,0},y_{\phi_{j,0}}\Big)\Big)a^j_\phi\right]\right]\\
&= \mathsf{E}\left[\sum_{\phi\in\mathcal{H}_T}\sum_{j=1}^{n}\Big(v^j_\phi b^j_\phi - U(\phi,y_\phi) + U\Big(\phi_{j,0},y_{\phi_{j,0}}\Big)\Big)a^j_\phi\right]\\
&\geq \mathsf{E}\left[\sum_{\phi\in\mathcal{H}_T}\sum_{j=1}^{n}\Big(v^j_\phi b^j_\phi - U(\phi,y_\phi)\Big)a^j_\phi\right]\\
&= \mathsf{E}\left[\sum_{\phi\in\mathcal{H}_T}\sum_{j=1}^{n}\left(v^j_\phi b^j_\phi - \left(\sum_{j'=1}^{n} v^{j'}_\phi b^{j'}_\phi - p_\phi\right)\right)a^j_\phi\right]\\
&= \mathsf{E}\left[\sum_{\phi\in\mathcal{H}_T}\sum_{j=1}^{n} p_\phi a^j_\phi\right]\\
&= \mathsf{E}\left[\sum_{\phi\in\mathcal{H}_T} p_\phi\right]\\
&= V(y),
\end{aligned}
$$

where the third equality follows from Proposition 1(b); the first inequality follows the (IR) constraint that $U(\phi_{j,0},y_{\phi_{j,0}}) \geq 0$ and the property that $a^j_\phi \geq 0$; the eighth equality follows from that $\sum_{j=} 1^n a^j_\phi = 1$, or $p_\phi = 0$ if $\sum_{j=} 1^n a^j_\phi = 0$. We conclude that $J^{\text{md}} \leq \hat{J}$. ∎

We can now apply the Myersonian approach to the relaxed problem above to obtain an upper bound to its optimal value $\hat{J}$. In particular, for any $\chi \in [0,1]$, consider the problem

$$
\begin{aligned}
\text{maximize}\quad & \mathsf{E}\left[\sum_{\phi\in\mathcal{H}_T}\sum_{j=1}^{n}\left(v^j_\phi - \frac{\bar{F}_j\Big(v^j_\phi\Big)}{f_j\Big(v^j_\phi\Big)}\right)\tilde{b}^j_\phi\right]\\
\text{subject to}\quad & \mathsf{E}_\phi\left[\sum_{j=1}^{n}\tilde{b}^j_\phi\right] \leq \chi\\
& \mathsf{E}\left[\sum_{\phi\in\mathcal{H}_T}\sum_{j=1}^{n}A_{ij}\tilde{b}^j_\phi\right] \leq x^i_0 \quad \forall\, i\\
& \tilde{b}^j_\phi \in [01] \quad \forall\, \phi j.
\end{aligned}
\tag{5}
$$

and let $\tilde{J}^\chi$ denote its the optimal value.

Lemma 2. We have $\hat{J} \leq \tilde{J}^1$. *Furthermore, for any* $\chi \in [0,1]$, $\chi\tilde{J}^1 \leq \tilde{J}^\chi$.

The final step is to derive an upper bound to $\tilde{J}^\chi$ using the optimal value of an optimization problem that is motivated by a fluid-type approximation of an NRM model in which customers were to behave myopically. Specifically, for any $\chi \in [0,1]$ consider

$$
\begin{aligned}
\text{maximize}\quad & \lambda T\sum_{j=1}^{n} q^j\bar{F}_j(q^j)\\
\text{subject to}\quad & \sum_{j=1}^{n}\bar{F}_j(q^j) \leq \chi\\
& \lambda T\sum_{j=1}^{n}A_{ij}\bar{F}_j(q^j) \leq x^i_0, \quad \forall\, i
\end{aligned}
\tag{6}
$$

and let $\bar{J}^\chi$ be its optimal value and $\boldsymbol{q}_\chi$ an optimal solution. To prove our result, we also enforce hereafter a standard assumption on the product valuation marginal distribution functions:

Assumption 1. *The virtual value function* $v - \frac{\bar{F}_j(v)}{f_j(v)}$ *is nondecreasing for all* $v \geq 0$, $j \in \{1,\ldots,n\}$.

Lemma 3. *For any* $\chi \in [0,1]$, $\tilde{J}^\chi \leq \bar{J}^\chi$.

To summarize, by combining the results in this section, we obtain

$$
J^\pi \leq J^\star \leq J^{\text{md}} \leq \hat{J} \leq \tilde{J}^1 \leq \frac{1}{\chi}\tilde{J}^\chi \leq \frac{1}{\chi}\bar{J}^\chi, \tag{7}
$$

which leads to the following.

Theorem 1 (Revenue Bound). *For any dynamic pricing policy $\pi \in \Pi$ and $\chi \in (0,1]$,*

$$J^{\pi} \le J^{\star} \le \frac{1}{\chi}\bar{J}^{\chi}.$$

The decomposition approach we introduced in our analysis and enabled us to provide a first characterization of optimal revenues in NRM with forward-looking customers with product choice does not come without its limitations. First, it necessitates the assumption of independent valuations. Second, it provides as an upper bound the optimal value of (6), an optimization problem in which fluid customer streams are also decomposed.

4. Performance Analysis of Static Pricing

We now leverage the upper bound we derived to conduct in our model a novel performance analysis of static pricing, that is, policies that post a constant price for each product throughout the selling season. Our analysis is twofold. First, we derive an analytical result, specifically guarantees for the performance of static pricing in the classical fluid-type regime where inventory and demand grow large. Second, we conduct a numerical analysis wherein we explore the performance of static pricing under realistic ranges for problem parameters and commonly used demand distributions.

4.1. Theoretical Performance Analysis

Formally, we shall refer to an admissible pricing policy as *static* if it posts a constant price for each product as long as inventory is available. That is, for a static pricing policy $\bar{\pi} \in \Pi$, there exist prices $\bar{\boldsymbol{q}} \in \mathbb{R}^n$ such that for all $t \in [0,T]$ and $j \in \{1,\dots,n\}$

$$\bar{\pi}_t^j = \begin{cases} \infty & \text{if } \exists\, i: \ X_{t-}^i = 0 \text{ and } A_{ij} = 1, \\ \bar{q}^j & \text{otherwise.} \end{cases}$$

We are now going to construct a static pricing policy backed by a performance guarantee in a fluid-type approximation regime. Our analysis of static pricing is at some points similar and at some other points different from analyses of static pricing in the literature, such as Gallego and Van Ryzin (1994) or Talluri and Van Ryzin (1998). For example, different from these papers, we consider a parameterized bound on the number of items purchased in expectation that is lower than 1. In particular, we use the parameter χ introduced previously to derive a meaningful bound. To see this, fix some $\chi \in (0,1]$ and consider the static pricing policy $\bar{\pi}_{\chi} \in \Pi$ that posts prices $\boldsymbol{q}_{\chi} \in \mathbb{R}^n$ (recall that $\boldsymbol{q}_{\chi}$ is an optimal solution to Problem (6)). Note that under any static pricing policy $\bar{\pi}$, the customers' best-response actions $y^{\bar{\pi}}$ can be readily seen to be myopic. Formally, under a static pricing policy $\bar{\pi}$, customer ϕ's dominant equilibrium is to behave myopically, that is, for all $\phi \in \Phi$, (a) $\tau_{\phi}^{\bar{\pi}} = t_{\phi}$; (b) $\mathbf{1}^{\mathsf{T}}\boldsymbol{a}_{\phi}^{\bar{\pi}} = 1$ if and only if $\max_j\{v_{\phi}^j - \bar{\pi}_{t_{\phi}}^j\} \ge 0$; and, (c) $a_{\phi}^{\bar{\pi},j} = 1$ only if $v_{\phi}^j - \bar{\pi}_{t_{\phi}}^j \ge v_{\phi}^{j'} - \bar{\pi}_{t_{\phi}}^{j'}$ for all $j' \ne j$. We then have the following performance guarantee for $\bar{\pi}_{\chi}$.Finally, consider a sequence of systems where in the Nth system we have $\lambda^{(N)} = N\lambda$ and $\boldsymbol{x}_0^{(N)} = N\boldsymbol{x}_0$. That is, we proportionally scale up the customer arrival rate and the DM's initial inventory for all resources. Let $J^{\pi}(\lambda^{(N)}, \boldsymbol{x}_0^{(N)})$ denote the DM's expected revenues under policy $\pi \in \Pi$ for the Nth system, and $J^{\star}(\lambda^{(N)}, \boldsymbol{x}_0^{(N)})$ denote the associated optimal revenues. For this high-volume regime, it can be readily seen that

$$\frac{J^{\bar{\pi}_{\chi}}\left(\lambda^{(N)}, \boldsymbol{x}_0^{(N)}\right)}{J^{\star}\left(\lambda^{(N)}, \boldsymbol{x}_0^{(N)}\right)} \ge \chi\left(1 - \min\left\{\chi, \frac{\sum_{i=1}^m x_0^i}{\lambda T r}\right\}\right) - O\left(\frac{1}{\sqrt{N}}\right),$$

for all $\pi \in \Pi$ and $\chi \in (0,1]$. By appropriately assigning a value on χ, we obtain the following.

Proposition 3 (Performance Guarantee). *For the static pricing policy $\bar{\pi}_{\chi}$,*

$$\frac{J^{\bar{\pi}_{\chi}}}{J^{\star}} \ge \chi\left(1 - \min\left\{\chi, \frac{\sum_{i=1}^m x_0^i}{\lambda T r}\right\}\right) - \frac{\max_j q_{\chi}^j}{\sum_{j=1}^n q_{\chi}^j \bar{F}_j\left(q_{\chi}^j\right)} \frac{m\chi}{\sqrt{2\lambda T}},$$

where $r \triangleq \min_j \sum_{i=} 1^m A_{ij}$ denotes the minimum number of resources required for any product.

Proof of Proposition 3. First, we establish the following useful property:

$$1 - \prod_{j=1}^{k}(1 - \zeta^j) \le \sum_{j=1}^{k} \zeta^j, \quad \forall\, k \in \mathbb{N}, \text{and} \zeta \in \mathbb{R}^k \quad \text{such that} 0 \le \zeta \le 1. \tag{8}$$

We prove it via induction. It is trivially true for $k = 1$. Suppose it is true for some $k \ge 1$. Then,

$$1 - \prod_{j=1}^{k+1}(1 - \zeta^j) \le 1 - \left(1 - \sum_{j=1}^{k} \zeta^j\right)(1 - \zeta^{k+1}) = \sum_{j=1}^{k+1} \zeta^j - \left(\sum_{j=1}^{k} \zeta^j\right)\zeta^{k+1} \le \sum_{j=1}^{k+1} \zeta^j.$$

For each customer $\phi \in \Phi$ and $j \in \{1,\dots,n\}$, let $\beta_{\phi}^j \triangleq \begin{cases} 1 & \text{if } v_{\phi}^j - q_{\chi}^j \ge \max_{j' \ne j}(v_{\phi}^{j'} - q_{\chi}^{j'})^+ \\ 0 & \text{otherwise.} \end{cases}$ Denoting the indicator function with $\mathbb{1}\{\cdot\}$, we have

$$\begin{aligned} J^{\bar{\pi}_{\chi}} &= \mathsf{E}\left[\sum_{\phi \in \mathcal{H}_T} \sum_{j=1}^{n} q_{\chi}^j \mathbb{1}\left\{a_{\phi}^{\bar{\pi}_{\chi},j} = 1\right\}\right] \\ &\ge \sum_{j=1}^{n} q_{\chi}^j \mathsf{E}\left[\sum_{\phi \in \mathcal{H}_T} \mathbb{1}\left\{\beta_{\phi}^j = 1\right\}\right] \\ &\quad - \left(\sum_{j=1}^{n} q_{\chi}^j\right) \mathsf{E}\left[\sum_{\phi \in \mathcal{H}_T} \mathbb{1}\left\{\sum_{j=1}^{n} \beta_{\phi}^j \ge 2\right\}\right] \\ &\quad - \left(\max_j q_{\chi}^j\right) \sum_{i=1}^{m} \mathsf{E}\left[\left(\sum_{\phi \in \mathcal{H}_T} \sum_{j=1}^{n} A_{ij} \times \mathbb{1}\left\{\beta_{\phi}^j = 1\right\} - x_0^i\right)^+\right]. \end{aligned} \tag{9}$$

Now, we analyze the three terms in (9) separately. For the first term in (9), we have

$$\begin{aligned}
\sum_{j=1}^{n} q_\chi^j \mathsf{E}\left[\sum_{\phi\in\mathcal{H}_T} \mathbb{1}\left\{\beta_\phi^j = 1\right\}\right]
&= \sum_{j=1}^{n} q_\chi^j \mathsf{E}[|\mathcal{H}_T|]\mathsf{E}\left[\mathbb{1}\left\{\beta_\phi^j = 1\right\}\right] \\
&= \lambda T \sum_{j=1}^{n} q_\chi^j \mathsf{E}\left[\mathbb{1}\left\{\beta_\phi^j = 1\right\}\right] \\
&\geq \lambda T \sum_{j=1}^{n} q_\chi^j \mathsf{P}\left(v_\phi^j \geq q_\chi^j \text{ and } v_\phi^{j'} < q_\chi^{j'}, \forall j' \neq j\right) \\
&= \lambda T \sum_{j=1}^{n} q_\chi^j \bar{F}_j\left(q_\chi^j\right) \prod_{j'\neq j} F_{j'}\left(q_\chi^{j'}\right) \\
&\geq \lambda T \sum_{j=1}^{n} q_\chi^j \bar{F}_j\left(q_\chi^j\right) \prod_{j'=1}^{n} F_{j'}\left(q_\chi^{j'}\right) \\
&= \lambda T \left(\sum_{j=1}^{n} q_\chi^j \bar{F}_j\left(q_\chi^j\right)\right)\left(\prod_{j=1}^{n} F_j\left(q_\chi^j\right)\right) \\
&\geq \lambda T \left(\sum_{j=1}^{n} q_\chi^j \bar{F}_j\left(q_\chi^j\right)\right)\left(1 - \sum_{j=1}^{n} \bar{F}_j\left(q_\chi^j\right)\right) \\
&= \left(1 - \sum_{j=1}^{n} \bar{F}_j\left(q_\chi^j\right)\right)\bar{J}^\chi,
\end{aligned}$$

where the first equality follows from Wald's identity; the third equality follows from the independence of customers' valuations for different products; the second inequality follows from $F_j(q_\chi^j) \leq 1$; the third inequality follows from (8).

Now, we establish two upper bounds of $\sum_{j=} 1^n \bar{F}_j(q_\chi^j)$. The first is simply the first constraint in (6), $\sum_{j=} 1^n \bar{F}_j(q_\chi^j) \leq \chi$. The second follows from the second set of constraints in (6),

$$\begin{aligned}
\sum_{i=1}^{m} x_0^i \geq \sum_{i=1}^{m}\left(\lambda T \sum_{j=1}^{n} A_{ij}\bar{F}_j\left(q_\chi^j\right)\right)
&= \lambda T \sum_{j=1}^{n} \bar{F}_j\left(q_\chi^j\right)\left(\sum_{i=1}^{m} A_{ij}\right) \geq \lambda T r \sum_{j=1}^{n} \bar{F}_j\left(q_\chi^j\right).
\end{aligned}$$

These two upper bounds of $\sum_{j=} 1^n \bar{F}_j(q_\chi^j)$ jointly imply

$$\sum_{j=1}^{n} q_\chi^j \mathsf{E}\left[\sum_{\phi\in\mathcal{H}_T} \mathbb{1}\left\{\beta_\phi^j = 1\right\}\right] \geq \left(1 - \min\left\{\chi, \frac{\sum_{i=1}^m x_0^i}{\lambda T r}\right\}\right)\bar{J}^\chi.$$

For the second term in (9), we have

$$\begin{aligned}
\left(\sum_{j=1}^{n} q_\chi^j\right)\mathsf{E}\left[\sum_{\phi\in\mathcal{H}_T} \mathbb{1}\left\{\sum_{j=1}^{n} \beta_\phi^j \geq 2\right\}\right]
&= \left(\sum_{j=1}^{n} q_\chi^j\right)\mathsf{E}[|\mathcal{H}_T|]\mathsf{E}\left[\mathbb{1}\left\{\sum_{j=1}^{n} \beta_\phi^j \geq 2\right\}\right] \\
&= \lambda T\left(\sum_{j=1}^{n} q_\chi^j\right)\mathsf{E}\left[\mathbb{1}\left\{\sum_{j=1}^{n} \beta_\phi^j \geq 2\right\}\right] \\
&= 0,
\end{aligned}$$

where the second equality follows from Wald's identity and the third equality follows from the property that for any j, j' with $j \neq j'$, $\mathsf{P}(v_\phi^j - q_\chi^j = v_\phi^{j'} - q_\chi^{j'}) = 0$.

For the third term in (9), we first establish upper bounds to the mean and the variance of $\sum_{\phi\in\mathcal{H}_T}\sum_{j=} 1^n A_{ij}\mathbb{1}\{\beta_\phi^j = 1\}$. To ease notation, we denote this term by B^i. For the mean of B^i,

$$\begin{aligned}
\mathsf{E}\left[B^i\right] &= \mathsf{E}[|\mathcal{H}_T|]\mathsf{E}\left[\sum_{j=1}^{n} A_{ij}\mathbb{1}\left\{\beta_\phi^j = 1\right\}\right] \\
&= \lambda T \mathsf{E}\left[\sum_{j=1}^{n} A_{ij}\mathbb{1}\left\{\beta_\phi^j = 1\right\}\right] \\
&= \lambda T \mathsf{E}\left[\sum_{j=1}^{n} A_{ij}\mathbb{1}\left\{\beta_\phi^j = 1\right\}\middle| \sum_{j=1}^{n} A_{ij}\,\mathbb{1}\left\{\beta_\phi^j = 1\right\} \leq 1\right] \\
&\leq \lambda T\left(1 - \prod_{j=1}^{n} F_j\left(q_\chi^j\right)^{A_{ij}}\right) \\
&\leq \lambda T\left(\sum_{j=1}^{n} A_{ij}\bar{F}_j\left(q_\chi^j\right)\right) \\
&\leq x_0^i,
\end{aligned}$$

where the first equality follows from Wald's identity; the third equality follows from the property that for any j, j' with $j \neq j'$, $\mathsf{P}(v_\phi^j - q_\chi^j = v_\phi^{j'} - q_\chi^{j'}) = 0$; the second inequality follows from (8); the third inequality holds because $\boldsymbol{q}_\chi$ satisfies the inventory constraint in (6). For the variance of B^i,

$$\begin{aligned}
\mathsf{Var}\left[B^i\right] &= \mathsf{E}[|\mathcal{H}_T|]\mathsf{Var}\left[\sum_{j=1}^{n} A_{ij}\mathbb{1}\left\{\beta_\phi^j = 1\right\}\right] \\
&\quad + \left(\mathsf{E}\left[\sum_{j=1}^{n} A_{ij}\mathbb{1}\left\{\beta_\phi^j = 1\right\}\right]\right)^2 \mathsf{Var}[|\mathcal{H}_T|] \\
&= \mathsf{E}[|\mathcal{H}_T|]\mathsf{Var}\left[\sum_{j=1}^{n} A_{ij}\mathbb{1}\left\{\beta_\phi^j = 1\right\}\right| \\
&\quad \left.\times \sum_{j=1}^{n} A_{ij}\mathbb{1}\left\{\beta_\phi^j = 1\right\} \leq 1\right] \\
&\quad + \left(\mathsf{E}\left[\sum_{j=1}^{n} A_{ij}\mathbb{1}\left\{\beta_\phi^j = 1\right\}\middle| \sum_{j=1}^{n} A_{ij}\right.\right. \\
&\quad \left.\left.\times \mathbb{1}\left\{\beta_\phi^j = 1\right\} \leq 1\right]\right)^2 \mathsf{Var}[|\mathcal{H}_T|] \\
&\leq 2\lambda T,
\end{aligned}$$

where the first equality follows from the property that if N is a random variable and $X_1, \ldots, X_N$ are independent and identically distributed (i.i.d.), then $\mathsf{Var}[\sum_{n=} 1^N X_n] = \mathsf{E}[N]\mathsf{Var}[X_n] + (\mathsf{E}[X_n])^2\mathsf{Var}[N]$. The second equality follows from $\mathsf{P}(v_\phi^j - q_\chi^j = v_\phi^{j'} - p_\chi^{j'}) = 0$, for any j, j' with

$j \neq j'$. The inequality follows from $\mathsf{E}[|\mathcal{H}_T|] = \mathsf{Var}[|\mathcal{H}_T|] = \lambda T$ and the condition $\sum_{j=1}^{n} A_{ij} \mathbb{1} \times \{\beta_\phi^j = 1\} \in [0,1]$. Therefore, we have

$$\mathsf{E}\left[(B^i - x_0^i)^+\right] \leq \frac{1}{2}\left(\sqrt{\mathsf{Var}[B^i] + (x_0^i - \mathsf{E}[B^i])^2} - (x_0^i - \mathsf{E}[B^i])\right) \leq \sqrt{\frac{\lambda T}{2}},$$

where the first inequality follows from equation (18) in Gallego and Van Ryzin (1994); the second inequality follows the bounds $\mathsf{E}[B^i] \leq x_0^i$ and $\mathsf{Var}[B^i] \leq 2\lambda T$. Using this property, we can now bound the third term in (9)

$$\left(\max_j q_\chi^j\right) \sum_{i=1}^{m} \mathsf{E}\left[\left(\sum_{\phi \in \mathcal{H}_T} \sum_{j=1}^{n} A_{ij} \mathbb{1}\left\{\beta_\phi^j = 1\right\} - x_0^i\right)^+\right] \leq \left(\max_j q_\chi^j\right) m \sqrt{\frac{\lambda T}{2}}.$$

Using (7), (9), and the bounds we derived on the three terms of the latter, we conclude that

$$\frac{J^{\bar{\pi}_\chi}}{J^\star} \geq \frac{\chi J^{\bar{\pi}_\chi}}{\bar{J}^\chi} \geq \chi\left(1 - \min\left\{\chi, \frac{\sum_{i=1}^{m} x_0^i}{\lambda T r}\right\}\right) - \frac{\max_j q_\chi^j}{\sum_{j=1}^{n} q_\chi^j \bar{F}_j\left(q_\chi^j\right)} \frac{m\chi}{\sqrt{2\lambda T}}. \quad \blacksquare$$

Corollary 1 (Asymptotic Performance Guarantee)**.** *For the static pricing policy* $\bar{\pi}_\chi$*,*

$$\frac{J^{\bar{\pi}_\chi}\left(\lambda^{(N)}, \boldsymbol{x}_0^{(N)}\right)}{J^\star\left(\lambda^{(N)}, \boldsymbol{x}_0^{(N)}\right)} \geq \frac{1}{4} - O(N^{-1/2}).$$

Furthermore, if $\sum_{i=} 1^m x_0^i \lambda T r \leq \frac{3}{4}$*, we have*

$$\frac{J^{\bar{\pi}_\chi}\left(\lambda^{(N)}, \boldsymbol{x}_0^{(N)}\right)}{J^\star\left(\lambda^{(N)}, \boldsymbol{x}_0^{(N)}\right)} \geq 1 - \frac{\sum_{i=1}^{m} x_0^i}{\lambda T r} - O(N^{-1/2}).$$

The first part of Corollary 1 provides an asymptotic constant factor guarantee for static pricing performance, namely 1/4. The second part is relevant for problem instances in which available inventory is low relative to demand. In particular, if we interpret $\frac{\sum_{i=1}^{m} x_0^i}{r}$ as the *maximum* number of customers that available inventory can serve, Corollary 1 says that if that number is 3/4 of the number of expected customers (λT) or less, then we obtain a stronger bound. Namely, we then get an asymptotic guarantee that is equal to $1 - \frac{\sum_{i=1}^{m} x_0^i}{\lambda T r}$, that is, loosely speaking, the ratio of unserved customers to total number of customers. As inventory availability is reduced, more customers tend to be unserved, and performance of static pricing becomes closer to being optimal.

Finally, we note that the guarantee we provide here for static pricing is worse than that of Chen et al. (2018). We hypothesize the following two reasons: first, we deal with customers who strategize both on when and what to buy, instead of only on when. Dealing with *more strategic* customers could then undermine the efficacy of static pricing. Second, herein we deal with an intractable multidimensional mechanism design problem. Consequently, we have to use a decomposition approach, which, being approximate, could also introduce some loss.

4.2. Numerical Performance Analysis

Our theoretical treatment yielded a uniform guarantee that static pricing is capable of capturing a certain fraction of revenues relative to dynamic pricing in our model. As is often the case with guarantees of this kind, we expect, in realistic settings, the revenues captured by static pricing to exceed what is prescribed by the guarantee. This is because the guarantee, being applied to all possible problem instances, tends to be driven by worst-case instances.

We conduct numerical studies that enable us to quantify how well static pricing policies perform under realistic parameter choices. In particular, we generate multiple problem instances, for which we simulate the selling process and record the DM's revenues when using a judiciously chosen static pricing policy. By comparing the average recorded revenues with one of the upper bounds we derived, we find static pricing policies capable of capturing *at least* 75%–90% of the optimal revenues. That the simplest pricing policy captures such a high percentage of our bound consistently across all experiments provides some evidence about the bound's practical relevance and also supports our claim about the robustness of static prices across parameter regimes.

Furthermore, considering a wide range of parameter values enables us to conduct a sensitivity analysis. We find that static pricing policies tend to perform better for lower load factors, which is consistent with Corollary 1, and higher product proliferation. Their performance remains almost constant as product complexity, product differentiation and market scale vary. Details on the experimental setup and the results are included in the e-companion.

5. Concluding Remarks

In this paper, we made a first step toward connecting two very broad streams of literature within revenue management: papers dealing with forward-looking customers and papers dealing with customer product choice. Today's electronic commerce landscape facilitates customers' strategic responses to firms' pricing policies in choosing *when to buy* and *what to buy* (often among a plethora of alternatives). We expect, therefore, research in revenue management that accounts for both these important facets of customer behavior to become increasingly more relevant and popular. Herein, we started off from a canonical network revenue management model and extended it to allow customers to strategize on when and what to buy, guided by heterogeneous product valuations and heterogeneous decay rates. After pointing out the unique technical challenges that merging forward-looking behavior with customer product choice represents, we presented a novel decomposition approach within dynamic mechanism design that enabled us to provide a first characterization of optimal revenues, namely an upper bound.

Our decomposition approach did not come without its limitations. First, it precluded us from modeling the interactions between products in an exact way. Instead, the fluid customer streams were decoupled midway in our analysis. This approximation could have introduced some loss in our performance guarantee. Second, the approach necessitated the assumption of independent customer valuations across products. How to circumvent these limitations represents interesting challenges and could help tighten the analysis. However, it will also require new approaches because coupling customer streams and/or allowing correlated valuations are likely to bring one back in front of the technical barriers of multidimensional mechanism design that our approach got round.

Our upper bound analysis could form the basis for subsequent research, as it may serve as a tool for benchmarking candidate (dynamic) pricing policies. Indeed, we provided a performance analysis of static pricing policies, by comparing expected revenues within that class of policies with our upper bound. This enabled us to derive guarantees for the performance of static pricing, and an insightful numerical analysis.

Endnotes

[1] The website keepa.com, for example, provides retail customers seeking to purchase goods on the amazon.con platform with unprecedented detail in historical price and stock information for their desired product, alongside information on related alternative product offerings.

[2] To ease notation, we use bold font to denote vectors that suppress dummy product or resource superscript indices.

[3] We do not prove existence in general. We only demonstrate existence of such an equilibrium stopping rule for a specific class of pricing policies, and they exist trivially for static prices.

[4] We restrict ourselves to direct mechanisms.

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