Scaling with Fractional Calculus

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Normal vs anomalous relaxation

Maxwell-Debye relaxation

\[ \frac{d\varepsilon}{dt} = K_1 \nabla^2 \varepsilon(x,t) \]

\[ \varepsilon(x,t) = e^{ik \cdot x} \varepsilon(k,t) \]

\[ \varepsilon(k,t) = e^{-K_1 k^2 t} \]

Anomalous relaxation

\[ \frac{d\varepsilon}{dt} = K_\alpha \, \mathcal{D}_t^{1-\alpha} \nabla^2 \varepsilon(x,t) \]

\[ \varepsilon(k,t) = E_\alpha \left( -K_\alpha k^2 t^\alpha \right) \]

Fractional integration & derivation

\[ _0\mathcal{D}_t^{-\alpha} (W(x,t)) \equiv \frac{1}{\Gamma(\alpha)} \int_0^t \frac{W(x,t)}{(t-t')^{1-\alpha}} dt' \]

\[ _0\mathcal{D}_t^{1-\alpha} (W(x,t)) \equiv \frac{d}{dt} [ _0\mathcal{D}_t^{-\alpha} (W(x,t))] \]

Mittag-Leffler function

\[ E_\alpha (-z) \equiv \sum_{n=0}^{\infty} \frac{(-z)^n}{\Gamma(1+\alpha z)} \]

At short times, \[ E_\alpha \left( -K_\alpha k^2 t^\alpha \right) \sim \exp \left( -\frac{K_\alpha k^2 t^\alpha}{\Gamma(1+\alpha)} \right) \]

Stretched exponential / KWW

At long times, \[ E_\alpha \left( -K_\alpha k^2 t^\alpha \right) \sim \left( K_\alpha k^2 t^\alpha \Gamma(1-\alpha) \right)^{-1} \]

Inverse power-law / Nutting

Commutativity of the operator $0D_t^\alpha [[]]$

In general, NOT commutative

Explanation:

Let $A(\alpha)$ be the space of constant functions for the operator of order $\alpha$

Example:

$$0D_t^{1/2}(t^{-1/2}) = 0$$  \hspace{1cm} \text{$t^{-1/2}$ belongs to $A(1/2)$}

$$0D_t^{1/2}(C) = \frac{C}{\sqrt{\pi t}}$$  \hspace{1cm} \text{Constant functions don’t belong to $A(1/2)$}

Practical rule

Let $f(t)$ have a power series expansion

We have

$$0D_t^\alpha(f(t)) = 0D_t^{\mu+\nu}(f(t)) = 0D_t^\mu\left[0D_t^\nu(f(t))\right] = 0D_t^\nu\left[0D_t^\mu(f(t))\right]$$

if $f(t)$ doesn’t contain any function which is constant for the operators $0D_t^\nu$ & $0D_t^\mu$

*Short Introduction to fractional calculus*, Mauro Bologna

A. Helal & M.A. Fardin  Scaling and fractional calculus
Fractional rheological models: Zener model

### Standard Zener model

\[
\tau + \lambda \dot{\epsilon} = (G_m + G_e) \lambda \dot{\epsilon} + G_e \epsilon \\
G_0 = G_m + G_e \quad \lambda = \frac{\eta}{G_m}
\]

### Fractional Zener Model

\[
\tau(t) + \lambda^q \int_0^t D_t^q \tau(t) = (G_e + G_m) \lambda^q \int_0^t D_t^q \epsilon(t) + G_e \lambda^{q-\mu} \int_0^t D_t^{q-\mu} \epsilon(t)
\]

\[q, \mu \in (0, 1) \quad q \geq \mu\]

(Adapted from original article because of typos)

### Stress relaxation

\[
G(t) = G_0 E_{q,1}(- (t/\lambda)^q) + G_e (t/\lambda)^\mu E_{q,1+\mu}(- (t/\lambda)^q)
\]

\[E_{\alpha,\beta}(-z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{\Gamma(\beta + \alpha z)}\]

### Asymptotic behaviour

\[
G(t) \sim \frac{G_0(t/\lambda)^{-q}}{\Gamma(1-q)} + \frac{G_e(t/\lambda)^{\mu-q}}{\Gamma(1+\mu-q)}
\]

### Generalized Mittag-Leffler function

Experimental results & Cole-Cole plots

Experimental fitting for Carbon filled natural rubber

Carbon filled natural rubber (0, 30, 60 phr from bottom to top)

Oscillatory testing

\[
g_1 = \frac{G'(\omega) - G_e}{G_0 - G_e} = \frac{\tilde{\omega}^{q-\mu} \left( \cos(\pi(q - \mu)/2) + \tilde{\omega}^q \cos(\pi(\mu/2)) \right)}{1 + 2\tilde{\omega}^q \cos(q\pi/2) + \tilde{\omega}^{2q}}
\]

\[
g_2 = \frac{G''(\omega)}{G_0 - G_e} = \frac{\tilde{\omega}^{q-\mu} \left( \sin(\pi(q - \mu)/2) + \tilde{\omega}^q \sin(\pi(\mu/2)) \right)}{1 + 2\tilde{\omega}^q \cos(q\pi/2) + \tilde{\omega}^{2q}}
\]

Time scales & Deborah number I

Linear Viscoelasticity

\[ \tau(t) = \int_{-\infty}^{t} G(t - t') \dot{\gamma}(t') dt' \]

Maxwell-Debye relaxation

\[ G(t) = G_0 e^{-t/\lambda} \]
\[ \lambda = \frac{\int_{0}^{\infty} G(s) ds}{\int_{0}^{\infty} G(s) s ds} \]
\[ De = \frac{\lambda}{T_{obs}} \]

Anomalous relaxation

\[ G(t) = G_0 \left( \frac{t}{\lambda} \right)^{-\alpha} \]
\[ \lambda = \frac{\int_{0}^{\infty} G(s) ds}{\int_{0}^{\infty} G(s) s ds} \rightarrow \infty \]
\[ De = ? \]
Time scales & Deborah number II

Differential definition of the average relaxation time

\[ \lambda = -\frac{1}{[d \log G/ dt]} \]

Application

\[ G(t) = G_0 e^{-t/\lambda} \quad \lambda = -\frac{1}{[d \log G/ dt]} = \text{constant} \]
\[ G(t) = G_0 \left( \frac{t}{\lambda} \right)^{-\alpha} \quad \lambda = -\frac{1}{[d \log G/ dt]} = t / \alpha \]

Dynamic Deborah number

\[ De(T_{\text{obs}}) = \frac{\lambda}{T_{\text{obs}}} \quad \text{Maxwell-Debye} \]
\[ De(T_{\text{obs}}) = \frac{\lambda^\alpha}{\alpha T_{\text{obs}}^\alpha} \quad \text{KWW} \]
\[ De(T_{\text{obs}}) = \frac{1}{\alpha} \quad \text{Nutting} \]
Scaling & Buckingham-Pi Theorem

$\mathcal{S}$ Relevant parameters space

Normal situation

$[K] = [M]^a [L]^b [T]^c$

$\mathcal{S} \rightarrow \mathbb{Q}^3$

$K \rightarrow \{a, b, c\}$

$\mathcal{S}$ is a vector space over $\mathbb{Q}$

Anomalous situation

$\mathcal{S} \rightarrow \mathbb{R}^3$

$K \rightarrow \{a, b, c\}$

$\mathcal{S}$ is a vector space over $\mathbb{R}$

Example

$[K_1] = [M]^0 [L]^2 [T]^{-1} \quad m^2 / s$

$[K_\alpha] = [M]^0 [L]^2 [T]^{-\alpha} \quad m^2 / s^\alpha$

On physically similar systems; Illustrations of the use of dimensional equations, E. Buckingham, 1914
What are the relevant parameters?
Do they depend on rescaling (theory) /resampling (experiment)?

\[ k = M \cdot T^{-2} \]
\[ m = M \]
\[ l_0 = L \]
\[ \omega = T^{-1} \]
\[ \hbar = M \cdot L^2 \cdot T^{-1} \]

\[ \pi_1 = \omega^2 \frac{m}{k} \]
\[ \pi_2 = \frac{\hbar}{l_0^4 km} \]
\[ \omega = \sqrt{\frac{k}{m}} f \left( \frac{\hbar}{l_0^4 km} \to 0 \right) = \sqrt{\frac{k}{m}} \]
What are the relevant parameters?
Do they depend on rescaling (theory) / resampling (experiment) ?

Examples:
- Average size of a cluster?
- Density?
- Diffusion coefficient?
- Distribution in X, Y...

Note:
Even for a “linear system” finite size and time introduce non trivial coarse-graining
Avoiding the question: The tradition of “Linear” (and infinite) systems


Why do we love linear systems?

- Calculus (fr: Analyse): Taylor exp., approx. for “smooth” functions, solving equations...
- Mechanics: Model systems, spring, dashpot...
- Physics: Free systems, mode decomposition, superposition principle, input/output...
- Statistics: LLN, Central limit theorem, independent probability, Ergodicity...
- Philosophy: Traditional causality, traditional reductionism, Occam's razor...

Introduction to Nonlinear Physics, By Lui Lam (2003) 11
Rescaling a Gaussian system: introduction to coarse-graining

- Polymer in theta solvent / random walk...

\[ R \sim a N^{1/2} \]

\[ F[r] = \int_0^N ds \frac{1}{2a^2} \left( \frac{dx}{ds} \right)^2 \]

\[ F' = F \]

\[ F'[r] = \int_0^{rg^{-1/2}} ds \frac{1}{2a'^2} \left( \frac{dx}{ds} \right)^2 \]

→ The parameter \( a \) (monomer or mean free path) keeps its “bare” value
→ The coarse-grained and original statistics are the same

- Integer fractal dimension – pseudo classical space

Hausdorff dimension: \( \mathcal{H}(r) \sim r^{-D} \)

\[ D = \frac{\log g}{\log \sqrt{g}} = \frac{\log g}{1/2 \log g} = 2 \]
Rescaling a non Gaussian system: non trivial coarse-graining

- (more) realistic polymer chain / self avoiding random walk

\[ R \sim aN^\nu \]

\[ F[r] = \frac{1}{F} \int_0^N ds \left[ \frac{1}{2a^2} \left( \frac{dr}{ds} \right)^2 + \int_0^N ds \int_0^N ds' \nu \delta(r(s) - r(s')) \right] \]

\[ F \neq 'F \]

The parameter \( a \) and \( \nu \) are renormalized. They don't keep their bare value.

The coarse-grained and original statistics are the same only if \( F \) is renormalized.

- Non-integer fractal dimension – pure fractal space

Hausdorff dimension: \( D = \nu^{-1} = 1.56... \)

Scaling concepts in polymer physics, PGG (1979)
Renormalization: a powerful method to find relevant parameters

Quantities at fixed points have Levy stable distributions

Fixed-point=Self-similar, it maps to itself upon coarse-graining=“universality class”

The difficulties are in computing the flow, i.e. finding how $F$ is renormalized (diagrams)

The method is well established only when $F$ exists, i.e. at equilibrium (no $T$)

Extremely relevant in an empirical approach: measures involve coarse-graining

Scaling concepts in polymer physics, PGG (1979)
Lectures on phase transitions and the renormalization group, N. Goldenfeld (1993)
Renormalization group theory: its basis and formulation in statistical physics, Rev. Mod. Phys. 70, 653 (1998)
Geometrical interpretation of anomalous (spatial) dimensions

→ Geometric self-similarity and fractal (Chris—Fractal gels...)
→ Link between fractal and fractional calculus (1 to 1 correspondence still debated)
→ Finite (incomplete) fractals and asymptotic fixed points ("phases")
→ Some psychophysics roots too...

How to interpret fractional time?

→ Some out-of-equilibrium renormalization (surface growth process...)
→ Anomalous diffusion (Jason—diffusion on fractal network, trap models...)
→ Fractal representation of fractional elements
→ Link with temporal correlations

\[ T^\mathbb{Z} \text{ on } L^\mathbb{R} \rightarrow T^\mathbb{R} \]

Levy stable

Time correlation \( \rightarrow T^\mathbb{R} \)

No time correlation \( \rightarrow T^\mathbb{Z} \)

C. Friedrich et al. in Advances in the Flow and Rheology of Non-Newtonian Fluids (1999)
Non Newtonian time measure

Indeed, how do we measure time intervals? Only by observing some processes, which we consider as regularly repeated. G. Clemence wrote:

“The measurement of time is essentially a process of counting. Any recurring phenomenon whatever, the occurrences of which can be counted, is in fact a measure of time.”

Non Newtonian time measure: another insight to experiments

\[ \dot{J}(t) = \frac{dJ(t)}{dt} = \int P(\tau) \exp\left(-\frac{t}{\tau}\right) d\tau = B\Gamma(\alpha - 1)t^{\alpha-1} \quad P(\tau) = B\tau^{\alpha-2} \]

Geometric and physical interpretation of fractional integration and fractional differentiation, Igor Podlubny (2002)
M. Balland et al. *PRE* 74, 021911 (2006)
Fundamental question of Non Newtonian/Riemannian time measure

\[ P(\tau) d\tau \]

Figure 6: Homogeneous time axes.

Figure 7: Time slowing down.

<table>
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<th>uniform measure</th>
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Frame transformations?? (Arezoo)

There is a very famous example of frame transformation with non uniform measure...
Back to the Deborah number

Differential/local definition of the average relaxation time

\[ \dot{\lambda} = - \left[ \frac{d \log G}{dt} \right]^{-1} \]

Dynamic Deborah number

\[ \text{De}(T_{\text{obs}}) = \frac{\lambda}{T_{\text{obs}}} \]

Maxwell-Debye

\[ \text{De}(T_{\text{obs}}) = \lambda^\alpha \frac{T_{\text{obs}}}{\alpha} \]

KWW

\[ \text{De}(T_{\text{obs}}) = \frac{1}{\alpha} \]

Nutting

Local definition of the Deborah number, measures the stretching of time \[ P(\tau) d\tau \]
Keep the balance between empiricism and rationalism

→Don't get lost in lingo

→Watch for sloppiness

In a variety of contexts, physicists study complex, nonlinear models with many unknown or tunable parameters to explain experimental data. We explain why such systems so often are sloppy: the system behavior depends only on a few “stiff” combinations of the parameters and is unchanged as other “sloppy” parameter combinations vary by orders of magnitude.

http://www.lassp.cornell.edu/sethna/Sloppy/WhatAreSloppyModels.html