Instability of a cylinder of viscous liquid

Trush Majmudar
Summer Reading Group
Paper by Lord Rayleigh
Phil. Mag. XXXIV. pp 145-154, 1892

Lord Rayleigh

1842-1919

Electromagnetism

Fluid Dynamics

Promoting Science

done. Happy should I be if, through this visit of the Association, or by any words of mine, a larger measure of the youthful activity of the West could be drawn into this service. The work may be hard, and the discipline severe, but the interest never fails, and great is the privilege of achievement.

Theory of Sound

Discovery of Argon

Shape of Pebbles

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SCIENTIFIC PAPERS

BY

JOHN WILLIAM STRUTT,

BARON RAYLEIGH,

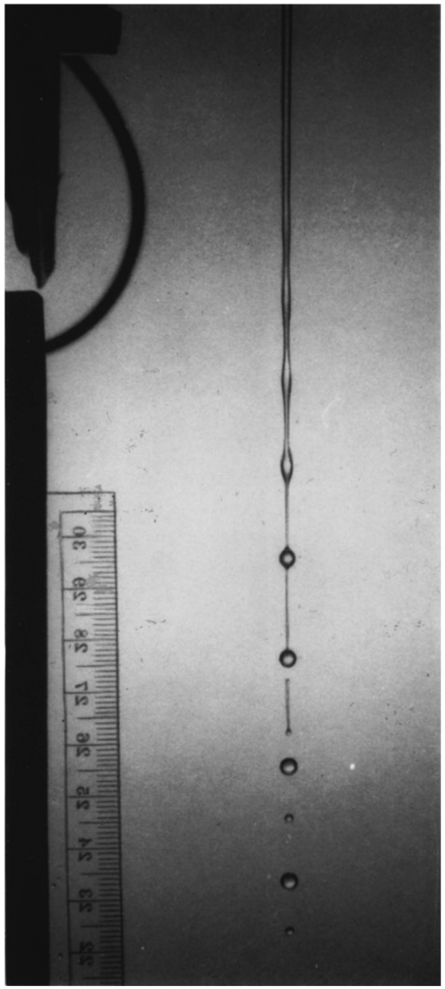
D.Sc., F.R.S.,

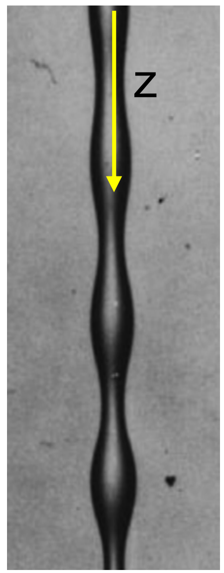
HONORARY FELLOW OF TRINITY COLLEGE, CAMBRIDGE,
PROFESSOR OF NATURAL PHILOSOPHY IN THE ROYAL INSTITUTION.

VOL. I.	VOL. II.	VOL. III.
1869—1881.	1881—1887.	1887—1892.

There's more! Scientific Papers by Stokes, Basset's Hydrodynamics….

Viscous Fluid Jet





- Long cylinder of viscous liquid
- Equilibrium surface deformed by perturbations:

$$r = a + \alpha \cos kz,$$

Inviscid Result:

- Stability of deformation depends on the value of *ka*
  - *ka > 1 :* Jet stable
  - *ka < 1 :* Jet unstable

=>

Wavelength of instability ($\lambda = 2\pi/k$) has to be greater than the circumference of the cylinder.

$$\lambda > 2\pi a$$

The unstable mode of wavelength λ, grows exponentially in time at a growth rate *q*.

(Donnelly and Glaberson,1966)

Rayleigh's intuition regarding viscous jets

which viscosity would predominate over inertia. Having in my mind some old experiments upon the behaviour of fine threads of treacle deposited upon paper, which slowly resolve themselves into drops having a very similar appearance to those obtained from a jet of water, I rather expected to find that under the influence of viscosity alone the mode of resolution would be nearly the same as under the influence of inertia alone. This anticipation proved to be wide of the mark, the result showing that under viscosity alone the value of λ for maximum instability would be very great. And a little

Inviscid result for a cylinder of radius *a*:

The most unstable mode corresponds to $\lambda = 4.51 \times 2\,a$

Rayleigh's notation

$$q^2 = \frac{T}{\rho a^3} \frac{ika\, J'_0(ika)}{J_0(ika)} (1 - k^2 a^2)$$

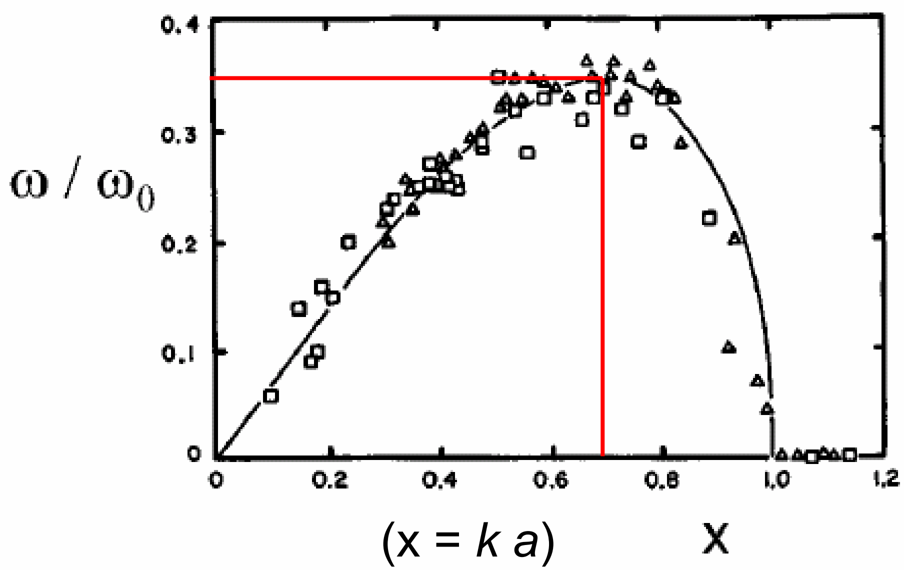
Eggers' review notation

$$\bar{\omega}^2 = \frac{x I_1(x)}{I_0(x)} (1 - x^2).$$

Frequency scaled with:

$$\omega_0 = (\gamma / (\rho r_0^3))^{1/2}$$

Plot of dispersion relation ($k\,a$ vs. ω/ω_0) for inviscid jet



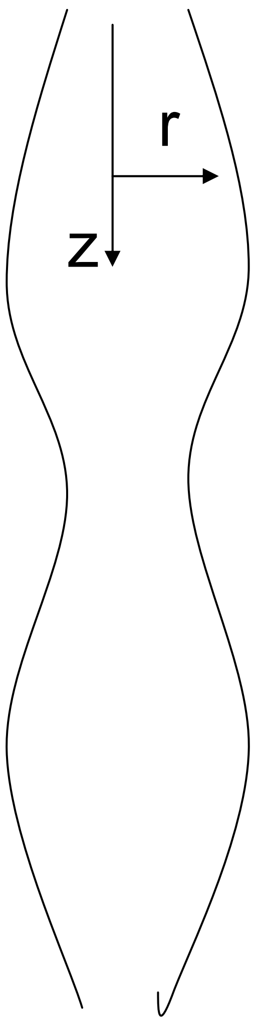
According to Rayleigh, the dispersion curve for the viscous case should be similar to the inviscid case.

$X_c = 0.697$
$\omega/\omega_0 = 0.343$

In the inviscid case, the dispersion curve is asymmetric.

Rayleigh’s Formulation for viscous jets

- Long, cylindrical thread of an incompressible, viscous liquid
  viscosity μ, density ρ, kinematic viscosity
  Sinusoidal perturbations along the length of the jet
- Axisymmetric jet – described by Stokes equation
- Equations of motion: (cylindrical coordinates)



$$\left.\begin{aligned}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}&=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}+\frac{\partial^2 u}{\partial z^2}\right),\\ \frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}&=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right),\end{aligned}\right\} \qquad 1$$

u, w: velocity components, p: pressure

Continuity equation: $$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}=0, \qquad 2$$

Velocity components satisfying continuity equation

$$u=\frac{1}{r}\frac{\partial \psi}{\partial z}, \qquad w=-\frac{1}{r}\frac{\partial \psi}{\partial r}, \qquad 3$$

where, ψ is the stokes current function

Eliminate p from Eq. 1, and use ψ:

$$\left(\frac{\partial}{\partial t} + \frac{1}{r}\frac{\partial \psi}{\partial z}\frac{\partial}{\partial r} - \frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial}{\partial z} - \frac{2}{r^2}\frac{\partial \psi}{\partial z}\right)\mathrm{D}\psi = \nu \mathrm{DD}\psi, \qquad (4)$$

where, $\mathrm{D} \equiv \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$

For small motions, ignoring squares and products of velocity components,

$$\left(\mathrm{D} - \frac{1}{\nu}\frac{\partial}{\partial t}\right)\mathrm{D}\psi = 0. \qquad (5)$$

ψ can be written in two parts ψ_1, and ψ_2 such that:

$$\mathrm{D}\psi_1 = 0, \qquad \mathrm{D}\psi_2 - \frac{1}{\nu}\frac{\partial \psi_2}{\partial t} = 0. \qquad (6)$$

$\psi$is assumed to be a function of z, and t of the form

$\psi \propto \exp[\, i\,(n\,t + k\,z)\,]$,

$$\frac{d^2\psi_1}{dr^2} - \frac{1}{r}\frac{d\psi_1}{dr} - k^2\psi_1 = r\,\frac{d}{dr}\left(\frac{1}{r}\frac{d\psi_1}{dr}\right) - k^2\psi_1 = 0,$$

$$\frac{d^2\psi_2}{dr^2} - \frac{1}{r}\frac{d\psi_2}{dr} - k'^2\psi_2 = r\,\frac{d}{dr}\left(\frac{1}{r}\frac{d\psi_2}{dr}\right) - k'^2\psi_2 = 0,$$

$$k'^2 = k^2 + in/\nu.$$

Boundary conditions:

No tangential forces at r=a:
(by symmetry)

$$2k^2\psi_1 + (k'^2 + k^2)\psi_2 = 0,$$

Normal Stresses balanced by variable part of surface tension at r = a:

Normal stress: $$P = -p + 2\mu\frac{du}{dr}$$

Variable pressure due to surface tension:
where T is the surface tension,
a is the unperturbed radius, and

$$\frac{T\xi\,(k^2a^2 - 1)}{a^2}.$$

$\xi = \int u\,dt = k\psi / na.$ is the radial displacement of the column

The forms of ψ_1, and ψ_2 satisfying boundary conditions are in terms of Bessel functions.

$$\psi_1 = ArJ_0'(ikr) \qquad \psi_2 = BrJ_0'(ik'r).$$

These functions can be plugged back in boundary conditions and constants A & B can be eliminated to give a transcendental equation relating growth rate and wave numbers.

Dispersion relation relating growth rate (*i n*) and wave mode (*k a*)

$$
\begin{aligned}
&\frac{T(1-k^2a^2)}{\rho a^3}\,\frac{ka}{n}\,\frac{k'^2-k^2}{k'^2+k^2}\,J_0'(ika) \\
&\quad = -2k^2\nu\left\{J_0''(ika)-\frac{2kk'}{k'^2+k^2}\,\frac{J_0'(ika)}{J_0'(ik'a)}\,J_0''(ik'a)\right. \\
&\qquad\qquad\left. -\frac{k'(k'^2-k^2)}{k(k'^2+k^2)}\,\frac{J_0'(ika)}{J_0'(ik'a)}\,J_0(ik'a)\right\} \\
&\qquad +\frac{n}{ka}\left\{ika\,J_0(ika)-\frac{2k^2}{k'^2+k^2}\,\frac{J_0'(ika)}{J_0'(ik'a)}\,ik'a\,J_0(ik'a)\right\}.
\end{aligned}
$$

Fortunately this equation can be simplified for the viscosity-dominated case.

$$
in = -\frac{T(1-k^2a^2)}{2\mu a\,.\,k^2a^2\left\{J_0^2/J_0'^2+1+1/k^2a^2\right\}}.
$$

Using $x = k\,a$, $i\,n = \omega$ and using:

$$F(x) = x^2\left\{J_0^2(ix)/J_1^2(ix) + 1 + 1/x^2\right\} = x^2 + 1 - x^2 I_0^2(x)/I_1^2(x),$$

we get:

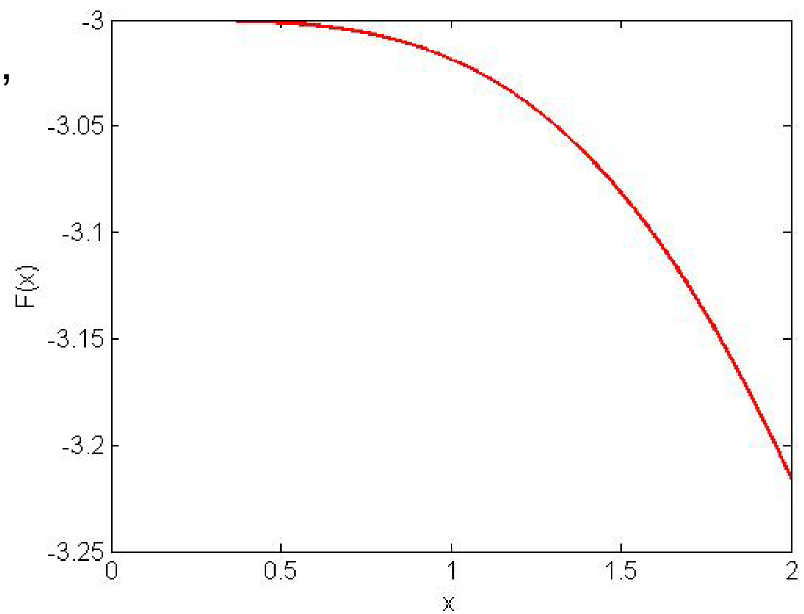
$$\omega = -\left(\frac{T}{\mu a}\right)\left(\frac{(1-x^2)}{2(x^2+1-x^2(I_0^2/I_1^2)}\right) \qquad \text{(A)}$$

Stability: $k\,a < 1$ Unstable: $k\,a > 1$

Rayleigh mentions that the function in the denominator does not deviate much from a value -3 for $0 < k\,a < 1$.

As an approximation to above form, in viscosity dominated jet:

$$\omega = \left(\frac{T}{6\mu a}\right)(1-x^2) \qquad \text{(B)}$$

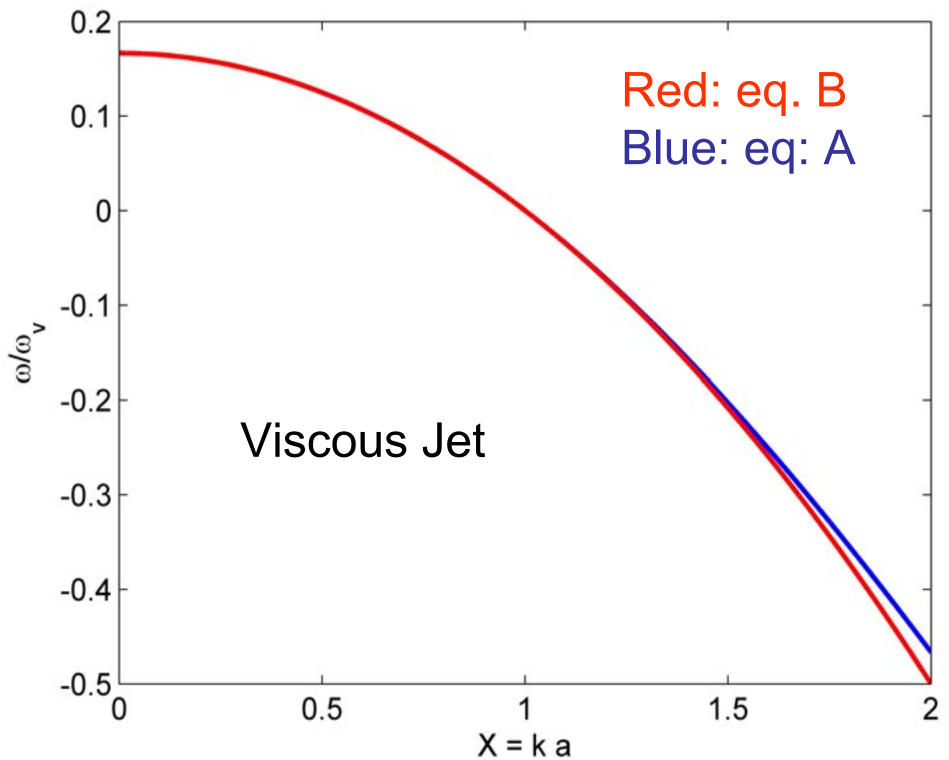


$$\omega = -\left(\frac{T}{\mu a}\right)\left(\frac{(1-x^2)}{2(x^2+1-x^2(I_0^2/I_1^2)}\right) \quad \text{(A)}$$

$$\omega = \left(\frac{T}{6\mu a}\right)(1-x^2) \quad \text{(B)}$$

$$\omega_v = \left(\frac{T}{\mu a}\right)$$

Frequency associated with viscous time scale

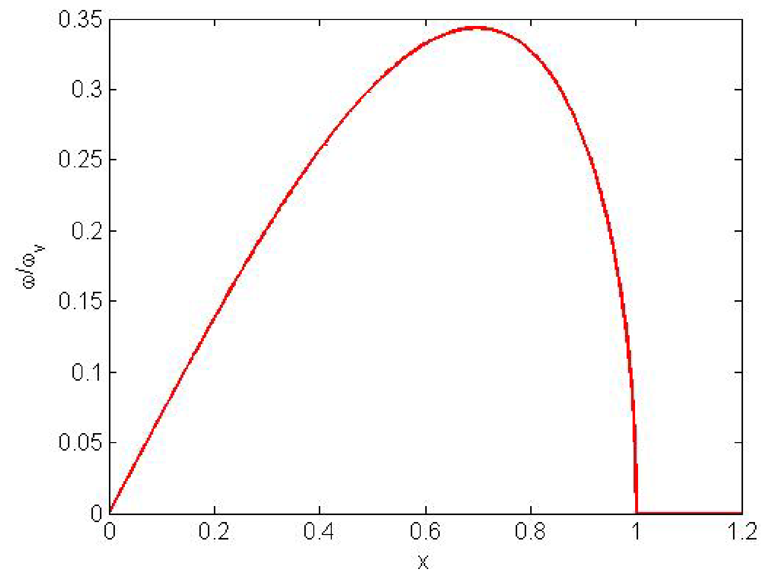


Result:

- The most unstable wave number is $k\ a = 0$

⇒ Infinite wavelength mode

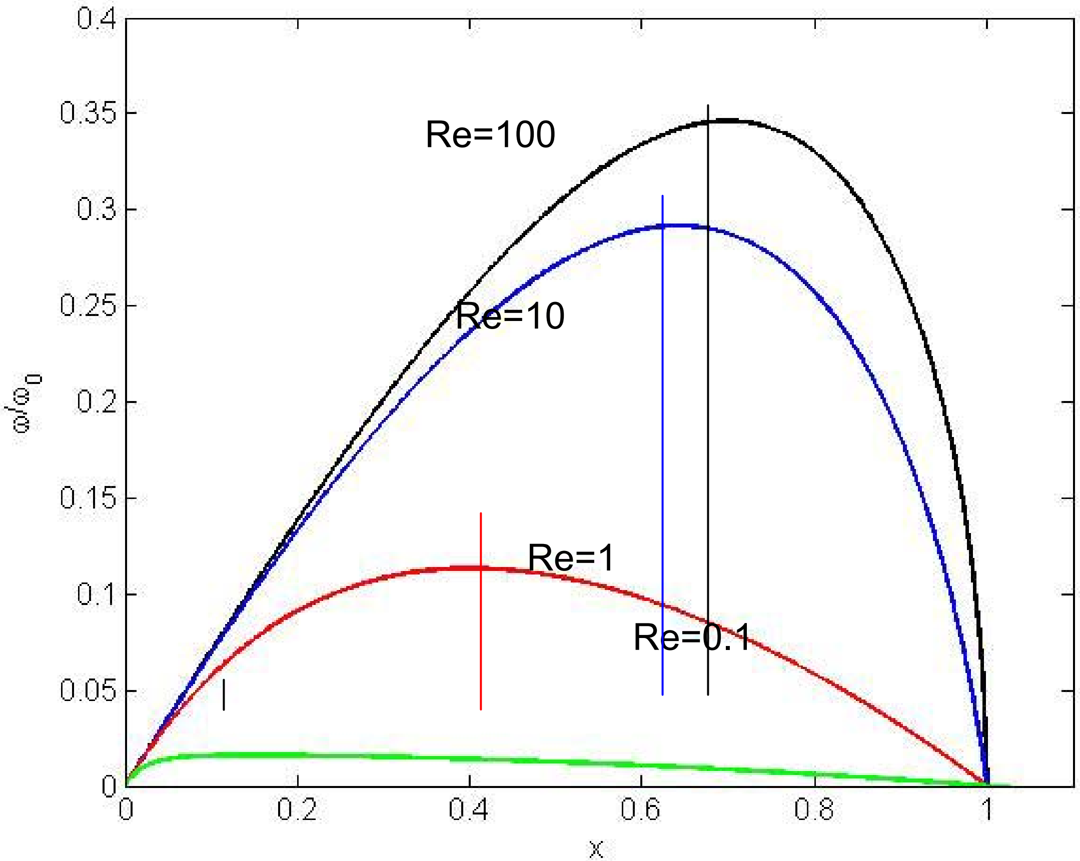
⇒ In practice the jet breaks at a few distant locations



Comparison with equations from Eggers' review: (Rev. Mod. Phys., Vol. 69, No. 3, July 1997)

Weber's simplified equation: (small *x* limit)

$$\omega=\omega_0\left\{\left[\frac{1}{2}x^2(1-x^2)+\frac{9}{4}\,\mathrm{Re}^{-2}x^4\right]^{1/2}-\frac{3}{2}\,\mathrm{Re}^{-1}x^2\right\}$$



Fastest growing mode:

$$x_R^2=\frac{1}{2+\sqrt{18/\mathrm{Re}}}$$

1) Re=100
x_R = 0.699

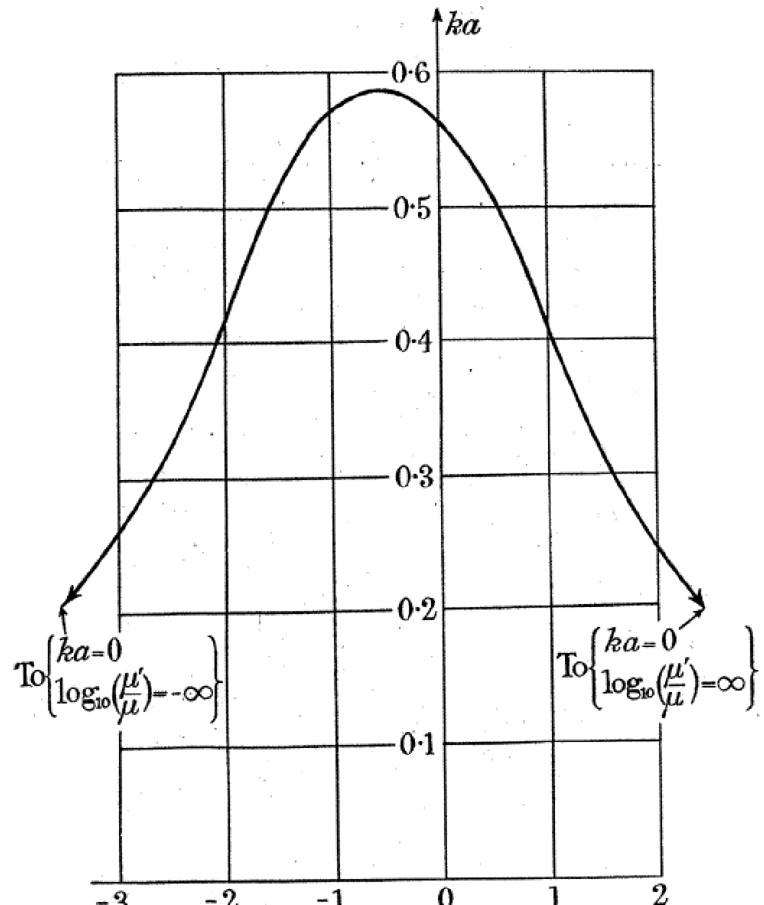
2) Re = 10
x_R = 0.642

3) Re = 1
x_R = 0.400

4) Re = 0.1
x_R = 0.150

Tomotika's work on a viscous jet surrounded by a viscous fluid (Proc. Royal Soc., 1935)

- Tomotika not only gets Raleigh's inviscid and viscous jets as special cases, he can calculate the change of maximally unstable mode with change in viscosity ratio.



μ' = jet viscosity

μ= surrounding fluid viscosity

low viscosity jet/viscous surrounding $\log_{10}\left(\frac{\mu'}{\mu}\right) \longrightarrow$ viscous jet/low viscosity surrounding

Other types of instabilities:

Highly viscous Newtonian, or viscoelastic fluids, at low Reynolds numbers, impinging on a plate can give rise to bending instabilities, resulting in coiling of the jet.

More about that from Matthieu, next week!