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On Metric Controllers and Observers for Nonlinear Systems

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Abstract

While observer and controller design is well understood and widely used for linear systems, extensions to nonlinear systems have lacked generality. Motivated by fluid dynamics, and following up on our earlier work [Lohmiller and Slotine, 1996], this paper further explores the use of so-called Euler coordinates (rather than the usual or Lagrange-based methods) in general nonlinear, non-autonomous systems, and the analysis and design tools they lead to. In this field formulation, a complex plant may be regarded as the superposition of simpler plants, each of them with very specific properties. There are two major classes of simple flow fields in fluid dynamics, namely potential flow and solenoidal flow. Both of them may be associated to an integral formulation of the system equations, leading in turn to a Lyapunov-like function, in the form of of a scalar potential Φ or of stream function $\Psi^\star$. These generalized distance functions can be regarded as a extension of the differential concept of length introduced in our earlier work.

1. Introduction

Nonlinear control system design has been very successfully applied to particular classes of systems and problems, but it still lacks generality, as e.g. in the case of feedback linearization, or explicitness, as e.g. in the case of Lyapunov theory [Isidori, 1995; Marino and Tomei, 1995; Khalil, 1995; Nijmeyer and Van der Schaft, 1992; Vidyasagar, 1992; Slotine and Li, 1991]. Motivated by fluid dynamics and electromagnetism, we show in this paper that the use of so-called Euler coordinates allows new design and analysis methods. [Lohmiller and Slotine, 1996a] analyze a general flow field with the help of the Jacobian of the system. This differential analysis leads to a sufficient stability concept. More general stabilty concepts can be expected with an integral or Lyapunov formulation of a flow field. This integral formulation in the form of a scalar or "vector" potential can be found for potential and solenoidal flows in a straightforward way.

The general deterministic setting can be written

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \\ \mathbf{y} &= \mathbf{h}(\mathbf{x}) \end{aligned} \tag{1}$$

where $\mathbf{f}$ is a $n \times 1$ nonlinear vector function, $\mathbf{x}$ is the $n \times 1$ state vector, $\mathbf{u}$ is the $p \times 1$ input vector,and $\mathbf{y}$ is the $m \times 1$ measurement vector. In this paper, all quantities are assumed to be smooth, by which is meant that any required derivative or partial derivative exists and is continuous.

A standard problem is the observer problem, *i.e.*, how to estimate the state $\mathbf{x}(t)$ given only the measurement $\mathbf{y}(t)$. Another standard problem is the tracking control problem, *i.e.* to make some k-dimensional vector $\mathbf{z}(\mathbf{x})$ follow a desired trajectory $\mathbf{z}_d(t)$, while preserving internal boundedness. This paper first reviews and extends some general results on representation and differential analysis (sections 2 and 3) based on [Lohmiller and Slotine, 1996a], then discusses integral analysis (sections 4 - 6) and provides a preliminary exploration of potential applications to the modelling of controllers and observers (sections 7-8).

2. Representation and Contraction Regions

Motivated by fluid dynamics, equation (1) can be thought of as an n-dimensional fluid flow, where $\dot{\mathbf{x}}$ is the n-dimensional "velocity" vector at the n-dimensional position $\mathbf{x}$ and time t. There are two ways of describing such a flow, namely Lagrange coordinates and Euler coordinates [Chung, 1988]. As discussed in [Lohmiller and Slotine, 1996a], simply correspond to describing the whole system flow *at each given point in the state space* rather than focusing on the paths of individual particles, allowing for instance to regard a given nonlinear plant $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ as the superposition of simpler plants $\dot{\mathbf{x}}_i = \mathbf{f}_i(\mathbf{x}, t)$. Euler coordinates will be systematically used in this paper.

[Lohmiller and Slotine, 1996a] show that the squared distance between two neighboring Lagrange particles changes with the eigenvalues of the rate of strain tensor (symmetric part of the Jacobian $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$). For strictly negative eigenvalues any length shrinks exponentially. Thus the radius of a sphere around an equilibrium point shrinks exponentially (Figure 1). This leads to the following stability theorem (Figure 1)

Theorem 1 *Given the system equations* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ *any plant, starting in a ball contained at at all times in a contraction region (region with uniformly negative definite Jacobian* $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$*) and centered at an equilibrium point, will converge exponentially to this equilibrium point.*

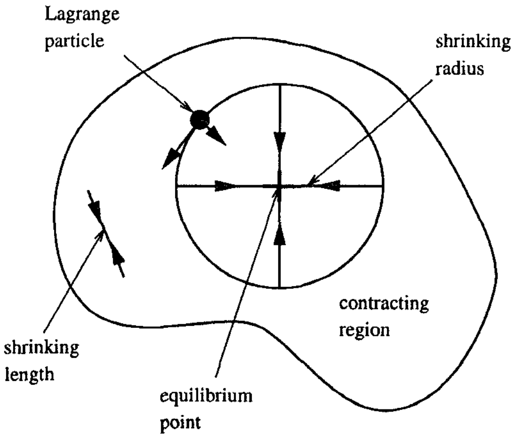

Figure 1: Stability of an equilibrium point

This result may be viewed as a generalization of Krasovskii's theorem on asymptotic convergence of nonlinear autonomous systems. Not surprisingly it only provides a *sufficient* condition since length is defined here in a spherical sense and not related to the actual plant dynamics. More general stability results can be expected with an integral or Lyapunov formulation of a flow field. According to the analysis of differential forms integral formulations of a flow field exist only for potential and solenoidal flow fields [Bronstein and Semendjajev, 1990].

Consider two Lagrange particles that remain within a convex contraction region at all times. The connecting line **s** in (Figure 2) shrinks exponentially so that the two Lagrange particles converge to one, in general time-varying, point exponentially. In fact any Lagrange particle that remains within that contraction region at all times will converge exponentially to this time-varying point.

Consider now a contraction region wrapped around a indifferent region, i.e., one in which the system Jacobian does not have a priori any negative definiteness property (Figure 3). Since any length shrinks exponentially the area of any surface contained in the contraction region shrinks exponentially. That means that this surface converges exponentially to the indifferent region. Any Lagrange particle contained in the volume of this surface thus converges exponentially to the indifferent region.

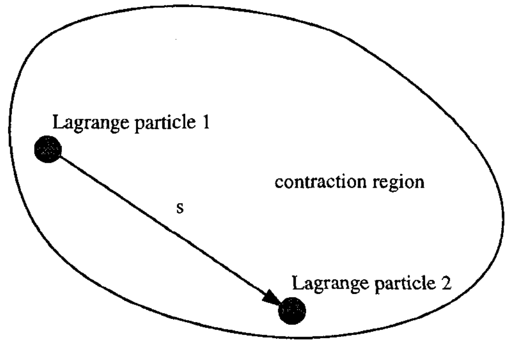

Figure 2: Two Lagrange particles in a contraction region

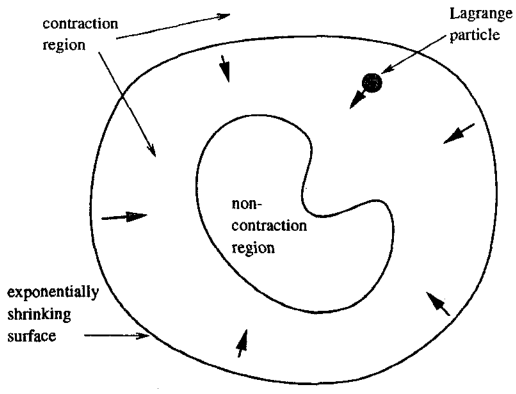

Figure 3: Surface in a contraction region around a indifferent region

Non-Cartesian Metric

Consider the coordinate tranformation

$$\mathbf{z} = \mathbf{z}(\mathbf{x})$$

where $\mathbf{z}$ represents a r-dimensional vector ($r \leq n$). A virtual displacement in $\mathbf{z}$ is given by

$$\delta \mathbf{z} = \frac{\partial \mathbf{z}}{\partial \mathbf{x}} \delta \mathbf{x}$$

Assuming this space to be Cartesian the squared distance between two Lagrange particles can be written as

$$(\mathbf{ds})^2 = \delta \mathbf{z}^T \delta \mathbf{z}$$

or equivalently

$$(\mathbf{ds})^2 = \delta \mathbf{x}^T \frac{\partial \mathbf{z}}{\partial \mathbf{x}}^T \frac{\partial \mathbf{z}}{\partial \mathbf{x}} \delta \mathbf{x}$$

where $\mathbf{g} = \frac{\partial \mathbf{z}}{\partial \mathbf{x}}^T \frac{\partial \mathbf{z}}{\partial \mathbf{x}}$ represents the metric tensor of the system [Fluegge, 1972]. The space in $\mathbf{x}$ is now non-Cartesian if the metric tensor $\mathbf{g}$ is not the idendity matrix.

The rate of change of any length in the flow field is thus of the form

$$\begin{aligned}
\frac{d}{dt}(\delta\mathbf{z}^T\delta\mathbf{z}) &= 2\,\delta\mathbf{z}^T\dot{\delta\mathbf{z}} \\
&= 2\,\delta\mathbf{x}^T\frac{\partial\mathbf{z}}{\partial\mathbf{x}}^T\left(\frac{\dot{\partial\mathbf{z}}}{\partial\mathbf{x}}\delta\mathbf{x} + \frac{\partial\mathbf{z}}{\partial\mathbf{x}}\dot{\delta\mathbf{x}}\right) \\
&= 2\,\delta\mathbf{x}^T\left(\frac{\partial\mathbf{z}}{\partial\mathbf{x}}^T\frac{\dot{\partial\mathbf{z}}}{\partial\mathbf{x}} + \frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right)\delta\mathbf{x} \\
&= 2\,\delta\mathbf{z}^T\left(\frac{\dot{\partial\mathbf{z}}}{\partial\mathbf{x}}\frac{\partial\mathbf{z}}{\partial\mathbf{x}}^{-1} + \frac{\partial\mathbf{z}}{\partial\mathbf{x}}\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\frac{\partial\mathbf{z}}{\partial\mathbf{x}}^{-1}\right)\delta\mathbf{z} \\
&= 2\,\delta\mathbf{z}^T\mathbf{F}_z\delta\mathbf{z} \\
&= 2\,\delta\mathbf{z}^T\mathbf{E}_z\delta\mathbf{z}
\end{aligned}$$

where $\mathbf{F}_z$ represents the rate of deformation tensor with respect to the Cartesian basis in $\mathbf{z}$. $\mathbf{E}_z$ is the rate of strain tensor with respect to to the Cartesian basis in $\mathbf{z}$. $\frac{\partial\mathbf{f}}{\partial\mathbf{x}}$ is the Jacobian of the system dynamics in $\mathbf{x}$, $\frac{\partial\mathbf{z}}{\partial\mathbf{x}}^T\frac{\dot{\partial\mathbf{z}}}{\partial\mathbf{x}}$ the Christoffel part due to the curvature of the space. Similarly to the earlier reasoning, exponential convergence to one point in $\mathbf{z}$ is guaranteed if $\mathbf{E}_z$ is uniformly strictly negative. If $\mathbf{z} = \mathbf{z}(\mathbf{x})$ is a invertible coordinate transformation it can be concluded that also $\mathbf{x}$ converges to one point.

EXAMPLE 1: Consider the system

$$\dot{x} = -tx$$

for $t \geq t_o > 0$ the rate of strain tensor is uniformly strictly negative and exponential convergence to the equilibrium point is guaranteed.

EXAMPLE 2: In the system

$$\dot{x} = -x + e^t$$

the rate of strain tensor is again uniformly strictly negative and exponential convergence to one time-varying point (a single trajectory) is guaranteed – trivially here, two different initial conditions will both exponentially converge to the same asymptotic behavior $x = e^t$.

EXAMPLE 3: Consider a LTI system of the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

The invertible coordinate transformation $\mathbf{z} = \mathbf{T}\mathbf{x}$, leads to the new dynamics:

$$\dot{\mathbf{z}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{z} = \mathbf{\Lambda}_{real}\,\mathbf{z}$$

Choosing a proper coordinate transformation $\mathbf{\Lambda}_{real}$ represents the Jordan matrix of the system, where the complex eigenvalues have been put to real form and the 1's on the second diagonal have been replaced with $\beta < \lambda_{real}$. $\mathbf{\Lambda}_{real}$ is uniformly strictly negative if and only if the LTI system is stable.

Assume now that $\mathbf{A}$ is in fact a linearization of a nonlinear non-autonomous system, $\mathbf{A} = \frac{\partial\mathbf{f}}{\partial\mathbf{x}}$. The rate of deformation tensor is now given by $\mathbf{F}_z = \mathbf{T}^{-1}\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\mathbf{T}$ and exponential convergence is guaranteed in the contraction region with uniformly strictly negative $\mathbf{F}_z$.

Example 4: Consider the general nonlinear non-autonomous system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$

Assume that this system is uniformly strictly negative definite only in some coordinates $\mathbf{x}_{reduced}$. Then length can be defined as:

$$(\mathbf{ds})^2 = \delta\mathbf{x}_{reduced}^T\,\delta\mathbf{x}_{reduced}$$

and exponential convergence is guaranteed to one point in this coordinate subset, i.e., to a manifold in the whole state space.

3. Potential and solenoidal flow

Any general flow field can be decomposed, non-uniquely, into the sum of two simpler flow fields with very specific structural properties, namely potential and solenoidal flow – this is the continuum mechanics equivalent of the fact that any square matrix can be written as the sum of a symmetric part and a skew-symmetric part.

3.1. Potential flow

A potential flow field $\dot{\mathbf{x}}$ can be expressed as the gradient field of a scalar potential Φ [Fluegge, 1972]. The following three statements can be used equivalently to define a potential field [Aris, 1962] [1]

$$\begin{aligned}
\dot{\mathbf{x}} &= grad\ \Phi \\
curl\ \dot{\mathbf{x}} &= 0 \\
\oint_C \dot{\mathbf{x}}\ dt &= 0 \qquad \text{for any closed curve C}
\end{aligned}$$

where $\mathbf{t}$ represents the unit tangent vector of C.

A typical example of a potential field is damping of the form $\ddot{x}_i = g(\dot{x}_i)$. In more general systems, many authors suggest using the scalar potential component Φ as a Lyapunov function candidate [Abd-Ali, et al., 1975].

3.2. Solenoidal flow

Solenoidal flow fields are used in continuum mechanics to model incompressible fluids. This incompressibility can be expressed with the following three equiv-

[1] In tensor analysis, the Cartesian gradient of a scalar Φ is generalized for n dimensions as the 1-form $grad\ \Phi = \frac{\partial\Phi}{\partial x^i}dx^i$ of the scalar potential Φ. The curl of a vector $\dot{\mathbf{x}}$ is defined as the 2-form $curl\ \dot{\mathbf{x}} = \frac{1}{2}\left(\frac{\partial\dot{x}_i}{\partial x^j} - \frac{\partial\dot{x}_j}{\partial x^i}\right)dx^j \wedge dx^i$. The Cartesian divergence of a vector $\dot{\mathbf{x}}$ is defined as the n-form $div\ \dot{\mathbf{x}} = \left(\frac{\partial\dot{x}_1}{\partial x^1} + ... + \frac{\partial\dot{x}_n}{\partial x^n}\right)dx^1 \wedge ... \wedge dx^n$ [Bronstein and Semendjajev, 1990].

alent definitions [Aris, 1962], which extend to the n-dimensional case[2]

$$\begin{aligned} \dot{\mathbf{x}} &= curl^\star\ \boldsymbol{\Psi}^\star \\ div\ \dot{\mathbf{x}} &= 0 \\ \int_S \dot{\mathbf{x}}^T \mathbf{n}\ dS &= 0 \qquad \text{for any closed surface S} \end{aligned}$$

where $\mathbf{n}$ represents the unit normal vector of S. The (non-unique) vector field $\boldsymbol{\Psi}^\star$ is then called a vector potential of the flow field.

Many important systems are solenoidal flow fields. For example, in a general Hamiltonian system [Arnold, 1984] with Hamiltonian $H(\mathbf{p}, \mathbf{q}, t)$

$$\begin{aligned} \dot{\mathbf{p}} &= -\frac{\partial H}{\partial \mathbf{q}} \\ \dot{\mathbf{q}} &= \frac{\partial H}{\partial \mathbf{p}} \end{aligned}$$

and thus $div \begin{pmatrix} \dot{\mathbf{p}} \\ \dot{\mathbf{q}} \end{pmatrix} = 0$.

The vector potential $\Psi^\star$ represents now the integral formulation of the solenoidal flow, that might be used as Lyapunov function. The vector potential $\Psi^\star$ is the sum of "general" and "specific" vector potentials [Bronstein and Semedjajev, 1990]

$$\Psi^\star = \Psi_{gen} + \Psi_{spec}$$

where Ψ_{gen} is a solution of the equation $curl^\star \Psi_{gen} = 0$, and

$$\Psi_{spec} = \frac{1}{n-2} x^i v_{ii_2 \dots i_{n-1}} dx^{i_2} \wedge \dots \wedge dx^{i_{n-1}}$$

with

$$v_{i_1 \dots i_{n-1}}(x^1, \dots, x^n) = \int_o^1 \dot{x}_{i_1 \dots i_{n-1}}(tx^1, \dots, tx^n) t^{n-2} dt$$

A solution is always guaranteed since $div\ \dot{\mathbf{x}} = 0$.

4. Stream functions

Consider the time-augmented $(n+1)$-dimensional form of a general non-autonomous system:

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{t} \end{pmatrix} = \begin{pmatrix} \mathbf{f}(\mathbf{x}, t) \\ 1 \end{pmatrix}$$

Let $\begin{pmatrix} \mathbf{dx} \\ dt \end{pmatrix}$ be an element of a system trajectory passing through point $\begin{pmatrix} \mathbf{x} \\ t \end{pmatrix}$, and let $\begin{pmatrix} \dot{\mathbf{x}} \\ 1 \end{pmatrix}$ denote the velocity vector at this point (Figure 4). Since $\begin{pmatrix} \mathbf{dx} \\ dt \end{pmatrix}$ is parallel to $\begin{pmatrix} \dot{\mathbf{x}} \\ 1 \end{pmatrix}$, one has

$$\begin{aligned} \dot{x}_i dx_j &= \dot{x}_j dx_i \\ dx_j &= \dot{x}_j dt \end{aligned}$$

Equations of this kind are known as 1-form or Pfaffian

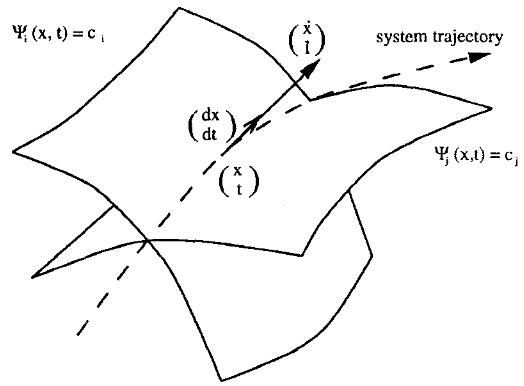

Figure 4: Two intersecting stream functions

differential equations [Sneddon, 1957]. A solution is given by n independent stream functions,

$$\Psi_i(\mathbf{x}, t) = c_i \qquad i = 1, \dots, n$$

Thus a system trajectory can be described as the intersection of n surfaces [Chung, 1988] such that

$$\dot{\mathbf{x}} \cdot grad\ \Psi_i = 0 \qquad i = 1, \dots, n$$

or [3]

$$\rho(\mathbf{x}, t)\dot{\mathbf{x}} = grad\ \Psi_1 \times \dots \times grad\ \Psi_n$$

were ρ is a scalar function (density) of position and time. It can be shown that $div(grad\ \Psi_1 \times \dots \times \Psi_n) = 0$. Thus $\rho(\mathbf{x}, t)$ must be chosen such that $div(\rho(\mathbf{x}, t)\dot{\mathbf{x}}) = 0$. For a solenoidal flow, $\rho = 1$.

Once a Lagrange particle is on a given manifold $\Psi_i(\mathbf{x}, t) = c_i$ it will never leave this manifold. The stream functions and their associated invariant manifolds are in general not uniquely defined. But some of them have a strong physical interpretation as e.g. the energy, angular, or linear momentum of a mechanical system. For autonomous systems it is not necessary to augment the state space with the time variable, and thus only $n - 1$ stream functions define the flow field.

In a time-invariant Hamiltonian system, the Hamiltonian itself represents a stream function, since

$$\dot{H} = \begin{pmatrix} \frac{\partial H}{\partial \mathbf{p}} \\ \frac{\partial H}{\partial \mathbf{q}} \end{pmatrix} \cdot \begin{pmatrix} \dot{\mathbf{p}} \\ \dot{\mathbf{q}} \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial \mathbf{p}} \\ \frac{\partial H}{\partial \mathbf{q}} \end{pmatrix} \cdot \begin{pmatrix} -\frac{\partial H}{\partial \mathbf{q}} \\ \frac{\partial H}{\partial \mathbf{p}} \end{pmatrix} = 0$$

[2] In n dimensions, the Cartesian vector potential $\Psi^\star$ is defined as the $(n-2)$-form $\Psi^\star = \frac{1}{(n-2)!}\Psi^\star_{i_1 \dots i_{n-2}} dx^{i_1} \wedge \dots \wedge dx^{i_{n-2}}$. $\dot{\mathbf{x}} = curl^\star\ \boldsymbol{\Psi}^\star$ is the differential of $\Psi^\star$ that is $\dot{\mathbf{x}} = curl^\star\ \boldsymbol{\Psi}^\star = \frac{1}{n-1} c^\star_{i_1 \dots i_{n-1}} dx^{i_1} \wedge \dots \wedge dx^{i_{n-1}}$ with $c^\star_{i_1 \dots i_{n-1}} = \frac{\partial \Psi_{i_2 \dots i_{n-1}}}{\partial x^{i_1}} - \frac{\partial \Psi_{i_1 i_3 \dots i_{n-1}}}{\partial x^{i_2}} + \dots + (-1)^{n-1} \frac{\partial \Psi_{i_1 \dots i_{n-2}}}{\partial x^{i_{n-2}}}$ [Bronstein and Semendjajev, 1990].

[3] In tensor analyis, the generalized cross-product is defined for n dimensions as $\epsilon_{1 \dots n} v^1 \dots v^m$, correponding to a tensor of order $n - m$.

It is straightforward to show that the "vector" potential $\Psi^\star$ of a solenoidal flow in 2 dimensions is the stream function Ψ of the flow field. For higher dimensions research is being done to show whether the stream functions Ψ can be computed from the "vector" potential $\Psi^\star$.

5. Generalized Lyapunov Functions

Consider now the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$$

and assume that a control input $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ can be chosen such that there exists a closed-loop stream function Ψ whose level surfaces (i.e.., the surfaces defined by $\Psi(\mathbf{x}, t) = constant$) represent at each time t a set of closed surfaces in a certain region around an equilibrium point. We further assume that the stream function Ψ has a global minimum at the equilibrium point and other local minima do not occur in the region of interest. We will refer to such stream functions in the following simply as closed stream functions. This is illustrated in Figure 5. This set of stream functions obviously represents a generalization of the spherical stream functions used in Figure 1 if only the tangential flow is regarded.

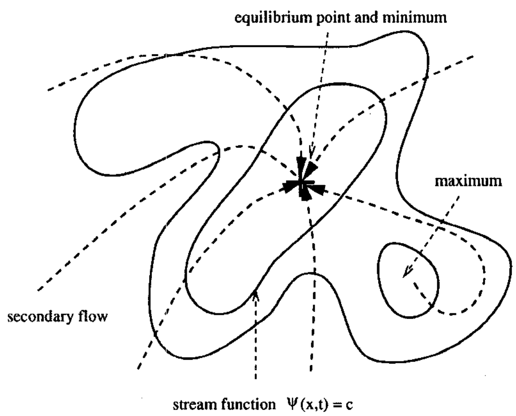

Figure 5: Closed stream functions and superimposed secondary flow

Consider now a secondary flow field $\dot{\mathbf{x}}$, representing e.g. a system part (damping) that was not modelled yet or an additional control input $\mathbf{u}$. This secondary flow field, represented by the dotted lines in Figure 5, always points to the interior of the stream function $\Psi(\mathbf{x}, t)$ or in the direction of decreasing Ψ. It is intuitively clear that this secondary flow field decreases the distance to the equilibrium point and makes the system converge to the equilibrium point, if strong enough. The dynamics of this generalized distance can be expressed in the following way:

$$\dot{\Psi} = grad\ \Psi \cdot \dot{\mathbf{x}} + \frac{\partial \Psi}{\partial t}$$

Using the generalized distance $\Psi(\mathbf{x}, t)$ as a Lyapunov function convergence to the equilibrium point is guaranteed if $grad\ \Psi \cdot \dot{\mathbf{x}} + \frac{\partial \Psi}{\partial t}$ is strictly negative. Exponetial convergence is guaranteed if $grad\ \Psi \cdot \dot{\mathbf{x}} + \frac{\partial \Psi}{\partial t} < \lambda \cdot \Psi$; $(\lambda \leq \beta < 0)$. This secondary flow represents the generalization of the normal flow in Figure 1 induced by the shrinking radius. [Lohmiller and Slotine, 1996] superimpose this secondary flow with the help of a coordinate error feedback in an observer design.

In general it has to be accepted that the secondary flow $\dot{\mathbf{x}}$ lies tangential to the stream function in some regions and can thus not decrease the distance to the equilibrium point or stabilize the system in this region. Such a region is characterized by:

$$grad\ \Psi(\mathbf{x}, t) \cdot \dot{\mathbf{x}} = 0 \qquad (2)$$

For $grad\ \Psi(\mathbf{x}, t) \neq 0$ this region is left by a Lagrange particle with velocity

$$\frac{\partial (grad\ \Psi(\mathbf{x}, t) \cdot \dot{\mathbf{x}})}{\partial \mathbf{x}}\Big|_o \cdot \dot{\mathbf{x}} \qquad (3)$$

If this leaving velocity is 0 a particle will not leave this region, the distance Ψ stays constant and convergence to the equilibrium point cannot be guaranteed. If the velocity in equation (3) is strictly positive or negative it can be guaranteed that a Lagrange particle moves in finite time Δt a finite distance $\Delta \mathbf{x}$ away from this unstabilizable region. If the shrinking velocity $(grad\ \Psi(\mathbf{x}, t) \cdot \dot{\mathbf{x}})$ is then strictly negative again and the Lagrange particle stays at least a finite distance $\Delta \mathbf{x}$ in this region convergence to the equilibrium point can still be guaranteed.

6. Closed stream functions

Since the velocity vectors $\dot{\mathbf{x}}$ have to lie tangential to the stream functions a necessary condition for a stream function to be closed is $\int \dot{\mathbf{x}}^T \mathbf{n}\ dS = 0$. That means it is in general easier to find a set of closed stream functions in a *solenoidal flow field*. That is the reason why we will split up a general flow field in the following in a solenoidal and non-solenoidal part first. A closed stream function is then searched in the solenoidal field. The non-solenoidal part (e.g. damping) is then superimposed as secondary flow. Since this secondary flow field often represesents a dissipative system it can be expected that it will stabilize the primary flow.

A closed stream function can only be found in a certain region if the system trajectories do not blow up in this region. That means it is desired to have e.g. closed particle trajectories (limit cycles) in the solenoidal part. If the natural system is unbounded (as, e.g., in a vehicle with constant velocity) the control input $\mathbf{u}$ or error-feedback in a observer design can be used to modell e.g. virtual springs to prevent the system from blowing up. Once it is assured that the system trajectories in a certain region can fit in a set of closed stream functions the stream function itself (e.g. the Hamiltonian) can be searched.

Consider for example a conservative multibody system, which represents a special case of a Hamiltonian system:

$$\begin{aligned} \dot{\mathbf{q}} &= \mathbf{p} \\ \dot{\mathbf{p}} &= \mathbf{f}(\mathbf{p}, \mathbf{q}, \mathbf{u}) \end{aligned}$$

where $\mathbf{q}$ represents the positons and $\mathbf{p}$ represents the velocities of the system. With the help of the control inputs $\mathbf{u}$ at the joint links the system is supposed to follow a desired trajectory $\mathbf{q}_d(t), \mathbf{p}_d(t) = \dot{\mathbf{q}}_d(t)$.

Natural reference frames for a multibody system are the desired trajectories of each link. These desired trajectory frames can be regarded as inertial frames if the control input $\mathbf{u} = \mathbf{u}^* + \mathbf{u}_{inertia}$ cancels all the inertial forces (Coriolis, centrifugal etc.) of these moving reference frames. A potential stream function is the total energy, that is the sum of the kinetic and potential energy of the system then. Since $\tilde{\mathbf{p}} = \mathbf{p} - \mathbf{p}_d$ are inertial velocities now the kinetic energy has its global minima for $\tilde{\mathbf{p}} = \mathbf{0}$. The control input $\mathbf{u}$ can be used to model virtual springs at the joints and to cancel e.g. the gravity forces to assure that even the potential energy has its global minima at $\tilde{\mathbf{q}} = \mathbf{q} - \mathbf{q}_d$. And a closed stream function is found. Damping elements as a function of $\tilde{\mathbf{p}} = \mathbf{p} - \mathbf{p}_d$ assure that this energy decreases to the desired postion. The cancelled inertia forces are the inertia forces of the desired movement. Since the desired movement is well known this design should not be as sensitive to parameter uncertainities as feedback linearization, that cancels the whole inertia forces. In principle this tracking controller generalizes PD energy-based controllers for stabilization. The same method could also be used to model an energy-based observer just with the help of the position measurements of a multibody system. Error-feedback in the velocities can be achieved with the coordinate error feedback illustrated in [Lohmiller and Slotine, 1996].

7. Concluding Remarks

Considering the flow of a nonlinear system at each given point in space rather than individual trajectories, leads to different analysis and design tools compared to the common Lagrange-based methods. [Lohmiller and Slotine, 1996] analyze a fluid flow with a differential concept conditioned by the Jacobian of the system. This method leads to a spherical concept of length. After summarizing and extending our earlier results, this paper shows thatimportant systems aare special classes of potential and solenoidal flows. These two flows indeed allow an integral formulation of a dynamical system. The concept of length can also be introduced with the help a stream function, that is closely related to a Lyapunov function. One important stream function is the energy of a system. For 2 dimensions is the stream function given by the vector potential. Current research focuses on a method that gives a general stream function as a function of the vector potential that itself can be found in a straightforward way in a solenoidal flow field. This would lead to a straightforward Lyapunov function for systems with mainly solenoidal or Hamiltonian dynamics.

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