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③ Capillary rise :

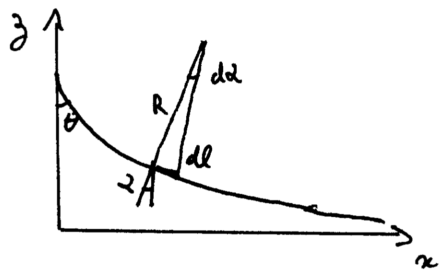
3.1 Menisci : (from greek "mênê" = moon).

demos :

non-dimensional analysis : $\underline{h = l_c f(\theta)}$

$90° > \theta > 0 \Rightarrow f(\theta) > 0$ (up)

$\theta > 90° \Rightarrow f(\theta) < 0$ (down)



Laplace pressure = gravity .

$$P_0 - \rho g z = P_0 - \frac{\sigma}{R} .$$

$$\Rightarrow \boxed{Rz = \frac{\sigma}{\rho g} = l_c^2}$$

signs : $\alpha > 0 \quad \frac{\pi}{2} - \theta \longrightarrow 0$

$\Rightarrow d\alpha < 0$

$$\begin{cases} dz < 0 \\ dx > 0 \\ dl > 0 \end{cases}$$

$$dl = -R\, d\alpha \quad \Rightarrow \quad R = -\frac{dl}{d\alpha}$$

$$dz = -dl \cdot \sin\alpha \quad \Rightarrow \quad dl = -\frac{dz}{\sin\alpha}$$

$$\Rightarrow R = \frac{dz}{\sin\alpha\, d\alpha}$$

$$\Rightarrow \frac{z\, dz}{\sin\alpha\, d\alpha} = l_c^2 \quad \rightarrow \quad z\, dz = l_c^2 \sin\alpha\, d\alpha$$

far away $\begin{cases} z = 0 \\ \alpha = 0 \end{cases} \rightarrow \underline{z^2 = l_c^2 (1 - \cos\alpha)}$

$\begin{cases} \alpha = \frac{\pi}{2} - \theta \\ z = h \end{cases} \rightarrow \boxed{h = l_c (1 - \sin\theta)^{1/2}}$ for $\theta < 90°$

$h = - l_c (1 - \sin\theta)^{1/2}$ for $\theta > 90°$

→ cartesian equations : nasty

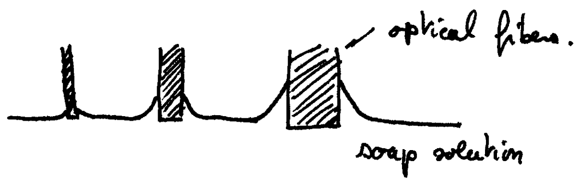
$$\frac{1}{R} = \frac{\ddot{z}}{(1+\dot{z}^2)^{3/2}}$$

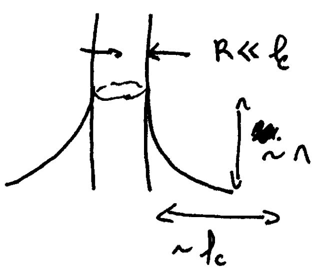
shape: $\quad x - x_0 = \ell_c \,\mathrm{Argsh}\, \frac{2\ell_c}{z} - \sqrt{2}\,\ell_c \left(2 - \frac{z^2}{2\ell_c^2}\right)^{1/2}$

x_0 to get $z = h$ for $x = 0$.

Meniscus on a fibre

demo:





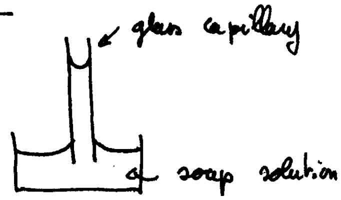
→ 2 curvatures ⇒ 2 sizes R and ℓ_c.

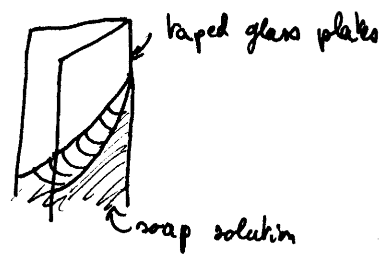
$$\boxed{h \simeq R \ln\left(\frac{2\ell_c}{R}\right)} \quad \text{for } \theta = 0 \text{ and } R \ll \ell_c$$

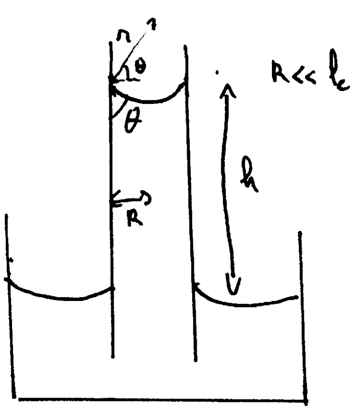
→ slide : experiment with etching liquid.

3.2 Rise in a tube

demo:







Laplace = hydrostatic

$$p_0 - \frac{2\sigma\cos\theta}{R} = p_0 - \rho g h$$

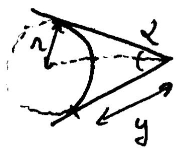
$$\Rightarrow \boxed{h_J = \frac{2\sigma\cos\theta}{\rho g R} = 2\,\frac{\ell_c^2}{R}\cos\theta} \quad \text{Jurin's law.}$$

→ rise if $\theta < 90°$.

→ question : what happens if tube shorter than h_J?

↳ <u>profile in the edge experiment:</u>

↳ ~~[crossed out]~~ important for microfluidic devices → angular channels.



for $\theta = 0$

$$R = y \tan \alpha/2$$

$$\Rightarrow \frac{\sigma}{R} = \rho g z \Rightarrow \frac{\sigma}{y \tan \alpha/2} = \rho g z$$

$$\boxed{y = \frac{l_c^2}{z \tan \alpha/2}}$$

3.3 <u>Capillary forces:</u>

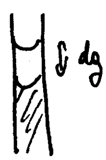
demo:

liquid train

perfluorodecalin ethyl acetate

conical tube.

<u>force in a tube:</u>



$$dE_s = (\sigma_{SL} - \sigma_{SV}) \, 2\pi R \, dz$$

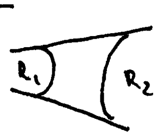
$$\boxed{f = \frac{dE_s}{dz} = 2\pi R \sigma \cos\theta}$$

perimeter of the contact line.

↳ <u>if $\sigma_{SL} - \sigma_{SV} > \sigma_{SL}$</u> ($S > 0$, $\theta = 0$ and $\cos\theta = 1$).

↳ <u>same</u> force : the difference $\sigma_{SL} - \sigma_{SV} - \sigma_{LV}$ is burned by viscous dissipation on the precursor film (de Gennes).

<u>Note:</u>



$R_2 > R_1$ → would be expected to move to the right.

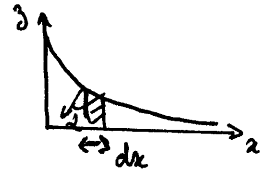
⇒ misleading : <u>work with pressures.</u>

force on a plate:

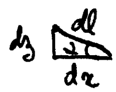
moving contact line → same argument

$$\underline{f = \sigma \underbrace{l}_{\text{perimeter}} \cos\theta}$$

weight of the meniscus:



$$dm = \rho l z\, dx$$



$$\tan\alpha = -\frac{dz}{dx}$$

$$\rightarrow dm = -\frac{\rho l z\, dz}{\tan\alpha}$$

→ from shape: $z\,dz = l_c^2 \sin\alpha\, d\alpha$.

$$\Rightarrow dm = -\rho l \cdot l_c^2 \underbrace{\frac{\sin\alpha}{\tan\alpha}}_{\cos\alpha\, d\alpha} d\alpha$$

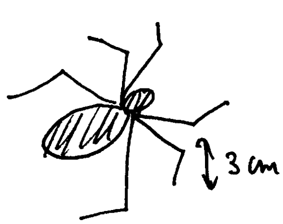
$$\rightarrow m = -\rho l\, l_c^2 \left[\sin\alpha\right]_{\frac{\pi}{2}-\theta}^{0} = \rho l \cdot l_c^2 \cos\theta$$

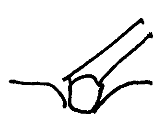
$$\Rightarrow \underline{mg = f}.$$

→ if $\underline{\cos\theta < 0}$ force upwards ⇒ dense object can float.

→ slide: water strider.

↳ force it can hold:



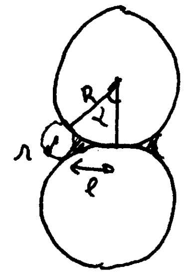


$$f \approx \underbrace{2}_{\text{2 sides}} \times 0.03 \times 0.072 \times 6 \simeq 0.03\,N \rightarrow 3g$$

(with $\theta = 180°$).

Contact force in between 2 grains:

→ slide sand castle.



for $\theta = 0$.

shape of the meniscus assumed to be toroidal. ($\alpha \ll 1$).

$r \ll \ell \Rightarrow$ 1 curvature.

$$R = (R+r)\cos\alpha \rightarrow r = R\,\frac{1-\cos\alpha}{\cos\alpha}$$

$$\Delta P = \frac{\sigma}{r} = \frac{\sigma}{R}\,\frac{\cos\alpha}{1-\cos\alpha}$$

$$f = \Delta P.\,\pi \ell^2 = \frac{\sigma}{R}\,\frac{\cos\alpha}{1-\cos\alpha}\,\pi R^2 \tan^2\alpha$$

$$\rightarrow f = \pi R\sigma\,\frac{1+\cos\alpha}{\cos\alpha}$$

$\alpha \ll 1 \Rightarrow \boxed{f \simeq 2\pi R\sigma}$ independent of volume of liquid!

→ for small V → roughness ⇒ $f \searrow$.