Electronic Companion

A Proofs

Proof of Theorem 1. The theorem requires solving the following convex optimization problem in variables $\boldsymbol{x}_i, \boldsymbol{x}_i^+, \boldsymbol{x}_i^-, \bar{\tau}_{ij}$ and $\bar{\tau}_i$:

$$
\begin{aligned}
\text{maximize}\quad & f\big(u_1(\boldsymbol{x}_1) - \bar{\tau}_1, \ldots, u_n(\boldsymbol{x}_n) - \bar{\tau}_n\big) \\
\text{subject to}\quad & \boldsymbol{x}_i \in \mathcal{C}_i, \quad \forall i \in \mathcal{I} \\
& \boldsymbol{x}_i = \boldsymbol{x}_i^+ - \boldsymbol{x}_i^-, \quad \forall i \in \mathcal{I} \\
& \boldsymbol{x}_i^+, \boldsymbol{x}_i^- \geq 0, \quad \forall i \in \mathcal{I} \\
& \bar{\tau}_i = \sum_{j \in \mathcal{J}} \bar{\tau}_{ij}, \quad \forall i \in \mathcal{I} \\
\text{(P)} \qquad & \mathbb{E}\big[\tilde{t}_j(x_{ij}^+, x_{ij}^-, \boldsymbol{\xi})\big] \leq \bar{\tau}_{ij}, \quad \forall i \in \mathcal{I},\, j \in \mathcal{J} \\
& \mathbb{E}\left[\tilde{t}_j\left(\sum_{a \in \mathcal{I}\setminus\{i\}} x_{aj}^+, \sum_{a \in \mathcal{I}\setminus\{i\}} x_{aj}^-, \boldsymbol{\xi}\right)\right] \leq \sum_{a \in \mathcal{I}\setminus\{i\}} \bar{\tau}_{aj}, \quad \forall i \in \mathcal{I},\, j \in \mathcal{J} \\
& \mathbb{E}\left[\tilde{t}_j\left(\sum_{a \in \mathcal{I}} x_{aj}^+, \sum_{a \in \mathcal{I}} x_{aj}^-, \boldsymbol{\xi}\right)\right] \leq \sum_{a \in \mathcal{I}} \bar{\tau}_{aj}, \quad \forall j \in \mathcal{J} \qquad \text{(23a)} \\
& u_i(\boldsymbol{x}_i) - \bar{\tau}_i \geq U_i^{\text{IND}}, \quad \forall i \in \mathcal{I},
\end{aligned}
$$

To understand the origin of problem (P), consider first a problem ($\widehat{\text{SP}}$), obtained from (SP) by replacing the a.s. constraints (17c) with the expectation constraints

$$
\mathbb{E}\left[\tilde{t}_j\left(\sum_{a \in \mathcal{I}} x_{aj}^+, \sum_{a \in \mathcal{I}} x_{aj}^-, \boldsymbol{\xi}\right)\right] \leq \sum_{a \in \mathcal{I}} \mathbb{E}\big[\tau_{aj}(\tilde{\boldsymbol{Z}})\big],\ \forall j \in \mathcal{J}.
$$

Clearly, the optimal value in ($\widehat{\text{SP}}$) is at least that in (SP), since any decisions that are feasible in (SP) remain feasible in ($\widehat{\text{SP}}$). Furthermore, note that in ($\widehat{\text{SP}}$), the policies τ_{ij} and τ_i only affect the objective and constraints through their expected values. By replacing these expected values with the static decision variables $\bar{\tau}_{ij}$ and $\bar{\tau}_i$, respectively, we arrive at problem (P). This reasoning also shows that the optimal value in (P) is at least that in (SP). Since, at optimality, the constraints (23a) always hold as equalities in (P), it can be readily checked that $\boldsymbol{x}_i^\star$, $(\boldsymbol{x}_i^\star)^+$, $(\boldsymbol{x}_i^\star)^-$ and the choice in (18) result in feasible decisions in (SP), which also yield the same objective as the optimal value of (P). Therefore, these decisions must be optimal for (SP). □

Proof of Theorem 2. First, note that any feasible solution in (15) results in a feasible solution in (19), by simply projecting out the variables τ_{ij}, i.e., by considering $\tau_i = \sum_{j \in \mathcal{J}} \tau_{ij}$, $\forall\, i \in \mathcal{I}$ in (19). Therefore, the feasible set of (19) contains the corresponding one in (15).

To prove the reverse inclusion, consider any feasible solution $\{\boldsymbol{x}_i, \boldsymbol{x}_i^+, \boldsymbol{x}_i^-, \tau_i\}_{i \in \mathcal{I}}$ in (19). Extending this into a feasible solution for (15) is equivalent to finding a set of τ_{ij} variables satisfying the following constraints:

$$
(*)\quad
\begin{aligned}
&\sum_{j\in\mathcal{J}} \tau_{ij} = \tau_i, \quad \forall i \in \mathcal{I}\\
&\quad \tau_{ij} \geq t_j(x_{ij}^+, x_{ij}^-), \quad \forall i \in \mathcal{I},\, j\in\mathcal{J}\\
&\quad \tau_{ij} \leq t_j\Big(\sum_{a\in\mathcal{I}} x_{aj}^+, \sum_{a\in\mathcal{I}} x_{aj}^-\Big) - t_j\Big(\sum_{a\in\mathcal{I}\setminus\{i\}} x_{aj}^+, \sum_{a\in\mathcal{I}\setminus\{i\}} x_{aj}^-\Big) \quad \forall i \in \mathcal{I},\, j\in\mathcal{J}\\
&\sum_{a\in\mathcal{I}} \tau_{aj} \geq t_j\Big(\sum_{a\in\mathcal{I}} x_{aj}^+, \sum_{a\in\mathcal{I}} x_{aj}^-\Big), \quad \forall j\in\mathcal{J}.
\end{aligned}
\tag{24}
$$

In writing this system, we imposed the externality constraints $(*)$ using the original expressions in (12), rather than the equivalent conditions in (15). It can be readily seen that this change is without loss of generality, as the feasible set is not altered.

To simplify this system, let us define the following new variables and parameters:

$$\Delta_i \overset{\text{def}}{=} \tau_i - \sum_{j\in\mathcal{J}} t_j(x_{ij}^+, x_{ij}^-), \quad \forall\, i \in \mathcal{I} \tag{25a}$$

$$\varepsilon_{ij} \overset{\text{def}}{=} t_j\Big(\sum_{a\in\mathcal{I}} x_{aj}^+, \sum_{a\in\mathcal{I}} x_{aj}^-\Big) - t_j\Big(\sum_{a\in\mathcal{I}\setminus\{i\}} x_{aj}^+, \sum_{a\in\mathcal{I}\setminus\{i\}} x_{aj}^-\Big) - t_j(x_{ij}^+, x_{ij}^-), \quad \forall\, i \in \mathcal{I},\, j\in\mathcal{J} \tag{25b}$$

$$s_j \overset{\text{def}}{=} t_j\Big(\sum_{a\in\mathcal{I}} x_{aj}^+, \sum_{a\in\mathcal{I}} x_{aj}^-\Big) - \sum_{a\in\mathcal{I}} t_j(x_{aj}^+, x_{aj}^-), \quad \forall\, j\in\mathcal{J} \tag{25c}$$

$$q_{ij} \overset{\text{def}}{=} \tau_{ij} - t_j(x_{ij}^+, x_{ij}^-), \quad \forall\, i \in \mathcal{I},\, j\in\mathcal{J}. \tag{25d}$$

Note that $\Delta_i \geq 0$ since τ_i is feasible in (19). We also claim that $\varepsilon_{ij} \geq 0$. To see this, recall that, by our standing assumption in Section 2, the functions $t_j : \mathbb{R}_+^2 \to \mathbb{R}$ are jointly convex and component-wise increasing. Therefore, t_j must exhibit increasing differences on the set $\mathbb{R}_+^2$, i.e., t_j is increasing in the first argument when the second is fixed, and vice-versa. This, in turn, implies that t_j are supermodular on $\mathbb{R}_+^2$ (see, e.g., Corollary 2.6.1 in Topkis (1998)), so that

$$t_j(\boldsymbol{y}+\boldsymbol{\delta}) - t_j(\boldsymbol{y}) \geq t_j(\boldsymbol{x}+\boldsymbol{\delta}) - t_j(\boldsymbol{x}), \quad \forall\, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\delta} \in \mathbb{R}_+^2 \text{ such that } \boldsymbol{x} \leq \boldsymbol{y}. \tag{26}$$

Applying this to (25b) with $\boldsymbol{x} = \boldsymbol{0}$ and using the fact that $t_j(\boldsymbol{0}) = 0$ then readily yields that $\varepsilon_{ij} \geq 0$.

Returning to our original problem, note that finding a set of τ_{ij} feasible in (24) is then equivalent to the following linear program with variable $\boldsymbol{q}$ being feasible:

$$
\begin{aligned}
\text{minimize}\quad & 0\\
\text{subject to}\quad & \sum_{j\in\mathcal{J}} q_{ij} = \Delta_i,\ \forall\, i \in \mathcal{I} && \Leftarrow -\lambda_i\\
& q_{ij} \leq \varepsilon_{ij}, \forall\, i \in \mathcal{I},\, j\in\mathcal{J} && \Leftarrow -\eta_{ij}\\
& \sum_{i\in\mathcal{I}} q_{ij} \geq s_j, \forall\, j\in\mathcal{J} && \Leftarrow \mu_i\\
& \boldsymbol{q} \geq \boldsymbol{0}.
\end{aligned}
\tag{27}
$$

With the choice of dual variables $\boldsymbol{\lambda}$, $\boldsymbol{\eta}$, $\boldsymbol{\mu}$ as indicated above, the dual to program (27) becomes

$$\begin{aligned} \text{maximize} \quad & \sum_{j\in\mathcal{J}} \mu_j s_j - \sum_{i\in\mathcal{I}} \lambda_i \Delta_i - \sum_{i\in\mathcal{I}, j\in\mathcal{J}} \eta_{ij}\varepsilon_{ij} \\ \text{subject to} \quad & \mu_j - \lambda_i - \eta_{ij} \le 0, \, \forall\, i \in \mathcal{I},\, j \in \mathcal{J} \\ & \boldsymbol{\eta}, \boldsymbol{\mu} \ge \mathbf{0}. \end{aligned} \tag{28}$$

Since $\varepsilon_{ij} \ge 0$, it can be readily seen that in any optimal solution to the dual, we have $\eta_{ij} = (\mu_j - \lambda_i)^+$. The dual therefore simplifies to an (unconstrained) optimization over $\boldsymbol{\lambda}$ and $\boldsymbol{\mu} \ge \mathbf{0}$. In this context, note that feasible decisions τ_{ij} exist in (24) if and only if, for any $\boldsymbol{\lambda}$, we have

$$\max_{\boldsymbol{\mu}\ge\mathbf{0}} \Big[\sum_{j\in\mathcal{J}} \mu_j s_j - \sum_{i\in\mathcal{I}, j\in\mathcal{J}} (\mu_j - \lambda_i)^+ \varepsilon_{ij} \Big] \le 0.$$

In the above problem, if $s_j \le \sum_{i\in\mathcal{I}} \varepsilon_{ij}$, the optimal choice is to always set $\mu_j = \lambda_i$. Otherwise, by taking $\mu_j \to \infty$, the optimal value can be made arbitrarily large. Therefore, the optimal value in the problem above is (at most) zero if and only if $s_j \le \sum_{i\in\mathcal{I}} \varepsilon_{ij}$, $\forall\, j \in \mathcal{J}$. By using (25b) and (25c) to express these, we arrive at the following set of conditions:

$$\sum_{i\in\mathcal{I}} t_j\Big(\sum_{a\in\mathcal{I}\setminus\{i\}} x^+_{aj}, \sum_{a\in\mathcal{I}\setminus\{i\}} x^-_{aj} \Big) \le (n-1)\cdot t_j\Big(\sum_{a\in\mathcal{I}} x^+_{aj}, \sum_{a\in\mathcal{I}} x^-_{aj} \Big),\ \forall\, j \in \mathcal{J}. \tag{29}$$

These conditions, however, are always true, due to the following reasoning:

$$\begin{aligned} t_j\Big(\sum_{a\in\mathcal{I}} x^+_{aj}, \sum_{a\in\mathcal{I}} x^-_{aj}\Big) &= \sum_{i=2}^{n} \Bigg[t_j\Big(\sum_{a=1}^{i} x^+_{aj}, \sum_{a=1}^{i} x^-_{aj}\Big) - t_j\Big(\sum_{a=1}^{i-1} x^+_{aj}, \sum_{a=1}^{i-1} x^-_{aj}\Big) \Bigg] + t_j(x^+_{1j}, x^-_{1j}) \\ &\le \sum_{i=1}^{n} \Bigg[t_j\Big(\sum_{a\in\mathcal{I}} x^+_{aj}, \sum_{a\in\mathcal{I}} x^-_{aj}\Big) - t_j\Big(\sum_{a\in\mathcal{I}\setminus\{i\}} x^+_{aj}, \sum_{a\in\mathcal{I}\setminus\{i\}} x^-_{aj}\Big) \Bigg] \\ &= n\cdot t_j\Big(\sum_{a\in\mathcal{I}} x^+_{aj}, \sum_{a\in\mathcal{I}} x^-_{aj}\Big) - \sum_{i\in\mathcal{I}} t_j\Big(\sum_{a\in\mathcal{I}\setminus\{i\}} x^+_{aj}, \sum_{a\in\mathcal{I}\setminus\{i\}} x^-_{aj}\Big). \end{aligned}$$

The first equality above comes from telescoping the sum, and the inequality is a direct result of applying (26) to every term in the summation over i. □

B Extensions

B.1 Cross-trading of Assets

The model of trading introduced in Section 2 explicitly forbade the possibility of crossing trades, i.e., the practice of offsetting buy and sell orders of separate clients for the same asset "in-house," without recording the trade on the exchange. While cross-trading is outlawed at most exchanges, it has been traditionally permitted under rule 206(3)-2 of the Advisers Act (Securities and Exchange Commission) under selective circumstances. As of 2008, the US Department of Labor also finalized regulations that allowed cross-trading for retirement plans in excess of \$100M (see (U.S. Department of Labor 2008), which amended section

408(b)(19)(H) of the Employee Retirement Income Security Act).

Furthermore, note that the market impact model in Section 2 could potentially overestimate the actual market impact costs in case all orders are sent to the open market. In practice, buys and sells offset each other to some extent, though less so than if an internal cross of the trades is made.

With this motivation, we now argue that our framework, and particularly the main formulation (15), extend to the case where cross-trading is allowed. Recall that the trading model we have adopted thus far (introduced in Section 2), effectively forbade cross-trading by assuming that the (aggregate) buy and sell orders are submitted for execution without offsetting. More formally, let

$$z_j^+ \stackrel{\text{def}}{=} \sum_{a \in \mathcal{I}} x_{aj}^+, \quad z_j^- \stackrel{\text{def}}{=} \sum_{a \in \mathcal{I}} x_{aj}^-$$

denote the total buy and sell orders in the jth asset, respectively. Under the trading model discussed thus far, these aggregate orders are submitted for execution and the associated market impact costs are $t(z_j^+, z_j^-)$ (see equation (2)). When cross-trading is allowed, the manager first nets a buy and sell order for the same security in-house, and then places a single market order for the remainder of the bigger trade. The expression in (2) for the total market impact costs then becomes

$$\sum_{j \in \mathcal{J}} \Big[\, t_j\big((z_j^+ - z_j^-)^+, \, 0\big) + t_j\big(0, \, (z_j^- - z_j^+)^+\big) \, \Big]. \tag{30}$$

The first term of the summands above corresponds to the market impact cost of a net buy order for the jth asset, while the second corresponds to a net sell order. Effectively, the cost separates into buy and sell impact costs, since the manager never places both a buy and a sell order for the same security at the same time.[16] Note also that since $\max(\cdot)$ is a convex function, $t_j\big((z_j^+ - z_j^-)^+, \, 0\big)$ and $t_j\big(0, \, (z_j^- - z_j^+)^+\big)$ are convex functions of z_j^+ and z_j^- (Boyd and Vandenberghe 2004).

We use the same variables τ_{ij} to denote the amounts charged to the ith account for trading activity in the jth asset (for all $i \in \mathcal{I}$ and $j \in \mathcal{J}$). Also let $\boldsymbol{\tau} \in \mathbb{R}^{mn}$ be the vector containing all these values, and $\tau_i \stackrel{\text{def}}{=} \sum_{j \in \mathcal{J}} \tau_{ij}$, $\forall i \in \mathcal{I}$, denote the total amount charged to the ith account.

We now argue that the constraints (11)-(13) are directly applicable to the present model. By using expression (30) for the market impact costs, the former constraints can be written as follows:

$$(11) \Leftrightarrow t_j\big((x_{ij}^+ - z_j^-)^+, \, 0\big) + t_j\big(0, \, (x_{ij}^- - z_j^+)^+\big) \le \tau_{ij}, \qquad \forall i \in \mathcal{I}, \, j \in \mathcal{J} \tag{31a}$$

$$\begin{aligned}(12) \Leftrightarrow \tau_{ij} \le & \Big[t_j\big((z_j^+ - z_j^-)^+, \, 0\big) - t_j\big((z_j^+ - z_j^- - x_{ij}^+)^+, \, 0\big) \Big] \\ & + \Big[t_j\big(0, \, (z_j^- - z_j^+)^+\big) - t_j\big(0, \, (z_j^- - z_j^+ - x_{ij}^-)^+\big) \Big], \qquad \forall i \in \mathcal{I}, \, j \in \mathcal{J}\end{aligned} \tag{31b}$$

$$(13) \Leftrightarrow \sum_{a \in \mathcal{I}} \tau_{aj} = t_j\big((z_j^+ - z_j^-)^+, \, 0\big) + t_j\big(0, \, (z_j^- - z_j^+)^+\big), \qquad \forall j \in \mathcal{J}. \tag{31c}$$

Constraint (31a) reflects that the amount charged to an account for trading a particular quantity in an asset is at least the market impact cost of trading only *what remains of that quantity*, after first netting against opposing trades by other accounts. Note that this is a conservative lower bound, in that it corresponds to

[16] Note that although the summands in (30) can be rewritten as $t\big((z_j^+ - z_j^-)^+, \, (z_j^- - z_j^+)^+\big)$, we keep the formulation in (30) for the simplicity of exposition.

the most favorable treatment the ith account could hope for (since it is the first one to obtain the netting, before any other accounts).

Constraint (31b) places an upper bound on τ_{ij} that corresponds to the least favorable treatment of the ith portfolio, where its activity is the last one to be accounted for. The constraint can be understood by separately interpreting the terms in the brackets on the right-hand side. The first term reflects the externality imposed by an account i seeking to buy the jth security on to the aggregate market impact cost of buying the jth security. Similarly, the second bracket is the externality imposed by selling on the total market impact cost of selling.

Constraint (31c) simply states that the aggregate charge to all the accounts for trades in a particular asset equals the aggregate market impact cost for that asset.

In this context, the manager can determine the trades $\boldsymbol{x}$ and the split of market impact costs $\boldsymbol{\tau}$ by solving the following optimization problem, in variables $\boldsymbol{x}$, $\boldsymbol{x}^+$, $\boldsymbol{x}^-$, $\boldsymbol{z}^+$, $\boldsymbol{z}^-$ and $\boldsymbol{\tau}$:

$$
\begin{aligned}
\text{maximize} \quad & f(u_1(\boldsymbol{x}_1) - \tau_1, \ldots, u_n(\boldsymbol{x}_n) - \tau_n) \\
\text{subject to} \quad & \boldsymbol{x}_i \in \mathcal{C}_i, \quad \forall i \in \mathcal{I} \\
& \boldsymbol{x}_i = \boldsymbol{x}_i^+ - \boldsymbol{x}_i^-, \quad \forall i \in \mathcal{I} \\
& \boldsymbol{x}_i^+, \boldsymbol{x}_i^- \geq 0, \quad \forall i \in \mathcal{I} \\
& \boldsymbol{z}^+ = \sum_{a \in \mathcal{I}} \boldsymbol{x}_a^+ \\
& \boldsymbol{z}^- = \sum_{a \in \mathcal{I}} \boldsymbol{x}_a^- \\
& \tau_i = \sum_{j \in \mathcal{J}} \tau_{ij}, \quad \forall i \in \mathcal{I} \\
& t_j\big((x_{ij}^+ - z_j^-)^+, 0\big) + t_j\big(0, (x_{ij}^- - z_j^+)^+\big) \leq \tau_{ij}, \quad \forall i \in \mathcal{I}, j \in \mathcal{J} \\
& t_j\big((z_j^+ - z_j^- - x_{ij}^+)^+, 0\big) + t_j\big(0, (z_j^- - z_j^+ - x_{ij}^-)^+\big) \leq \sum_{a \in \mathcal{I} \setminus \{i\}} \tau_{aj}, \quad \forall i \in \mathcal{I}, j \in \mathcal{J} \\
& t_j\big((z_j^+ - z_j^-)^+, 0\big) + t_j\big(0, (z_j^- - z_j^+)^+\big) \leq \sum_{a \in \mathcal{I}} \tau_{aj}, \quad \forall j \in \mathcal{J} \\
& u_i(\boldsymbol{x}_i) - \tau_i \geq U_i^{\text{IND}}, \quad \forall i \in \mathcal{I}.
\end{aligned}
$$

As noted, the problem remains convex since all the functions t_j are convex in the x_{ij}^+ and x_{ij}^- variables.

B.2 Models with Permanent Price Impact

The formulations we analyzed so far involved general transaction cost models that depended entirely on the amounts bought or sold over a particular trading period. Despite their generality, such models are unable to capture permanent price impact effects, which frequently occur when large amounts are traded (see Carlin et al. (2007)). This is because losses (or gains) due to permanent price impact also depend on the holdings that a portfolio maintains, and not just on trading activities.

In the context of MPO, permanent price impact effects introduce further interactions between the multiple portfolios that can be potentially problematic. For instance, liquidation of a position in a particular asset held by one portfolio might permanently reduce the price of the asset, thus permanently devaluing long positions that other portfolios under management hold. Such interactions need to be properly accounted for in order for the manager to jointly optimize the portfolios, and fairly distribute costs and gains.

In this section, we show how our MPO model can be extended to capture the permanent price impact of trading. For illustration purposes, we provide our analysis in the context of the portfolio liquidation problem studied in Brown et al. (2010), which we extend to a multiportfolio setting. For simplicity of exposition and to ease notation, we limit attention to the case of $n = 2$ portfolios under management; the extension for $n > 2$ is straightforward. We first discuss the price and trading model, then formulate the MPO counterpart and draw our conclusions.

Price and Trading Model. For completeness, we present only the basic elements of the model here, and refer the reader to Brown et al. (2010) for details and justifications of underlying assumptions.

A financial adviser managing two distinct portfolios of m assets wishes to (partly) liquidate their holdings in continuous time over a finite horizon. Let $\boldsymbol{w}_i(0) \in \mathbb{R}^m_+$ be the initial holdings of the ith portfolio and l_i its liabilities. At any time $t \in [0, T]$, $\boldsymbol{y}_i(t) \in \mathbb{R}^m_-$ is the rate at which the manager trades its assets.[17] Consequently, its holdings at time t are given by $\boldsymbol{w}_i(t) = \boldsymbol{w}_i(0) + \int_0^t \boldsymbol{y}_i(s)ds$.

The prices of the assets at time t are denoted by $\boldsymbol{p}(t) \in \mathbb{R}^m_+$, and are determined by:

$$\boldsymbol{p}(t) = \boldsymbol{q} + \Gamma(\boldsymbol{w}_1(t) + \boldsymbol{w}_2(t)) + \Lambda(\boldsymbol{y}_1(t) + \boldsymbol{y}_2(t)). \tag{33}$$

Here, $\boldsymbol{q} \in \mathbb{R}^m$ is an intercept, and $\Gamma \in \mathbb{R}^{m\times m}$, $\Lambda \in \mathbb{R}^{m\times m}$ are positive definite, diagonal matrices capturing the effects of the permanent and temporary price impact, respectively. We refer the reader to Brown et al. (2010) for a thorough discussion of this pricing equation.

We denote the cumulative trades of the ith portfolio by $\boldsymbol{x}_i = \boldsymbol{w}_i(T) - \boldsymbol{w}_i(0)$, the cumulative trades of the manager by $\boldsymbol{x} = \boldsymbol{x}_1 + \boldsymbol{x}_2$ and the trading rate by $\boldsymbol{y}(t) = \boldsymbol{y}_1(t) + \boldsymbol{y}_2(t)$. The cash that is generated by trading over the horizon is

$$\kappa = -\int_0^T \boldsymbol{p}(t)^T \boldsymbol{y}(t)dt.$$

It is easy to see that a constant trading rate $\boldsymbol{y}(t) = \frac{1}{T}\boldsymbol{x}$ maximizes the cash generated, which in that case is equal to

$$\kappa = -\boldsymbol{p}(0)^T \boldsymbol{x}_1 - \boldsymbol{p}(0)^T \boldsymbol{x}_2 - \boldsymbol{x}^T \left(\Lambda + \frac{1}{2}\Gamma\right) \boldsymbol{x}. \tag{34}$$

In the above expression, the first two terms correspond to the cash generated by the sales of the two portfolios' holdings. The third term corresponds to the total transaction costs due to price impact.

Under this setting, the value of the assets of the ith portfolio at the beginning of the horizon is $a_i(0) = \boldsymbol{p}(0)^T \boldsymbol{w}_i(0)$. At the end of the trading horizon, the value becomes

$$\begin{aligned} a_i(T) &= \boldsymbol{p}(T)^T \boldsymbol{w}_i(T) \\ &= a_i(0) + \boldsymbol{p}(0)^T \boldsymbol{x}_i + \boldsymbol{w}_i(0)^T \Gamma \boldsymbol{x} + \boldsymbol{x}_i^T \Gamma \boldsymbol{x}. \end{aligned} \tag{35}$$

That is, the change in asset value $a_i(T) - a_i(0)$ is equal to the value of the liquidated assets $\boldsymbol{x}_i$, priced at $\boldsymbol{p}(0)$, plus the devaluation of the assets of the ith portfolio due to permanent price impact. The devaluation, which is driven by the price impact $\Gamma \boldsymbol{x}$ according to (33) and is captured by the sum of the last two terms

[17] Due to regulatory restrictions, only selling is allowed; see also Moallemi and Sağlam (2012).

in the expression above, $\boldsymbol{w}_i(0)^T\Gamma\boldsymbol{x} + \boldsymbol{x}_i^T\Gamma\boldsymbol{x}$. The terms have a slightly different origin: the first corresponds to a devaluation of all initial holdings $\boldsymbol{w}_i(0)$ (and would be incurred by the ith account even if it did not trade, due to permanent impact from the other accounts' trades), while the second is in fact related to the trading activity $\boldsymbol{x}_i$.

MPO formulation. We now use our methodology to formulate the MPO problem in this setting. We aggregate the transaction costs and the devaluation effects that are *exclusively due to trading*, and allow the MPO formulation to decide how to allocate them among the two portfolios. These costs amount to

$$\underbrace{\boldsymbol{x}^T\left(\Lambda + \frac{1}{2}\Gamma\right)\boldsymbol{x}}_{\text{transaction costs}} - \underbrace{\boldsymbol{x}_1^T\Gamma\boldsymbol{x}}_{\text{devaluation for pf 1}} - \underbrace{\boldsymbol{x}_2^T\Gamma\boldsymbol{x}}_{\text{devaluation for pf 2}} = \boldsymbol{x}^T\left(\Lambda - \frac{1}{2}\Gamma\right)\boldsymbol{x}.$$

The transactions costs term is obtained from (34), whereas the devaluation terms (related to trading) are obtained from (35). As in Brown et al. (2010), we henceforth assume that $\Lambda - \frac{1}{2}\Gamma$ is positive semi-definite.

Let τ_1 and τ_2 be the associated split decision variables. The utility of the ith portfolio at the end of the horizon is equal to its equity, i.e., value of its assets, plus cash generated, minus its liabilities l_i. Aggregating all the terms, we get

$$u_i = a_i(0) + \boldsymbol{w}_i(0)^T\Gamma\boldsymbol{x} - l_i.$$

Note that the equity or utility above is not adjusted for transaction and devaluation costs due to trading, in order to mimic our base formulation from Sections 2-3. The MPO problem is then to optimize over the cost-adjusted net utilities $u_i - \tau_i$ (using the welfare function f), subject to particular liquidation constraints[18] that are captured by the trade feasibility sets $\mathcal{C}_i$, and with decision variables $\boldsymbol{x}$, $\boldsymbol{x}_i$, τ_i, and u_i:

$$\begin{aligned}
\text{maximize} \quad & f(u_1 - \tau_1, u_2 - \tau_2) \\
\text{subject to} \quad & u_i = a_i(0) + \boldsymbol{w}_i(0)^T\Gamma\boldsymbol{x} - l_i, \quad i = 1,2 \\
& \boldsymbol{x}_i \in \mathcal{C}_i, \quad i = 1,2 \\
& \boldsymbol{x} = \boldsymbol{x}_1 + \boldsymbol{x}_2 \\
& \boldsymbol{x}_i^T\left(\Lambda - \frac{1}{2}\Gamma\right)\boldsymbol{x}_i \leq \tau_i, \quad i = 1,2 \\
& \boldsymbol{x}^T\left(\Lambda - \frac{1}{2}\Gamma\right)\boldsymbol{x} \leq \tau_1 + \tau_2.
\end{aligned} \tag{36}$$

Discussion. We now make several remarks about formulation (36), and compare it with formulation (15). Firstly, one can easily include coordination benefits constraints alike (14), by suitably adapting the independent scheme for this optimal liquidation problem.

Secondly, note that the lower bounds on τ_i exactly reflect the costs the would have been incurred by the manager, had she executed only the trades of the respective portfolio. Thus, the associated constraints

[18] Examples of such constraints are the exposure of a portfolio to a particular sector, restrictions on the liquidated amounts, possible regulatory constraints that enforce selling only and no short positions, etc.

correspond to constraints (11-12). In particular, in the "best-case" scenario for it, portfolio 1 is charged

$$\begin{aligned}\tau_1 &= \boldsymbol{x}_1^T\left(\Lambda - \frac{1}{2}\Gamma\right)\boldsymbol{x}_1\\ &= \boldsymbol{x}_1^T\left(\Lambda + \frac{1}{2}\Gamma\right)\boldsymbol{x}_1 + \boldsymbol{x}_1^T\Gamma\boldsymbol{x}_2 - \boldsymbol{x}_1^T\Gamma\boldsymbol{x}.\end{aligned}$$

In the expression above, the last term corresponds to the asset devaluation due to trading, as per (35). The first two terms correspond to portfolio 1's "share" of the incurred transaction costs due to price impact from equation (34). Accordingly, under the same scenario, portfolio 2 is charged

$$\begin{aligned}\tau_2 &= \boldsymbol{x}^T\left(\Lambda - \frac{1}{2}\Gamma\right)\boldsymbol{x} - \boldsymbol{x}_1^T\left(\Lambda - \frac{1}{2}\Gamma\right)\boldsymbol{x}_1\\ &= \boldsymbol{x}_2^T\left(\Lambda + \frac{1}{2}\Gamma\right)\boldsymbol{x}_2 + 2\boldsymbol{x}_1^T\Lambda\boldsymbol{x}_2 - \boldsymbol{x}_2^T\Gamma\boldsymbol{x},\end{aligned}$$

where one can note a similar break-down of the costs as for portfolio 1.

Thirdly, note that in the presence of permanent price impact, the utility of a portfolio depends on the overall trading activity through the term $\boldsymbol{w}_i(0)^T\Gamma\boldsymbol{x}$, and not just on its own activity $\boldsymbol{x}_i$. As discussed above, this term corresponds to the devaluation of the holdings of the ith portfolio due to the overall trading activity. Thus, it is possible that the portfolio will incur losses in its equity even under no trading activity. However, note that a portfolio that is not trading would never be charged further related costs, that is, τ_i would be 0.

Finally, note that for $\Gamma = 0$, there is no permanent price impact. Then, the model we considered here is identical to the base model from Sections 2-3, where the (temporary) market impact cost function takes the form $\boldsymbol{x}^T\Lambda\boldsymbol{x}$.